Recent Advances In Entropy: A New Class Of Generalized Entropy

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***Abstract*—In this paper different types of entropies have been discussed with their mathematical expressions and conditions. Shannon entropy, α-order entropy, Alpha/Beta order entropy, residual entropies, past entropies, weighted entropies and quantile based entropies have been studied in this paper. Some new entropies have been derived by introducing hazard rate as an information function in Shannon (1948), Renyi (1961), Havrda & Charvart (1967) and Tsallis (1988). These new entropies gives higher results as compare to the Renyi, Havrda & Charvat and Tsallis entropies when alpha less than 1.**

Keywords— Entropy; Residual entropy; Past entropy; weighted entropy; quantile based entropy

1. INTRODUCTION

The origin of information theory is connected with the mathematical theory of communication. A mathematician Shannon (1948) has the major contribution in the communication channels. He gave the idea of "measuring the amount of information". Shannon (1948) introduced the concept of information function using the logarithm reciprocal of the event's probability. Information function by Shannon (1948) is as follows:

$$h(\pi_i) = \log\left(\frac{1}{\pi_i}\right) \quad \text{or} \quad h(\pi_i) = -\log(\pi_i)$$

1.1 Entropy

An information function is the measure of the amount of information then entropy is the average of the amount of information. Let $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ be the events of a finite sample space Ω, having probabilities $\pi_1, \pi_2, \pi_3, \ldots, \pi_n$ respectively. The mathematical expression of the entropy is given below:

$$H(\pi_i) = -E\big(log(\pi_i)\big) \tag{1.2}$$

or

$$H(\pi_i) = -\sum_{i=1}^{n} \pi_i log(\pi_i)$$ (Discrete probability distribution)

$$H(X) = -\int_{-\infty}^{+\infty} f(x) ln(f(x)) dx$$ (Continues probability distribution)

The entropy of a communication system gives the quantitative amount of uncertainty and interpretation of the information. Entropy has different interpretations; thermodynamic interpretation and statistical interpretation. In a communication system, higher value of entropy describes the low information. Statistical entropy has a different interpretation; higher entropy of a data shows that there is high randomness in the data. Shannon entropy has some important properties which are given below:

i. **Non-negative:** Shannon entropy gives the non-negative results. i.e. $H(\pi_i) > 0$

ii. **Continuous:** $H(\pi_i)$ is a continuous function of π_i. So the small change in π_i will gives the small change in $H(\pi_i)$.

iii. **Symmetric:** Entropy has no change if the outcomes are reshuffling or reordering.
$$H(\pi_1, \pi_2) = H(\pi_2, \pi_1)$$

iv. **Maximized:** The measure gives the maximum result when all events have the equal probability of occurrence.

v. **Additive:** The joint entropy of two independent probability distributions is equal to the sum of their individual entropies.
$$H(X,Y) = H(X) + H(Y)$$

vi. **Normality:** The entropy measures give the result '1' when events have equal probability.
$$H(0.5,0.5) = 1.$$

The main objective of the paper is to provide the detail review of different types of entropies. The rest of the paper is based on the definition of residual and past entropy which is expressed in section 2. The generalization of entropies has been discussed in section 3. Literature on application of entropy in different filed is given in section 4. New generalization of Shannon entropy is derived in section 5. Finally, the concluding the remarks is given in section 6.

2. RESIDUAL ENTROPY AND PAST ENTROPY

The concept of the residual entropy was not too old. From the last two decades, researchers produced flood of papers on residual entropy. Ebrahimi (1996), first time introduced the residual entropy.

The uncertainty of any component of a system is measured by the information about its current age; Shannon (1948) entropy is not preferable in this scenario. Ebrahimi (1996) studied the situations of current age information and produced new entropy. The survival function of the system is involved in this entropy. This type of entropy is known as residual entropy in the literature. Shannon's entropy is the special case of this entropy, when the age 't' is zero.

Residual entropy is widely used in the life sciences and engineering. It is useful in the situation when uncertainty of any component of a system is measured by the information about its current age. Residual entropy is based on survival function or the distribution function of the life distributions. The formal expression of residual entropy is:

$$H(F,t)=-\int_t^{\infty}\frac{f(x)}{1-F(t)}\ln\left(\frac{f(x)}{1-F(t)}\right)dx \qquad (2.1)$$

where $f(x)$ and $F(t)$ are the density function and distribution function respectively.

Residual entropy deals with the current age of the component and past entropy is the opposite of the residual entropy. Most of the time in practical situations, researchers want to measure the uncertainty, related to past rather than the future. Let's assume, a specific system is observed for a specified time period '*t*', and uncertainty is to be measured for that period. In that situation past entropy is preferable rather than the residual entropy or Shannon entropy. Crescenzo & Longobardi (2002) have studied the residual entropy and provided the concept of past entropy.

$$\overline{H}(F,t)=-\int_0^{t}\frac{f(x)}{F(t)}\ln\left(\frac{f(x)}{F(t)}\right)dx \qquad (2.2)$$

where '*t*' is specified period of observation.

Rao et al. (2004) delivered a new concept of residual entropy. They derived cumulative residual entropy using the survival function instead of the probability density function. They replaced probability density function with survival function in Shannon's entropy and found a new form of entropy. Crescenzo & Longobardi (2002, 2004 and 2009), extended the Ebrahimi (1996) entropy and developed a new form of the residual entropy and its applications.

On some important aspects residuals entropy, Ebrahimi & Kirmani (1996) and Gupta & Nanda (2002) gave the detailed overview.

Abrham & Sankaran (2006) and Nanda & Paul (2006) introduced the theory of generalized residual entropy. They derived some important characteristics and properties of generalized residual entropy. Baig & Javid (2008a and 2008b) investigated the survival exponential entropies and produced a generalized survival exponential entropy with its application.

Sunoj&Lino (2010, 2017), Sunoj et al. (2018), Sati & Gupta (2015), Thapliyal&Taneja (2015), Thapliyal et al. (2013) produced their research on dynamic residual entropy, cumulative residual Tsallis entropy, Renyi entropy of order statistics and dynamic cumulative residual entropy. They also derived the characteristics of the entropies.

Ramdan (2018) developed weighted entropy with some distinct properties. He also extended the concept of residual and past entropy. In his study, he introduced the weighted residual entropy and weighted past entropy for the reliability theory.

Following residuals and past entropies are available in the literature.

i. **Rao et al. (2004) Entropy:**
Rao et al. (2004) derived a cumulative residual entropy by introducing the survival function in entropy. The formal expression of entropy is given below:

$$\varepsilon(X)=-\int_0^{\infty}\overline{F}(x)\ln\overline{F}(x)\,dx$$

ii. **Renyi (2004) Residual Entropy:**
Renyi residual entropy is the extension of Renyi (1961) entropy. This residual entropy is also generalized entropy.

$$H_\alpha(X)=\frac{1}{1-\alpha}\ln\int_t^{\infty}\left\{\frac{f(x)}{1-F(t)}\right\}^{\alpha}dx;\quad \alpha>0\ \&\ \alpha\neq1$$

iii. **Crescenze (2009) Past Entropy:**
Crescenze introduced the concept of past entropy and derived past entropy (2.2). He extended his own work, and further modified the past entropy.

$$\wp(F,t)=-\int_0^{t}\frac{F(x)}{F(t)}\ln\left(\frac{F(x)}{F(t)}\right)dx$$

iv. **Sunoj&Lino (2010) Residual Entropy:**
Sunoj&Lino derived a generalization of cumulative residual entropy. The mathematical expression is as follows:

$$H_\beta(X)=\frac{1}{1-\beta}\ln\int_0^{\infty}\left(1-F(x)\right)^{\beta}dx;\qquad \beta>0\ \&\ \beta\neq1$$

3. GENERALIZED ENTROPY MEASURES

In the literature various researchers provides the generalization of Shannon's entropy. They introduced an additional parameter in the entropy and generalized the Shannon's entropy. Renyi (1961) studied the uncertainty & randomness and introduced generalized entropy. In this mathematical model or entropy, he addressed the degree of randomness. Renyi's entropy is the generalization of the Shannon's entropy. He used a constant 'α' in entropy and studied the behavior of entropy. In his entropy constant 'α' has conditions in which entropy gave the positive result, other than these conditions entropy is negative which has no interpretation or it is meaningless.

Varma (1966) investigated the generalized entropy and derived another generalization by introducing two different constants α and β. In this entropy, he imposed different constraint on constant α and β which produced the better result of entropy. Generalized entropy did not converge when α and β have the same values. Kapur (1967) derived a more generalized form of the Varma's generalization.

Havrda&Charvat (1967) introduced the concept of structural α- entropy. It is also a generalization of Shannon's entropy, which differs from the Renyi's entropy. Havrda&Charvat's entropy was used as a quantitative measure of classification. Renyi's generalization and Havrda&Charvat generalization didn't converge when α is one. Arimoto (1971) enhanced the concept of structural α-entropy and derived the more generalized form of the structural α-entropy. Sharma & Mittal (1977) and Boekee& Lubbe (1980), also enhanced the concept of structural α-entropy and produced more generalized form of Arimoto's generalization. Sharma &Taneja (1975) introduced an additive function for the generalized entropy. In this quantitative model of measure of uncertainty, they used two constants α and β with some constraints.Awad et al. (1987) studied Shannon (1948), Renyi (1961) and Havrda&Charvat (1967) entropies and included the function sup f(x) in their entropies.

I. α-order entropies

i. **Renyi (1961) Entropy:**

Renyi introduced the generalization of Shannon's Entropy by introducing the parameter α, which is called α-order entropy.

$$H_\alpha(X)=\frac{1}{1-\alpha}\ln\left\{E\left(f^{\alpha-1}\right)\right\};\qquad \alpha>0\ \&\ \alpha\neq 1$$

ii. **Havrda&Charvat (1967) Entropy:**

Havrda&Charvat also introduced the generalized entropy with parameter α. The mathematical expression is given below:

$$H_\alpha(X)=\frac{1}{2^{1-\alpha}-1}\left\{E\left(f^{\alpha-1}\right)-1\right\};\qquad \alpha>0\ \&\ \alpha\neq 1$$

iii. **Arimoto (1971) Entropy:**

Arimoto extended the work of Havrda&Charvat and derived the further generalization which is given below:

$$A_\alpha(X)=\frac{1}{2^{\alpha-1}-1}\left\{\left(E\left(f^{1/\alpha-1}\right)\right)^{\alpha}-1\right\};\ \alpha>0\ \&\ \alpha\neq 1$$

iv. **Sharma & Mittal (1977) Entropy:**

Sharma & Mittal produced another generalization of Shannon's entropy with addition parameter.

$$H_\alpha(X)=\frac{1}{2^{1-\alpha}-1}\left(e^{(\alpha-1)E(\ln f)}-1\right);\qquad \alpha>0\ \&\ \alpha\neq 1$$

v. **Boekee&Lubbe (1980) Entropy:**

Another generalization of Shannon entropy by introducing Boekee&Lubbe is given below:

$$H_\alpha(X)=\frac{\alpha}{\alpha-1}\left\{1-\left(E\left(f^{\alpha-1}\right)\right)^{1/\alpha}\right\};\qquad \alpha>0\ \&\ \alpha\neq 1$$

vi. **Awad et al. (1987) Entropies:**

Awad et al. (1987) introduced the concept of super-mum in entropies and they further generalized the Renyi, Shannon and Havrda&Charvat entropies.

$$A_{w1}(X)=-E\left(\ln\left(\frac{f}{\delta}\right)\right)$$

where δ is sup-mum of '*f*'

$$A_{w2}(X)=\frac{1}{1-\alpha}\ln\left(E\left(\frac{f^{\alpha-1}}{\delta^{\alpha-1}}\right)\right);\quad \alpha>0\ \&\ \alpha\neq 1$$

$$A_{w2}(X)=\frac{1}{2^{1-\alpha}-1}\left(E\left(\frac{f^{\alpha-1}}{\delta^{\alpha-1}}\right)-1\right);\qquad \alpha>0\ \&\ \alpha\neq 1$$

vii. **Tsallis (1988) Entropy:**

Tsallis gives another generalization of Shannon entropy.

$$H_\alpha(X)=\frac{1}{\alpha-1}\left\{1-E\left(f^{\alpha-1}\right)\right\};\quad \alpha>0\ \&\ \alpha\neq 1$$

II. α, β-Order Entropies

Some other generalization based on two additional parameters (α, β) are available in the literature.

i. **Varma (1966) Entropy:**

Varma has introduced generalized entropy having two additional parameters (α, β). This is the further generalization of Renyi (1961) entropy.

$$H_\alpha^\beta(X)=\frac{1}{\beta-\alpha}\ln\left\{E\left(f^{\alpha+\beta-2}\right)\right\};\ \alpha\ \&\ \beta>0\ \&\ \alpha\neq\beta$$

ii. **Kapur (1967) Entropy:**

Kapur also derived generalized entropy by introducing two parameters (α, β). In this entropy he used the ratio of two averages.

$$H_\alpha^\beta(X)=\frac{1}{1-\alpha}\ln\left\{\frac{E\left(f^{\alpha+\beta-2}\right)}{E\left(f^{\beta-1}\right)}\right\};\qquad \alpha>0\ \&\ \alpha\neq 1$$

iii. **Sharma &Taneja (1975) Entropy:**

In this generalized entropy, they used the subtraction of two averages. The mathematical expression of this entropy is as follows:

$$H_{\alpha}^{\beta}(X)=\frac{1}{2^{1-\beta}-2^{1-\alpha}}\left\{E\left(f^{\beta-1}\right)-E\left(f^{\alpha-1}\right)\right\};\quad \alpha\&\beta>0\&\alpha,\beta\neq 1$$

Dey et al. (2016) compared different entropies for the Rayleigh distribution using the relative loss function. Similarly, Mahdy&Eltelbany (2017) derived the entropy measures for Nakagami-μ distribution and size-biased Nakagami-μ. Basit et al. (2017) compared different entropies for the truncated weighted exponential distribution and weighted exponential distribution. They compared the entropies using the relative loss of entropies. Maszczyk&Duch (2008) also compared the Shannon, Renyi and Tsallis entropies. Majumdar&Sood (2012) studied properties of statistical entropies. Ren &Lichun (2002) derived a statistical entropy of Kerr black hole.

4. APPLICATION OF ENTROPY

Information theory is playing an important role in different fields of research. In the last few decades, its role growing positively in many other fields of sciences like Biology, Physics, Chemistry, Psychology, Mathematical Statistics, Fuzzy set theory, Economics, Operational research, Biotechnology, Mechanics, Genetics, Reliability Theory and etc..

Quastler (1953, 1954) introduced the information theory in Biology, Chemistry and Psychology, Theil (1967) introduced in Economics and Mathematical Statistics. Kerridge (1961) worked on information theory in statistical estimation, Zadeh (1996) used information theory in Fuzzy set theory and many other researcher used information theory in different fields. Kovalev (2016) described the role of entropy in economics and identified its misuse. The principle of maximum entropy was stated by Jaynes (1975), which is widely applicable in the information theory. Purvis et al. (2019) explained the three different definitions of entropy. They produced the application of entropy to urban system. Martos et al. (2018) studied stochastic process and derived some different entropies with their applications. Chen et al. (2019) studied the sample entropy and permutation entropy with reference to the measuring complexity of time series.

Lin &Oyapero (2012) used the entropy and wavelet variance for determine the sequence of DNA. They found that variance of wavelet and entropy has some similarities. They analyze the two regions of DNA (coding and non-coding) and found some similarities on the basis of entropy results and wavelet variance. They stated that variance and entropy are the measures of randomness, disordering, variation and uncertainty. The use of entropy as a measure of uncertainty is rapidly increasing.

Yang & Burn (1994) used the entropy method for evaluation of stream gauge network in southern Manitoba, Canada. Krstanovic& Sing (1992), Ozkul et al. (2000), Sarlak&Sorman(2006), Yoo et al. (2008), Jordan & Jason (2010), Ridolfi et al. (2011), Vivekanandan et al. (2012) applied the entropy approach for evaluation of different rain gauge networks and streamflow networks.

Sheraz et al. (2015), studied the market volatilities using the different entropy measures. In this study, they find out the numerical results of Shannon (1948), Renyi (1961) and Tsallis (1988) entropies for the weekly and monthly stock exchange of Hong Kong, Paris, Milan and Singapore index. They stated that, entropy measures are preferable for assessing the market volatility. Dionisio et al. (2007) also used the entropy for the uncertainty analysis of financial markets.

5. NEW CLASS OF GENERALIZED ENTROPY

By using the concept of Shannon and Tsallis entropy, a new entropy is introducing using the failure rate or hazard rate of the distribution. As Shannon entropy is the negative average of the logarithm of *f*, similarly new entropy is also negative average of the logarithm of hazard function.

$$H_{New1}(X)=1-E\left(\ln\left(r(x)\right)\right) \qquad (5.1)$$

where $r(x)$ is the hazard function of the *f*.

The new entropy $H_{New1}(X)$ produces the same results as the Shannon (1948) entropy measure.

By expanding (5.1):

$$H_{New1}(X)=1-E\left(\ln\left(\frac{f(x)}{1-F(x)}\right)\right)$$

$$H_{New1}(X)=1-E\left(\ln\left(f(x)\right)\right)+E\left(\ln\left(1-F(x)\right)\right)$$

$$H_{New1}(X)=1+H(X)+E\left(\ln\left(1-F(x)\right)\right)$$

Consider

$$E\left(\ln\left(1-F(x)\right)\right)=\int_{-\infty}^{+\infty} f(x)\ln\left(1-F(x)\right)dx$$

$$E\left(\ln\left(1-F(x)\right)\right)=-1$$

$$\Rightarrow$$

$$H_{New1}(X)=1+H(X)-1=H(X)$$

Following is the new class of generalized entropies which are based on the hazard rate of the probability density function. These are the extension of Renyi (1961), Havrda&Charvat (1967) and Tsallis (1988) entropies respectively. These entropies are preferable for the life sciences and reliability theory.

$$H_{New2}(X)=\frac{1}{1-\alpha}\ln\left[E\left(\left(\frac{r(x)}{e}\right)^{\alpha-1}\right)\right];\quad \alpha>0\ \&\ \alpha\neq 1$$

(5.2)

$$H_{New3}(X)=\frac{1}{2^{1-\alpha}-1}\left[E\left(\left(\frac{r(x)}{e}\right)^{\alpha-1}\right)-1\right];\quad \alpha>0\ \&\ \alpha\neq 1$$

(5.3)

$$H_{New4}(X)=\frac{1}{\alpha-1}\left[1-E\left(\left(\frac{r(x)}{e}\right)^{\alpha-1}\right)\right];\quad \alpha>0\ \&\ \alpha\neq 1$$

(5.4)

5.1 Characteristics of New Entropies

The above new entropy H_{New1} produces the Shannon entropy. This characteristic of H_{New1} leads us to introduce the hazard rate as an information function in Renyi, Havrda&Charvat and Tsallis entropies. These new generalized entropies H_{New2}, H_{New3} and H_{New4} gives the Renyi, Havrda&Charvat and Tsallis entropies respectively with some conditions.

Case 1:

The new entropy H_{New2} is described in (5.2):

$$H_{New2}(X)=\frac{1}{1-\alpha}\ln\left[E\left(\left(\frac{r(x)}{e}\right)^{\alpha-1}\right)\right];\quad \alpha>0\ \&\ \alpha\neq 1$$

$$H_{New2}(X)=\frac{1}{1-\alpha}\ln\left[E\left(r(x)^{\alpha-1}e^{1-\alpha}\right)\right]$$

$$=\frac{1}{1-\alpha}\ln\left[E\left(r(x)^{\alpha-1}\right)\right]+\frac{1}{1-\alpha}\ln\left(e^{1-\alpha}\right)$$

$$=1+\frac{1}{1-\alpha}\ln\left[E\left(r(x)^{\alpha-1}\right)\right]$$

$$=1+\frac{1}{1-\alpha}\ln\left[E\left(\left(\frac{f(x)}{1-F(x)}\right)^{\alpha-1}\right)\right]$$

$$=1+\frac{1}{1-\alpha}\ln\int_0^\infty f(x)^\alpha\left(1-F(x)\right)^{1-\alpha}dx$$

As $(1-z)^{-r}=\sum_{\beta=0}^{\infty}\frac{\Gamma(r+\beta)}{\Gamma r}\frac{z^\beta}{\beta!};\quad 0\le Z\le 1$

$$=1+\frac{1}{1-\alpha}\ln\int_0^\infty\left\{f(x)^\alpha\sum_{\beta=0}^{\infty}\frac{\Gamma(\alpha-1+\beta)}{\Gamma(\alpha-1)\beta!}\left(F(x)\right)^\beta\right\}dx$$

$$=1+\frac{1}{1-\alpha}\ln\int_0^\infty\left\{f(x)^\alpha+\sum_{\beta=1}^{\infty}\frac{\Gamma(\alpha-1+\beta)}{\Gamma(\alpha-1)\beta!}f(x)^\alpha\left(F(x)\right)^\beta\right\}dx$$

$$=1+\frac{1}{1-\alpha}\ln\left\{\int_0^\infty f(x)^\alpha dx+\sum_{\beta=1}^{\infty}\frac{\Gamma(\alpha-1+\beta)}{\Gamma(\alpha-1)\beta!}\int_0^\infty f(x)^\alpha\left(F(x)\right)^\beta dx\right\}$$

$$=1+\frac{1}{1-\alpha}\ln\left\{\int_0^\infty f(x)^\alpha dx+\sum_{\beta=1}^{\infty}\frac{\Gamma(\alpha-1+\beta)}{\Gamma(\alpha-1)\beta\Gamma\beta}\int_0^\infty f(x)^\alpha\left(F(x)\right)^\beta dx\right\}$$

$$=1+\frac{1}{1-\alpha}\ln\left\{\int_0^\infty f(x)^\alpha dx+\sum_{\beta=1}^{\infty}\frac{1}{\beta B(\alpha-1,\beta)}\int_0^\infty f(x)^\alpha\left(F(x)\right)^\beta dx\right\}$$

$$=1+\frac{1}{1-\alpha}\ln\left\{\int_0^\infty f(x)^\alpha dx+\sum_{\beta=1}^{\infty}\frac{1}{\beta B(\alpha-1,\beta)}\int_0^\infty f(x)^\alpha\left(F(x)\right)^\beta dx\right\}$$

$$H_{New2}(X)=1+\frac{1}{1-\alpha}\ln\left\{E\left(f(x)^{\alpha-1}\right)+\sum_{\beta=1}^{\infty}\frac{E\left(f(x)^{\alpha-1}F(x)^\beta\right)}{\beta B(\alpha-1,\beta)}\right\}$$

Let *'f'* represents the *f(x)* and *'F'* represents *F(x):*

$$H_{New2}(X)=1+\frac{1}{1-\alpha}\ln\left\{E\left(f^{\alpha-1}\right)\left(1+\frac{\sum_{\beta=1}^{\infty}\frac{E\left(f^{\alpha-1}F^\beta\right)}{\beta B(\alpha-1,\beta)}}{E\left(f^{\alpha-1}\right)}\right)\right\}$$

$$H_{New2}(X)=1+\frac{1}{1-\alpha}\ln E\left(f^{\alpha-1}\right)+\frac{1}{1-\alpha}\ln\left(1+\frac{\sum_{\beta=1}^{\infty}\frac{E\left(f^{\alpha-1}F^\beta\right)}{\beta B(\alpha-1,\beta)}}{E\left(f^{\alpha-1}\right)}\right)$$

$$H_{New2}(X)=1+\mathrm{Renyi}+\frac{1}{1-\alpha}\ln\left(1+\frac{\sum_{\beta=1}^{\infty}\frac{E\left(f^{\alpha-1}F^\beta\right)}{\beta B(\alpha-1,\beta)}}{E\left(f^{\alpha-1}\right)}\right)$$

$$H_{New2}(X)=1+\mathrm{Renyi}+\frac{1}{1-\alpha}\ln\left(1+\frac{\sum_{\beta=1}^{\infty}\frac{E\left(f^{\alpha-1}F^\beta\right)}{\beta B(\alpha-1,\beta)}}{E\left(f^{\alpha-1}\right)}\right)$$

(5.5)

$$\alpha\ \&\ \beta>0,\ \alpha\neq 1\ \&\ \beta\neq 0$$

Case 2:

The new entropy H_{New3} is described in (5.3):

$$H_{New3}(X)=\frac{1}{2^{1-\alpha}-1}\left[E\left(\left(\frac{r(x)}{e}\right)^{\alpha-1}\right)-1\right]$$

$$H_{New3}(X)=\frac{1}{2^{1-\alpha}-1}\left[E\left(r(x)^{\alpha-1}e^{1-\alpha}\right)-1\right]$$

$$=\frac{1}{2^{1-\alpha}-1}\left[e^{1-\alpha}E\left(r(x)^{\alpha-1}\right)-1\right]$$

From H_{New2}

$$E\left(r(x)^{\alpha-1}\right)=E\left(f^{\alpha-1}\right)\left(1+\frac{\sum_{\beta=1}^{\infty}\frac{E\left(f^{\alpha-1}F^{\beta}\right)}{\beta B(\alpha-1,\beta)}}{E\left(f^{\alpha-1}\right)}\right)$$

$$\Rightarrow H_{New3}(X)=\frac{1}{2^{1-\alpha}-1}\left[E\left(f^{\alpha-1}\right)\left(1+\frac{\sum_{\beta=1}^{\infty}\frac{E\left(f^{\alpha-1}F^{\beta}\right)}{\beta B(\alpha-1,\beta)}}{E\left(f^{\alpha-1}\right)}\right)e^{1-\alpha}-1\right]$$

(5.6)

$$\alpha\,\&\,\beta>0,\alpha\neq 1\,\&\,\beta\neq 0$$

Case 3:

The new entropy H_{New4} is described in (5.4):

$$H_{New4}(X)=\frac{1}{\alpha-1}\left[1-E\left(\left(\frac{r(x)}{e}\right)^{\alpha-1}\right)\right];\qquad \alpha>0\,\&\,\alpha\neq 1$$

$$\Rightarrow H_{New4}(X)=\frac{1}{\alpha-1}\left[1-E\left(f^{\alpha-1}\right)\left(1+\frac{\sum_{\beta=1}^{\infty}\frac{E\left(f^{\alpha-1}F^{\beta}\right)}{\beta B(\alpha-1,\beta)}}{E\left(f^{\alpha-1}\right)}\right)e^{1-\alpha}\right]$$

(5.7)

$$\alpha\,\&\,\beta>0,\alpha\neq 1\,\&\,\beta\neq 0$$

Example 5.1

Considering exponential distribution with pdf $f=f(x)=\theta e^{-\theta x},\ x>0,\theta>0,$ and $F=F(x)=1-e^{-\theta x}$, the results of these new entropies are as follows:

Entropies	New Entropies
$Shannon1948=1-\ln(\theta)$	$H_{New1}(X)==1-\ln(\theta)$
Comparison: No Change	
$Renyi1961=\frac{1}{1-\alpha}\ln\frac{\theta^{\alpha-1}}{\alpha}$	$H_{New2}(X)=1-\ln(\theta)$
Comparison: $R>H_{New2};iff\ \frac{\alpha}{\alpha-1}>1$	
$HC1967=\frac{1}{2^{1-\alpha}-1}\left(\frac{\theta^{a}}{\alpha\theta}-1\right)$	$H_{New3}(X)=\frac{1}{2^{1-\alpha}-1}\left[\left(\frac{\theta}{e}\right)^{\alpha-1}-1\right]$
Comparison: $HC>H_{New3};iff\ \frac{1}{\alpha}<e^{1-\alpha}$	
$Tsallis1988=\frac{1}{\alpha-1}\left\{1-\frac{\theta^{\alpha-1}}{\alpha}\right\}$	$H_{New4}(X)=\frac{1}{\alpha-1}\left[1-\left(\frac{\theta}{e}\right)^{\alpha-1}\right]$
Comparison: $T>H_{New4};iff\ \frac{1}{\alpha}<e^{1-\alpha}$	

6. CONCLUSION

In the life sciences and reliability theory, hazard rate or failure rate has an important position. The new entropy describe that logarithmic ratio of hazard rate and exponential function i.e. $-\ln\left(\frac{r(x)}{e}\right)$ is an information function. The result of this information function is same as the result of Shannon (1948). The new information function used in Renyi, Havrda&Charvat and Tsallis entropies and derived the results for exponential distribution. The new entropies H_{New2}, H_{New3}, and H_{New4} gives the higher value of entropies when $\alpha<1$and Renyi, Havrda&Charvat and Tsallis has higher value when $\alpha>1$.

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