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A Note On Beta Expansion And Numbers Badly Aprroximable

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***Abstract*— Since antiquity, mathematics has been fundamental to advances in science, engineering, and philosophy. Mathematics and humanity itself are nearly at the same age. Sumerian and Babylonian mathematics was based on a sexegesimal, or base 60, numeric system. It is thought that the Egyptians introduced the earliest fully-developed base 10 numeration system at least as early as 2700 BCE. We now use the Indu-Arab systems in educations books, integer bases for computer science , binary systems, or others [18]. Among the different possible representations of real numbers in a real (or complex) base β, the β -expansion introduced by Renyi´ is very important for advance in different maths fields. The present paper focuses on β -expansion of an algebraic number in an algebraic base β, with a point of view from Diophantine approximation. We are going to study the point which are called the sets of badly approximable number. They have an important role in classical diophantine approximation [1]. For the set**

$$F(c,\beta)=\{x\in\mathbb{S}\,|\beta^n x\geq c\,(mod\ 1)\ for\ all\ n\geq 0\}$$

From now and all this papers beta numbers are Pisot or Salem numbers . The algebraic numbers $\beta>1$ and their conjugate are less or iqual in modul to 1. The expansions of 1 in base β turns out to be crucial for characterizing the $\beta-$ shift. The closure of the $\beta-$ shift is totally determined by the expansions of 1. Some results are to be given on this papers.

***Keywords*— Beta-integer, Diophantine approximation, Pisot numbers**

I. Introduction

There is different possible representations of real numbers in a real (or complex) base , the β-expansion introduced by Renyi´ is very important in different math fields. The present paper focuses on β -expansion of an algebraic number in an algebraic base , with a point of view from Diophantine approximation. We are going to study the point which are called the sets of badly approximable number. They have an important role in classical diophantine approximation[1]. Let take the set

$$F(c,\beta)=\{x\in\mathbb{S}\,|\beta^n x\geq c\,(mod\ 1)\ for\ all\ n\geq 0\}$$

Our beta numbers are Pisot or Salem numbers . we try to give same considerations for set F(c,β) in this papers .

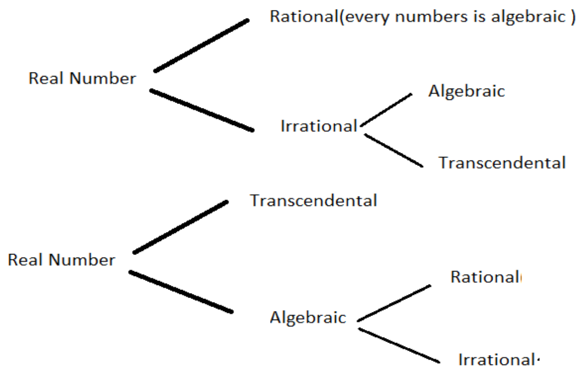
Definition 1. If a real number satisfies some equation of the the form

$$a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0=0$$

with integral coefficients, we say that it is an algebraic number.

If a real number satisfies no such equation, it is called a transcendental number.

So we can classificate numbers like the scheme [3]



Rational : $px-q=0$, where $(p,q)=1$; Irrational , Algebraic : $x^2-x-1=0$; Irrational , Transcendental : $\sqrt[3]{5}$, $\log_2 3,\pi$, Transcendental : $\sin 10^{\circ}$; Algebraic rational : $x^2-2x+1=0$; Algebraic Irrational : $x^3-ax^2+x-1=0, a\geq 2$.

The Gelfond-Schneider theorem establishes the transcendence of $\log r$, provided that r is rational and $\log r$ is irrational.

We give for a real number $x\geq 0$ by their decimal expansions

$$x=a_k10^k+a_{k-1}10^{k-1}+\cdots+a_1 10+a_0+a_{-1}10^{-1}+a_{-2}10^{-2}+\cdots$$

Where $a_k\in\{0,1,\cdots,9\}$ and $k\geq 0$.

Let $\beta>1$ be a real number that is not an integer.[6] A representation in base β of a positive real number x is in form :

$$x=a_k\beta^k+a_{k-1}\beta^{k-1}+\cdots+a_1\beta+a_0+a_{-1}\beta^{-1}+a_{-2}\beta^{-2}+\cdots$$

It is denoted by : $x=a_ka_{k-1}\cdots a_1a_0.a_{-1}a_{-2}\cdots$

A greedy expansions in base β of a positive real number x is in form

$$x=\sum_{i=-k}^{\infty}a_{-i}\beta^{-i}$$

With $a_{-i} \in \mathcal{A}_\beta = [0, \beta) \cap \mathbb{Z}$ and greedy condition [4]

$$\left| x - \sum_{N_0 \le k \le N} a_k \beta^{-k} \right| < \beta^{-N}$$

For all $N \ge N_0$.

$.a_{-1}a_{-2}\cdots$ is the fractional part of x denote by $\{x\}$. And $a_k a_{k-1} \cdots a_1 a_0$ is the integer part of x denote by $[x]$. The digit a_k obtained by greedy algorithm are integer from the set

$\mathcal{A}_\beta = \{0, \cdots, \beta - 1\}$ if β is an integer or the set $\mathcal{A}_\beta = \{0, \cdots, [\beta]\}$ if β is not an integer. This expansion for $x \in [0,1)$ is produced by iterating the beta transform :

$T_\beta : x \longrightarrow \beta x - [\beta x]$ where $[\beta x] \in \mathcal{A}_\beta$

Let $1 = d_{-1}\,\beta^{-1} + d_{-2}\,\beta^{-2} + \cdots$ be an expansion of 1 defined by the algorithm

$c_{-i} = \beta c_{-i+1} - [\beta c_{-i+1}]$, $d_{-i} = [\beta c_{-i+1}]$, with $c_0 = 1$

where $[x]$ denoted the maximal integer not exceeding x [8]. This expansion is achived as a trajectory of e $T_\beta^n(1)$ $(n = 1,2, \dots \dots)$. $d_\beta(1) = d_{-1}, d_{-2}, \cdots$ is called $\beta -$ expansion of 1. Parry [7] has shown that a sequence $x = a_1, a_2, \cdots$ of non negative integers give a $\beta -$ expansion of pozitiv real number iff satisfies lexicographical condition:[6]

$\forall p \ge 0$, $\sigma^p(d) < lex\ d^*(1)$

with

$$d^*(1) = \begin{cases} \mathrm{d}_\beta(1) & \text{if } \mathrm{d}_\beta(1) \text{ is infinite} \\ (d_{-1}, d_{-2} \dots, d_{-n+1}, (\mathrm{d}_{-n} - 1))^\omega & \text{if } \mathrm{d}_\beta(1) \\ & = \mathrm{d}_{-1}, \cdots \mathrm{d}_{-n} \end{cases}$$

where the string of simbols w , w^ω is a periodic expansions $w, w, \cdots$ and σ is the shift defined by

$$\sigma((a_i)_{i \le M}) = (a_{i-1})_{i \le M}$$

so the sequence $x = a_1, a_2, a_3 \dots$ is called admissible [8],[10].

A. *Pisot Number*

A Pisot- Vijayaraghavan number is a real algebraic integer $\beta > 1$, all of whose other Galois conjugates have absolut value less than 1. That is β satisfies a polynomial equarion of the form $P(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$, where the a_k are integer , $a_n \ne 0$ and the roots of $P(x)$ other than β all lie in the open unit circle $|x| < 1$ The set of this numbers in major case are denoted by S . Every pozitiv integer $n > 1$ is a Pisot number but a more interesting examples are that Pisot numbers like golden rate $\varphi = \frac{1+\sqrt{5}}{2}$. Every real number field $\mathbb{K}$ contains infinitely many Pisot numbers of degree equal to $[\mathbb{K} : \mathbb{Q}]$, and, in fact, every real number field $\mathbb{K}$ can be generated by Pisot numbers, even by Pisot units ($a_n = 1$). The Pisot numbers have the interesting property that if $0 < \lambda \in \mathbb{Z}(\beta)$ then $\|\lambda \beta^n\| \to 0$ as $n \to \infty$, where here $\|x\| = dist(x, \mathbb{Z})$ denotes the distance from x to the nearest integer. It is an open question whether this property characterizes S among the real numbers $\beta > 1$ (Pisot's conjecture). An important result of Ch. Pisot in this direction is that if $\beta > 1$ and $\lambda > 0$ are real numbers for which $\sum_{n=0}^{\infty} \|\lambda \beta^n\|^2 < \infty$ then $\beta \in S$ and $\lambda \in \mathbb{Q}(\beta)$. The unusual behaviour of the powers of Pisot numbers leads to applications in harmonic analysis,[13] , [15] , dynamical systems theory [15]] and the theory of quasi-crystals [13] . For example, if $\beta > 1$, then the set of powers $\{1, \beta, \beta^2, \cdots\}$ is harmonious if and only if β is a Pisot number or a Salem Number [15] . The Bragg spectrum of the diffraction pattern of a self-similar tiling is non-trivial if and only if the scaling factor of the tiling is a Pisot number[14] . A surprising fact is that S is a closed and hence nowhere-dense subset of the real line [15]. The derived sets $S^{(n)}$ are all non-empty and $\min_{n\to\infty} S^{(n)} \to \infty$. (Here $S' = S^{(1)}$ denotes the set of limit points of S , $S'' = S^{(2)}$ the set of limit points of S' .The order type of S is described in [12] . The smallest elements of S , S' and S'' are explicitly known [11]

There is an intimate relationship between the set S of Pisot numbers and the set T of Salem numbers. It is known that $S \subset T'$, It seems reasonable to conjecture that $S \cup T$ is closed and that $S = T'$, but it is not yet known whether or not T is dense in $[1, \infty)$

B. *Diophantine approximation*

On a metric space (X, d) we take sequences a_n , from X and sequences of positive real numbers b_n , and take in kosideration thes tow sets $M = \{m \in X | d(m, a_n) < b_n\ infinitely\ often\}$ and $N = X \setminus M$

We are interested on $a_{n,m} = \frac{m}{\beta^n}$ and $b_n = \frac{c}{\beta^n}$ for m integer and $0 \le c \le 1$

Theorem 1. [1](Dirichlet, 1842). For every irrational number α there are infinitely many rational numbers $\frac{r}{s}$ such that $0 < \left|\alpha - \frac{r}{s}\right| < \frac{1}{r^2}$

Theorem 2 [2](Liouville, 1844) .Let β be an algebraic number of degree $d \ge 1$. Then there is a constant

$c = c(\beta) > 0$ such tha t $\left|\alpha - \frac{r}{s}\right| > \frac{1}{r^d}$ for every choice of integers r and s

By the notation Diophantine approximation we shall mean the stady of the sets M and N. Let us make the following remark: if the sequence $\{a_n\}$ is dense in X then M is not empty and hence a residual set in the sence of Baire [2]. For the sequences $\{a_{n,m}\}_{n \in \mathbb{N}}$, $0 \le m < n$ with $a_{n,m} = \frac{m}{n}$ where $\gcd(m, n) = 1$ and with the particular choise of the sequence $a_n = \frac{c}{n^\alpha}$. For the special choise of $\{a_n\}$ and $\{b_n\}$ we are in the case of the classical diophantine approximation with rational numbers. It is well-known fact that if $\alpha > 2$ then the set $\mathcal{N}$ is non - empty set, but for $\alpha < 2$ it is empty. $\mathcal{N}(\alpha) = \left\{m \in X | d(m, a_n) < \frac{1}{n^\alpha}\ finitely\ often\right\}$. It have to look at this set for getting an α_0 such that $\mathcal{N}(\alpha) = \emptyset$ for $\alpha_0 < \alpha$ and $\mathcal{N}(\alpha) \neq \emptyset$ for $\alpha_0 > \alpha$. For this special value , we say that the set $\mathcal{N}(\alpha_0)$ is the set of Badly Approximable Numbers, BAN. We to put the dependence on an extra parameter, c , $\mathcal{N}_c(\alpha) = \left\{m \in X | d(m, a_n) < \frac{C}{n^\alpha}\ finitely\ often\right\}$. This leads in one-dimensional to continued fraction studied by Hygens while constracting a model of our solar systems. Continued fraction is $y = [a_0, a_1, a_2, \cdots]$

We are interested to beta – expansions of an algebraic numbers in an algebraic base beta with a point of Diophantine approximation. It is not an easy task to get beta –expansions for cases that this expansions is not finite or periodic. Let $\beta > 1$ be a real number that is not an integer.

Theorem 3 [1]. An irrational x is a BAN if and only if its partial denominators are bounded.

C. One- sided $\beta -$ shift Badly Approximable Numbers.

Recall the definition from [8] for $\beta > 1$ to the set $F(c, \beta)$ [1].

$F(c, \beta) = \{x \in \mathbb{S}\ | \beta^n x \geq c\ (mod\ 1)\ for\ all\ n \geq 0\}$

We have β-expansion of real number , and we study the set of

$F_\beta(c) = \{x \in S_\beta\ | \sigma^n\ (x) < d(1 - c, \beta) for\ all\ n \geq 0\}$

Where $\sigma -$ is a left shift $\sigma(x_0, x_1, x_2, \cdots) = (x_1, x_2, \cdots)$ and $\sigma^n = \sigma \circ \sigma^{n-1}$.

$d(x, \beta)$ is β-expansion of x, so x written on base β. We know that the β-expansion of 1, is very important for $d(x, \beta)$ [4].

Definition 1 [8]. The closure of the set of all β-expansion of $x \in [0,1]$ is called β –shift , denote by S_β.

For a fine sequences $v = d(1 - c, \beta)$ with w unique minimal prefix of $(\overline{w})^*$ define

$l(c, \beta) = 1 - \sum_{j=1}^{|w|} \frac{((\overline{w})^*)_j}{\beta^j}$ and

$L(c, \beta) = 1 - \sum_{j=1}^{\infty} \frac{((w^\infty)^*)_j}{\beta^j}$

Theorem 4 [1]. For any $c \in [0,1]$ the real numbers $l(c, \beta)$ and $L(c, \beta)$ are unique and such that $F_\beta(c) = F_\beta(l(c, \beta))$ and $d(1 - l(c, \beta), \beta) = d(1, B(c, \beta))$ with a unique $B(c, \beta) \leq \beta$.

Let take same examples of Pisot and their β-expansion of 1.

Examples 1 [9]. Polynomial $P(x) = x^3 - x - 1$ one of the root $\beta \approx 1{,}325$

$$d(1, \beta) = 10001$$

Greedy expansion of

$$x = \frac{1}{5}, d\left(\frac{1}{5}, \beta\right) = (000001000000001000000000)^\infty$$

Example 2 [17]. $P(x) = x^4 - 3x^3 + x^2 - 2x - 1$

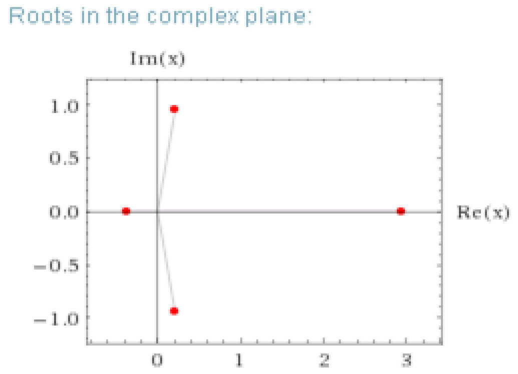


Figure 2. Roots of $P(x)$ in the compex plane

And a greedy $\beta -$ expansions of 1 is given by

$$1 = \frac{3}{\beta} - \frac{1}{\beta^2} + \frac{2}{\beta^3} + \frac{1}{\beta^4}$$

$$= \frac{2}{\beta} + \frac{2}{\beta^2} + \frac{1}{\beta^3} + \frac{3}{\beta^4} + \frac{1}{\beta^5}$$

$$= \frac{2}{\beta} + \frac{2}{\beta^2} + \frac{1}{\beta^3} + \frac{2}{\beta^4} + \frac{4}{\beta^5} - \frac{1}{\beta^6} + \frac{2}{\beta^7} + \frac{1}{\beta^8}$$

$$= \frac{2}{\beta} + \frac{2}{\beta^2} + \frac{1}{\beta^3} + \frac{2}{\beta^4} + \frac{4}{\beta^5} + \frac{2}{\beta^6} + \frac{1}{\beta^7} + \frac{3}{\beta^8} + \frac{1}{\beta^8}$$

$d_\beta(1) = 2, 2, 1, 3, 1 = 2, 2, 1, 2, 4, 2, 1\ 3, 1\ \cdots$

Example 3. Let root $\beta \approx 2{,}9$ and $c = 0.25$ we have $l(c, \beta) = L(c, \beta) = 0{,}25$

And the graph $c \mapsto dim_H F(0.25\ , 2.9)$

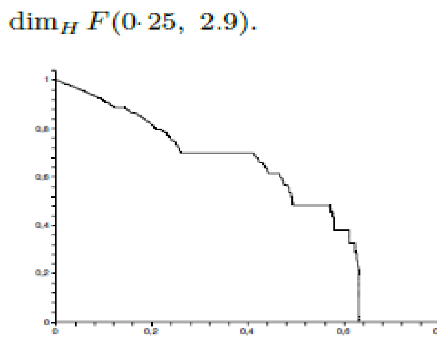


Figure 3 The graph of $c \mapsto \dim_H F(0{\cdot}25,\ 2.9)$.

Characterising the dimension of $F(0.25\ , 2.9)$via the expansion of 1 in a β-shift gives a neat way of calculating an approximative picture of the graph of $c \mapsto dim_H F(c\ , \beta)$.

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