Piecewise Linear Yield Surfaces In The Framework Of An Internal Variable Approach

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***Abstract*—The paper is centered on a non-traditional formulation that makes use of *internal variables* and *dissipation functions* in order to describe the response of elastic-plastic systems. A special role is played by the *dissipation functions*, which eventually enforce the *constitutive law* without making use of *yield functions*. The main consequence is that the governing equations eventually imply a solution of the incremental elastic-plastic problem, which corresponds to the minimum point of an *non-constrained convex function*, say ω, while classical approaches require the presence of *inequality constraints*, as suggested by the presence of *yield functions*.**

Next, an algorithm is briefly discussed, which is based on the so-called *backward-difference* iterative scheme and guarantees convergence to the solution when the material is *stable in Drucker's sense*, since it tends to reduce the value of the function ω iteration by iteration.

Finally, it is shown that the entire theoretical framework can be applied to the case of elastic-plastic systems for which *piecewise-linear yield surfaces* are assumed, so that the range of possible applications is practically unlimited, while (to the authors' knowledge) the existing literature on the *internal variable* approach discussed here has only considered *Mises' yield condition*, which leads to a straightforward definition of the corresponding *dissipation function*.

Keywords—backward-difference; convergence; convex analysis; dissipation functions; elastic-plastic materials; finite element method; internal variables; iterative schemes; piecewise-linear yield surfaces; yield functions

I. Introduction

This work deals with a non-traditional approach to the incremental analysis of elastic-plastic systems, which is due to Martin [1] and exploits the concept of the so-called *internal variables* (*i.e.*, non-measurable variables), which represent non-reversible plastic strains or displacements. The most interesting feature, however, is that adequate *dissipation functions* are introduced in order to enforce the constitutive law, without making use of *yield functions*.

The main consequence is that the governing equations (both at the material level and at the structural level) lead to the solution of the so-called *incremental elastic-plastic problem*, which coincides with the minimum point of a convenient *non-constrained convex function*, when the equilibrium equations are written by considering the initial, undeformed configuration (*small displacement hypothesis*) and the material is *stable in Drucker's sense* (*i.e.*, the *yield surfaces* are convex and the *flow rule* is associated).

It should be observed that, in this context, the expression *incremental elastic-plastic problem* is generally concerned with the discrete model of an *elastic-plastic system*, whose load history has been subdivided into a finite number of time-steps. In addition, an input vector (usually load increments) is known at the beginning of each time-step, as well as the initial nodal displacements, total strains, plastic strains and stresses (strains and stresses computed at convenient *strain points* or *stress points*), while the increments of the nodal displacements, total strains, plastic strains and stresses (again, increments of strains and stresses at the same *strain points* or *stress points*) are to be determined.

It can be shown [2, 3] that the value of the *non-constrained convex function* mentioned above progressively decreases and, hence, convergence towards the correct solution of the incremental problem is guaranteed, when the *backward-difference* iterative scheme is applied. Such *backward-difference* scheme is essentially aimed at satisfying the constitutive law by determining, at each *strain point*, an incremental plastic strain vector $\Delta\boldsymbol{\varepsilon}^P$, such that its direction coincides with the direction of the gradient $\partial\varphi/\partial\boldsymbol{\sigma}$ of the relevant *yield function* $\varphi(\boldsymbol{\sigma})=0$ computed for $\boldsymbol{\sigma}=\boldsymbol{\sigma}^o+\mathbf{D}\,(\Delta\boldsymbol{\varepsilon}-\Delta\boldsymbol{\varepsilon}^P)$, if $\boldsymbol{\sigma}^o$ is the stress vector at the beginning of the current time-step, $\mathbf{D}$ the material elastic stiffness matrix and $\Delta\boldsymbol{\varepsilon}$ the total incremental strain vector.

In the literature, further details can be found on the general features of this *internal variable* approach [4], for which all applications to the most general two-dimensional and three-dimensional problems, so far, have been concerned with materials for which *Mises' yield condition* is applicable (to the best of the authors' knowledge), since the relevant *dissipation function* can be defined in a straightforward way.

Here, we will discuss the possible application of the same procedure to the more general case of *piecewise linear* approximations of *yield surfaces*, which virtually allow one to consider any material model.

Elastic-plastic structural systems described by introducing *piecewise linear yield surfaces* have represented a major research topic mostly due to Maier [5-8] and have been applied to a large spectrum of topics (ranging from classical *elastic-plastic analysis* to *limit analysis*, from *shakedown theory* to *inverse problems*), in contexts in which the *yield functions* have always maintained their primary role and have led to inequality constraints. Here, the possible application of *piecewise linear yield surfaces* to the *internal variable approach* is discussed with the aim of deriving the governing equations of any *elastic-plastic system*, by considering convenient *dissipation functions*, without making use of inequality constraints.

II. An Internal Variable Approach: Some Basic Concepts

We will start by considering possible mechanical models of *elastic-plastic structural elements* subjected to uniaxial stress states. The fundamental component of these models is a *slip device*, which essentially consists of a *rigid, perfectly-plastic element*, which remains rigid unless the load (say χ) attains a *limit value* χ^+ or χ^-. When this condition is satisfied, it is assumed that unlimited plastic displacements λ can occur, which imply a dissipated energy $D=\chi^+\lambda$ or $D=\chi^-\lambda$.

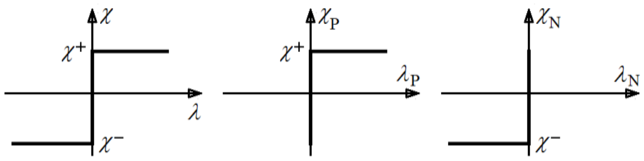

Fig. 1. *Typical χ-λ plots for slip devices.*

Typical χ-λ plots are reported in Fig. 1, while the corresponding *dissipation functions* are shown in Fig. 2. Here, the slopes of the straight lines passing through the origin are obviously in the range between χ^- and χ^+. In consequence, it is possible to establish a relationship between a function such as $D(\lambda)$ and the force χ acting on the slip device. Namely, χ equals the derivative of $D(\lambda)$ if $\lambda\neq0$, while $\chi\in\partial D(\lambda)$ if $\lambda=0$. Therefore, in this second case, χ is a subgradient of $D(\lambda)$ and, hence, an element of the subdifferential of $D(\lambda)$.

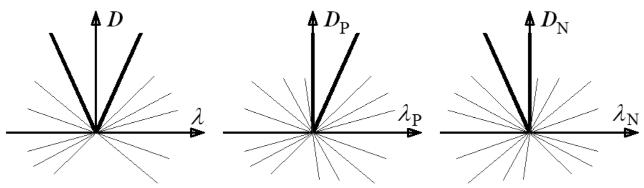

Fig. 2. *Typical dissipation functions for slip devices.*

Note that the symbols λ_P, χ_P, D_P and λ_N, χ_N, D_N in the above figures have been used to identify special slips, which can only be subjected to positive or negative displacements.

The slip devices can be combined with adequate springs in order to generate mechanical models, which describe the response of *elastic perfectly-plastic* or *hardening* materials. In what follows, we will focus on *linear* kinematic and isotropic hardening, but the extension to other materials (especially, *nonlinear hardening materials*) is quite straightforward.

The *elastic perfectly-plastic* model is simply obtained by connecting a slip device and a linear spring in series, while it is possible to describe the response of *kinematic hardening* materials by introducing another linear spring (*cf.* Fig. 3). Instead, two slips and a torsion spring are needed in the case of *isotropic hardening*, as shown in Fig. 4.

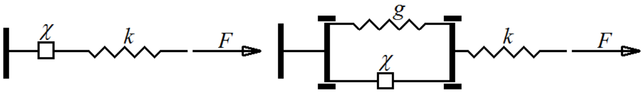

Fig. 3. *Mechanical models for elastic perfectly-plastic materials and elastic-plastic materials subjected to kinematic hardening.*

It can be easily checked that the response of the models in Fig. 3 is governed by the following equations

$$F = k\,u - k\,\lambda \quad , \quad -\chi = -k\,u + k\,\lambda + g\,\lambda \tag{1}$$

where u denotes the displacement of the free end, while g represents the stiffness of the second spring. Of course, we shall set g=0 for the model on the left hand side of Fig. 3, while we can introduce a convenient function $\psi(\lambda)$ such that $\chi=k(u-\lambda)-\mathrm{d}\psi/\mathrm{d}\lambda$, if we must deal with systems characterized by *nonlinear hardening*. In any case, the mechanical models in Fig. 3 will always feature an elastic range given by $\chi^+-\chi^-$.

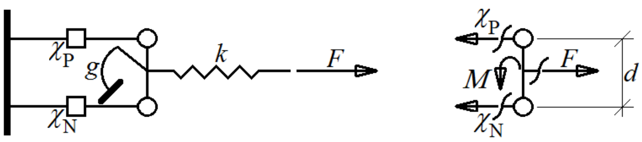

Fig. 4. *Mechanical model for elastic-plastic materials subjected to isotropic hardening.*

When we consider the mechanical model in Fig. 4, we have two different slip devices, which are activated only if $\chi_P=\chi^+$ or $\chi_N=\chi^-$, as suggested by the symbols placed near the squares that denote these particular slips. Eventually, the torsion spring enforces *isotropic hardening* in consequence of the obvious equations $F=\chi_P+\chi_N$ and $M=½d(\chi_N-\chi_P)=g\theta=g(\lambda_P-\lambda_N)/d$, which can be immediately derived if we remember that $\lambda_N\leq0$ and have a look at the sketch on the right hand side of Fig. 4 (assuming that displacements are sufficiently *small*, so that the rotation θ of the vertical bar is practically equal to the tangent of θ).

Therefore, if F exceeds the value $2\chi^+$ (which represents an initial yield load), the lower slip will be subjected to the force $\chi_N=F-\chi^+$, while the load acting on the upper slip will be $\chi_P=\chi^+$. Simultaneously, there will be a positive inelastic displacement λ_P and the torsion spring will react with a moment $g\theta$, where $\theta=\lambda_P/d$ is the rotation of the spring in this particular case. Hence, the forces acting on the slips become $\chi_P=\chi^+=½F–g\lambda_P/d^2$, $\chi_N=½F+g\lambda_P/d^2$.

Next, if we unload the system by enforcing the load increment $\Delta F=-F$, we have the residual forces $-g\lambda_P/d^2$ and $g\lambda_P/d^2$ acting on the upper and lower slip, respectively. Thus, further inelastic displacements can only occur in the presence of a new force F, which exceeds the thresholds $2(\chi^+ + g\lambda_P/d^2)$ or $2(\chi^- – g\lambda_P/d^2)$, which represent the updated yield loads. Of course, the increments of the yield loads (whose absolute values are exactly the same) are typical of systems subjected to *isotropic hardening*.

It is also obvious that we can consider equations such as $\chi_P=½F–(d\psi^*/d\lambda)$ and $\chi_N=½F+(d\psi^*/d\lambda)$ if we intend to describe the response of *nonlinear hardening* systems.

In the end, the governing equations read (with reference to the *linear hardening* case)

$$F = k\,u – ½\,k\,(\lambda_P + \lambda_N) \tag{2a}$$

$$-\chi_P = -½\,k\,u + ¼\,k\,(\lambda_P + \lambda_N) + (g/d^2)\,(\lambda_P – \lambda_N) \tag{2b}$$

$$-\chi_N = -½\,k\,u + ¼\,k\,(\lambda_P + \lambda_N) + (g/d^2)\,(\lambda_N – \lambda_P) \tag{2c}$$

Let us now consider a *truss structure*, whose members are subjected to *uniaxial stress states*. For the sake of brevity, we will only focus on *elastic perfectly-plastic* materials and systems subjected to *linear kinematic hardening*.

For a *truss structure* consisting of *e* structural elements, $F_i=k_i\,(u_i–\lambda_i)$ is the axial force acting on the *i-th* bar (i=1,…,e). Therefore, by introducing the vectors $\mathbf{F}$, $\mathbf{Q}$, $\mathbf{u}$, $\mathbf{U}$, $\boldsymbol{\lambda}$ (which collect the axial forces, the nodal loads, the elongations u_i, the nodal displacements referred to a global coordinate system and the inelastic elongations λ_i), we can write the equations

$$\mathbf{u} = \mathbf{C}\,\mathbf{U}\ ,\ \mathbf{Q} = \mathbf{C}^T\,\mathbf{F} = \mathbf{C}^T\,\mathbf{S}\,\{\mathbf{u} – \boldsymbol{\lambda}\} = \mathbf{K}\,\mathbf{U} + \mathbf{L}\,\boldsymbol{\lambda} \tag{3a,b}$$

where $\mathbf{C}$ is a compatibility matrix and $\mathbf{C}^T$ the consequent equilibrium matrix, in view of the principle of virtual works (since $\delta\mathbf{U}^T\mathbf{Q}=\delta\mathbf{u}^T\mathbf{F}$). In addition, we have set $\mathbf{S}$=diag[k_i], $\mathbf{K}=\mathbf{C}^T\mathbf{S}\,\mathbf{C}$ and $\mathbf{L}=-\mathbf{C}^T\mathbf{S}$.

Similarly, after collecting the forces χ_i acting on the *e* slips into a vector $\boldsymbol{\chi}$ and after introducing the matrix $\mathbf{G}$=diag[g_i] that collects the hardening parameters g_i, we can write the equation

$$-\boldsymbol{\chi} = -\mathbf{F} + \mathbf{G}\,\boldsymbol{\lambda} = -\mathbf{S}\,\{\mathbf{C}\,\mathbf{U}–\boldsymbol{\lambda}\} + \mathbf{G}\,\boldsymbol{\lambda} = \mathbf{L}^T\mathbf{U} + \mathbf{S}\,\boldsymbol{\lambda} + \mathbf{G}\,\boldsymbol{\lambda} \tag{4}$$

Of course, we shall set $\mathbf{G}=\mathbf{0}$ if we deal with an *elastic perfectly-plastic material*. More importantly, it should be noted that the governing equations (3b) and (4) imply that the solution of the elastic-plastic problem for a given force vector $\mathbf{Q}$ corresponds to the minimum point of the *non-constrained convex function*

$$\omega(\mathbf{u},\boldsymbol{\lambda}) = ½\,\mathbf{U}^T\,\mathbf{K}\,\mathbf{U} + ½\,\boldsymbol{\lambda}^T\,\mathbf{S}\,\boldsymbol{\lambda} + \\ + ½\,\boldsymbol{\lambda}^T\,\mathbf{G}\,\boldsymbol{\lambda} + \mathbf{U}^T\,\mathbf{L}\,\boldsymbol{\lambda} + D(\boldsymbol{\lambda}) – \mathbf{Q}^T\,\mathbf{U} \tag{5}$$

Here, $D(\boldsymbol{\lambda})$ denotes the dissipation function of the entire structure. In consequence, the optimality conditions are represented by the governing equations $\mathbf{K}\,\mathbf{U}+\mathbf{L}\,\boldsymbol{\lambda}–\mathbf{Q}=\mathbf{0}$ and $\mathbf{L}^T\mathbf{U}+\mathbf{S}\,\boldsymbol{\lambda}+\mathbf{G}\,\boldsymbol{\lambda}+\boldsymbol{\chi}=\mathbf{0}$. As for the *i-th* entry of $\boldsymbol{\chi}$ (i=1,…,e), we obviously have $\chi_i=\partial D/\partial\lambda_i$ if $\lambda_i\neq0$ and $\chi_i\in\partial D(\boldsymbol{\lambda})$ if $\lambda_i=0$.

III Internal Variables and Multiaxial Stress States

So far, to the best of the authors' knowledge, the *internal variable approach* discussed here has been applied to multiaxial stress states only in the case of materials for which it is possible to assume *Mises' yield condition*.

Indeed, if this condition is met and we reason in terms of *deviatoric stresses*, it is quite easy to generalize eqn. (1b) and introduce a dissipation function $d(\boldsymbol{\lambda})$ per unit volume, which will eventually lead to a function such as $D(\boldsymbol{\lambda})$ in eqn. (5).

The process is relatively simple. First, we can define *Mises' yield surface* by imposing a limit value, say $\mathcal{E}_d$, to the distortion energy per unit volume $½s_{ij}e_{ij}=¼s_{ij}s_{ij}/G$, so that we obtain the spherical elastic domain of Fig. 5, in view of the equation $s_{ij}s_{ij}=k^2$, with $k^2=4G\mathcal{E}_d$. Next, we can assume a *fictitious slip device* that remains rigid until the deviatoric stresses attain *critical* values, say $\hat{s}_{ij}=2Ge_{ij}$, which correspond to a point along the *initial yield surface*.

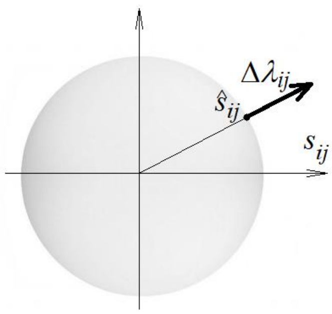

Fig. 5. *Elastic domain in the space of deviatoric stresses according to Mises' yield condition.*

Hence, in the case of *elastic perfectly-plastic* materials, *incremental plastic strains* $\Delta\lambda_{ij}$ may occur, such that $\hat{s}_{ij}=2Ge_{ij}–2G\Delta\lambda_{ij}$. Instead, *hardening* materials would lead to the relationship $\hat{s}_{ij}=2Ge_{ij}–2G\Delta\lambda_{ij}–2G'\Delta\lambda_{ij}$, where the parameter $2G'$ plays the role of the stiffness parameter *g* for the mechanical models concerned with linear *isotropic* or *kinematic* hardening. In other words, the product $2G'\Delta\lambda_{ij}$ gives the components Δs_{ij} of a vector, which quantifies the increment of the radius of the *yield surface* for *isotropic hardening* materials or the displacement of the center of the *yield domain* for *kinematic hardening* materials, as shown in Fig. 6.

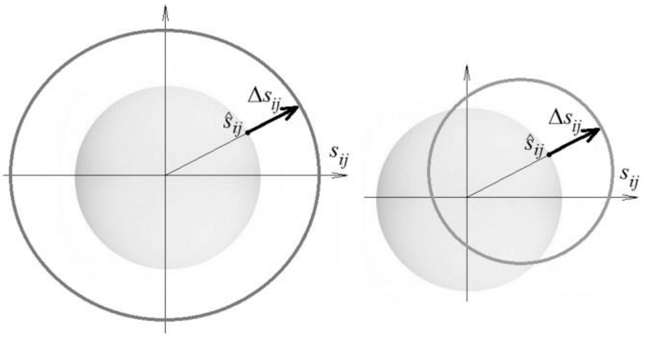

Fig. 6. *Mises' yield condition: isotropic and kinematic hardening.*

In any case, we can define the *dissipation function* per unit volume $d(\Delta\gamma)=k\Delta\gamma$ by introducing the parameter $\Delta\gamma=(\Delta\lambda_{ij}\Delta\lambda_{ij})^{½}$.

Next, it is possible to consider a structural system discretized by e finite elements for which (in view of the principle of virtual works) the following equation is satisfied:

$$\Sigma_i \int \boldsymbol{\sigma}^{\mathrm{T}}\, \delta\boldsymbol{\varepsilon}\, \mathrm{d}V = \Sigma_i \int \mathbf{b}^{\mathrm{T}}\, \delta\mathbf{u}\, \mathrm{d}V + \Sigma_i \int \mathbf{f}^{\mathrm{T}}\, \delta\mathbf{u}\, \mathrm{d}S \qquad (6)$$

with $i=1,\ldots,e$. Here, the vectors $\mathbf{u}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$, $\mathbf{b}$ and $\mathbf{f}$ refer to displacements, strains, stress, body forces and surface forces, respectively. Obviously, the above integrals are concerned either with the volume V_i or the surface S_i of each element (with $\mathbf{f}=\mathbf{0}$ if an element face is not loaded or does not belong to the surface of the continuum).

As typically happens in the case of finite element discrete models, we can make use of the relationships

$$\delta\mathbf{u} = \boldsymbol{\Phi}_i\, \delta\mathbf{u}_i \quad , \quad \delta\boldsymbol{\varepsilon} = \mathbf{B}_i\, \delta\mathbf{u}_i \qquad (7a,b)$$

$$\boldsymbol{\sigma} = \mathbf{D}_i\, \{\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}\} = \mathbf{D}_i\, \{\mathbf{B}_i\, \mathbf{u}_i - \boldsymbol{\Psi}_i\, \boldsymbol{\lambda}_i\} \qquad (7c)$$

where the displacements $\mathbf{u}$ and the plastic strains $\boldsymbol{\varepsilon}^{\mathrm{p}}$ depend on nodal displacements $\mathbf{u}_i$ and plastic strains $\boldsymbol{\lambda}_i$ of the *i-th* element at properly selected *strain points* (usually *Gauss points* in the case of quadrangles and hexahedrons) through matrices of shape functions ($\boldsymbol{\Phi}_i$ and $\boldsymbol{\Psi}_i$), while $\mathbf{D}_i$ is the element stiffness matrix and $\mathbf{B}_i$ a matrix that consists of convenient derivatives of the shape functions collected in the matrix $\boldsymbol{\Phi}_i$.

If there were no plastic strains $\boldsymbol{\lambda}_i$, we would derive the classical equation $\delta\mathbf{U}^{\mathrm{T}}\, \mathbf{K}\, \mathbf{U}=\delta\mathbf{U}^{\mathrm{T}}\, \mathbf{Q}$ (*i.e.*, $\mathbf{Q}=\mathbf{K}\, \mathbf{U}$) concerned with *linear elastic systems*, where $\mathbf{K}$, $\mathbf{U}$, $\mathbf{Q}$ denote the structure stiffness matrix, the nodal displacements and the equivalent nodal loads, respectively.

Instead, since we are dealing with elastic-plastic systems, we shall also consider the integrals

$$\Sigma_i \int \{-\boldsymbol{\varepsilon}^{\mathrm{p}}\}^{\mathrm{T}}\, \delta\boldsymbol{\varepsilon}\, \mathrm{d}V = \Sigma_i \int \{-\mathbf{D}_i\, \boldsymbol{\Psi}_i\, \boldsymbol{\lambda}_i\}^{\mathrm{T}}\, \delta\boldsymbol{\varepsilon}\, \mathrm{d}V = \qquad (8)$$
$$= \Sigma_i \int \{\delta\mathbf{u}_i\}^{\mathrm{T}}\, [-\mathbf{B}_i^{\mathrm{T}}\, \mathbf{D}_i\, \boldsymbol{\Psi}_i]\, \boldsymbol{\lambda}_i\, \mathrm{d}V = \Sigma_i\, \{\delta\mathbf{u}_i\}^{\mathrm{T}}\, \mathbf{L}_i\, \boldsymbol{\lambda}_i\, \mathrm{d}V$$

Therefore, by assembling the e submatrices $\mathbf{L}_i$ and subvectors $\boldsymbol{\lambda}_i$, we will eventually derive the relationship $\mathbf{Q}=\mathbf{K}\,\mathbf{U}+\mathbf{L}\,\boldsymbol{\lambda}$, which is formally identical to eqn. (3b).

Since the product $\{\mathbf{D}_i\ \mathbf{B}_i\ \mathbf{u}_i\}$ gives the stresses induced by the displacements $\mathbf{u}_i$ if the structural response is linear elastic, the products $\{\mathbf{L}_i^{\mathrm{T}}\mathbf{u}_i\}$ and $\{\mathbf{L}^{\mathrm{T}}\mathbf{U}\}$ represent *generalized forces* (with the sign changed), which are ideally applied at the *strain points* and are somehow equivalent to the stresses distributed in convenient volumes around each *strain point*, as discussed below with some details.

It is also worth noting that each integral on the left hand side of eqn. (6) can be written in the form

$$\Sigma_i \int \boldsymbol{\sigma}^{\mathrm{T}}\, \delta\boldsymbol{\varepsilon}\, \mathrm{d}V = \qquad (9)$$
$$= \Sigma_i \int \{\delta\mathbf{u}_i\}^{\mathrm{T}}\, [\mathbf{B}_i - \mathbf{B}_i^* + \mathbf{B}_i^*]^{\mathrm{T}}\, \mathbf{D}_i\, \{[\mathbf{B}_i - \mathbf{B}_i^* + \mathbf{B}_i^*]\, \mathbf{u}_i - \boldsymbol{\varepsilon}^{\mathrm{p}}\}\, \mathrm{d}V$$

where the matrix $\mathbf{B}_i^*$ consists of convenient derivatives of the entries of the matrix $\boldsymbol{\Phi}_i$ and the product $\mathbf{B}_i^*\mathbf{u}_i$ gives a vector, say $\boldsymbol{\varepsilon}^*$, whose significant entries are the *volumetric strain* $\varepsilon_{\mathrm{v}}=\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}$ divided by 3. Thus, the product $[\mathbf{B}_i-\mathbf{B}_i^*]\mathbf{u}_i=\mathbf{e}$ represents a vector of deviatoric strains.

At this stage, if we consider the parameter K (bulk modulus), the product $3K\mathbf{B}_i^*\mathbf{u}_i$ gives a vector $\boldsymbol{\sigma}^*=\boldsymbol{\sigma}-\mathbf{s}$, whose significant entries are the *mean stress* $\sigma_{\mathrm{m}}=(\sigma_{11}+\sigma_{22}+\sigma_{33})/3$, while the vector $\mathbf{s}$ collects deviatoric stresses.

In view of these remarks, it is possible to express the integral on the right hand side of eqn. (9) in a slightly different way:

$$\Sigma_i \int \boldsymbol{\sigma}^{\mathrm{T}}\, \delta\boldsymbol{\varepsilon}\, \mathrm{d}V = \Sigma_i \int \{\delta\mathbf{u}_i\}^{\mathrm{T}}\, \{[\mathbf{B}_i - \mathbf{B}_i^*]^{\mathrm{T}}\, 2G\, [\mathbf{B}_i - \mathbf{B}_i^*]\, \mathbf{u}_i +$$
$$+ \mathbf{B}_i^{*\mathrm{T}}\, 3K\, \mathbf{B}_i^*\, \mathbf{u}_i\} - \{\delta\mathbf{u}_i\}^{\mathrm{T}}\, [\mathbf{B}_i - \mathbf{B}_i^*]^{\mathrm{T}}\, 2G\, \boldsymbol{\Psi}_i\, \boldsymbol{\lambda}_i\, \mathrm{d}V = \qquad (10)$$
$$= \Sigma_i \int \{\delta\mathbf{u}_i\}^{\mathrm{T}}\, \{[\mathbf{B}_i - \mathbf{B}_i^*]\, 2G\, [\mathbf{B}_i - \mathbf{B}_i^*] +$$
$$+ \mathbf{B}_i^{*\mathrm{T}}\, 3K\, \mathbf{B}_i^*\}\, \mathbf{u}_i\, \mathrm{d}V + \Sigma_i\, \{\delta\mathbf{u}_i\}^{\mathrm{T}}\, \mathbf{L}_i^*\, \boldsymbol{\lambda}_i$$

Note that, as typical of *Mises' criterion*, an *associated flow rule* and *deviatoric plastic strains* have been assumed (in agreement with Figs. 5 and 6). Hence, in the above equations, where deviatoric and isotropic components of strains and stresses have been separated, we can recognize the presence of the deviatoric stresses $\mathbf{s}=2G\{\mathbf{e}-\boldsymbol{\varepsilon}^{\mathrm{p}}\}=2G\{\mathbf{e}-\mathbf{e}^{\mathrm{p}}\}$, since $\boldsymbol{\varepsilon}^{\mathrm{p}}=\mathbf{e}^{\mathrm{p}}$.

In addition, deviatoric components are normal to isotropic components (and, consequently, possible scalar products are equal to zero). Thus, at the end of the chain of equations (10) we find an integral (from which we can derive the usual element stiffness matrix) and a matrix $\mathbf{L}_i^*$ such that the product $\mathbf{L}_i^{*\mathrm{T}}\mathbf{u}_i$ represents *generalized forces* (with the sign changed) equivalent to *deviatoric stresses* acting on properly selected volumes around the *strain points* when the structural response is linear elastic.

By assembling the submatrices $\mathbf{L}_i^*$, once again we derive the governing equation $\mathbf{Q}=\mathbf{K}\,\mathbf{U}+\mathbf{L}\,\boldsymbol{\lambda}$. Next, it is definitely straightforward to introduce the *generalized forces* acting on the slip devices, which are given by the relationship $\boldsymbol{\chi}$=-$\mathbf{L}^T\mathbf{U}$–$\mathbf{S}\ \boldsymbol{\lambda}$–$\mathbf{G}\ \boldsymbol{\lambda}$, fully analogous to eqn. (4) even if they are now equivalent to *deviatoric stresses*.

As the matrix **L** is obtained by considering *deviatoric stresses*, **S** and **G** are simple diagonal matrices, whose entries depend on the stiffness parameters G (shear modulus) and G' (hardening parameter). Since we are dealing with *generalized forces*, not stress components, the significant entries of **S** and **G** actually depend on the stiffness parameter G or G' multiplied by convenient volumes $\tilde{V}_S$ (where the index S refers to the generic *strain point* S). More precisely, a significant entry will be equal to $2\tilde{V}_S G$ or $2\tilde{V}_S G'$ when we have to do with ratios of *deviatoric stresses* to *deviatoric strains*. Instead, when we need to consider ratios of *shear stresses* to *engineering shear strains* (as often happens in the field of computational mechanics), these terms shall be equal to $\tilde{V}_S G$ or $\tilde{V}_S G'$.

Clearly, the volumes $\tilde{V}_S$ play a significant role in this context. Thus, a comprehensive discussion of the topics concerned with this *internal variable formulation* requires some details about the volumes of the zones, which are around the *strain points* and are subjected to stresses that turn out to be equivalent to the *generalized forces* ideally applied at these *strain points*.

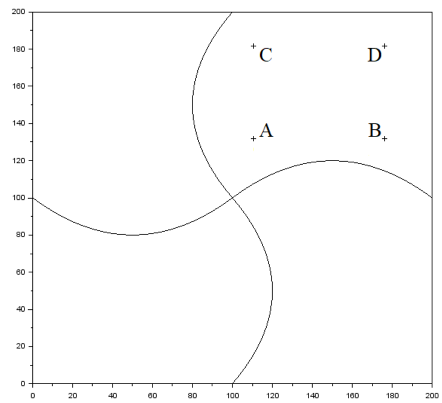

Fig. 7. *Discrete model of a plane membrane (total area:200x200mm^2).*

To this aim, it can be useful to briefly summarize some recent results [9] concerned with a square plane membrane discretized with four *isoparametric elements* with curved edges (*cf.* Fig. 7). It has been proved that the *generalized forces* ideally applied to a *strain point* are equivalent to the stresses acting on a volume $\tilde{V}_S$, which can be determined in a rather general way through the relationship $\tilde{V}_S$=|det[**J**]| $h\ w_\xi\ w_\eta$, in the case of plane systems (while the extension to the three-dimensional case is straightforward). In this formula, h is the thickness of the plane model, w_ξ and w_η represent the weights assigned to the *strain point* S (which must coincide with a *Gauss point*) according to Gauss integration method and |det[**J**]| denotes the absolute value of the determinant of the *Jacobian matrix* computed at the same *strain point*.

Of course, as typical of *isoparametric elements*, we assume that the usual cartesian coordinates of an element point (x and y in the case of plane systems) can be found by using convenient shape functions $f_k(\xi,\eta)$ that depend upon non-dimensional coordinates ξ and η. Therefore, we shall set $x(\xi,\eta)=\Sigma_k f_k(\xi,\eta)\ x_k$ and $y(\xi,\eta)=\Sigma_k f_k(\xi,\eta)\ y_k$, where x_k and y_k represent the coordinates of the k-th node. In consequence, the *Jacobian matrix* can be determined at each *strain/Gauss point* by computing the derivatives of the functions $x(\xi,\eta)$ and $y(\xi,\eta)$ with respect to ξ and η.

The above formula is immediately checked by assuming, for instance, the following mechanical properties and boundary conditions for the system in Fig. 7: unit thickness, linear elastic behavior (Young's modulus E=200,000 MPa and Poisson's ratio ν=0.3), zero horizontal displacements along the left vertical edge, horizontal displacements equal to 0.1 mm along the right edge, constrained vertical displacement of the mid-point of the lower edge to prevent free body motions. In this way, the system is subjected to uniform strain and stress distributions, so that the volume $\tilde{V}_S$ related to a *strain point* is necessarily correct if the *generalized forces* at that *strain point* coincide with the corresponding stresses multiplied by the volume $\tilde{V}_S$.

In actual fact, we obtain uniform normal horizontal stresses $\sigma_{11}=E\varepsilon_{11}=E\ 0.1/200=100$ MPa for *plane-stress* conditions. Similarly, in the presence of *plane-strain* conditions, the only significant (constant) stresses turn out to be $\sigma_{11}=E\varepsilon_{11}+\nu\sigma_{33}=E\varepsilon_{11}/(1\text{-}\nu^2)=109.89$ MPa and σ_{33}=32.967 MPa, since $\varepsilon_{11}=(\sigma_{11}-\nu\sigma_{33})/\mathrm{E}=0.1/200=0.005$ and $\varepsilon_{33}=(\sigma_{33}-\nu\sigma_{11})/E=0$.

In consequence, the *deviatoric stresses* are s_{11}=66.667 MPa and $s_{22}=s_{33}=-\sigma_h$=-33.333 MPa in the first case, while we obtain the *deviatoric stress components* s_{11}=62.271 MPa together with $s_{22}=-\sigma_h$=-47.619 MPa and s_{33}=-14.652 MPa in the second case (where σ_h denotes the *hydrostatic stress*).

These values are in agreement with the volumes $\tilde{V}_S$=|det[**J**]| $h\ w_\xi\ w_\eta$ and the *generalized forces* determined through the finite element analysis. In fact, the volumes concerned with the *strain points* A, B, C, D in Fig. 7 are $\tilde{V}_A$=2787.293 mm^3, $\tilde{V}_B=\tilde{V}_C$=2366.667 mm^3, $\tilde{V}_D$=2479.373 mm^3, while the relevant *generalized forces* turn out to be 185819.56 Nmm=$s_{11}\tilde{V}_A$, 157777.78 Nmm=$s_{11}\tilde{V}_B=s_{11}\tilde{V}_C$, 165291.55 Nmm=$s_{11}\tilde{V}_D$ in the case of a *plane stress* state. Instead, the results are 173567.72 Nmm =$s_{11}\tilde{V}_A$, 147374.85 Nmm =$s_{11}\tilde{V}_B=s_{11}\tilde{V}_C$, 154393.21 Nmm = $s_{11}\tilde{V}_D$, -40839.464 Nmm = $s_{33}\tilde{V}_A$, -34676.435 Nmm = $s_{33}\tilde{V}_B=s_{33}\tilde{V}_C$, -36327.813 Nmm = $s_{33}\tilde{V}_D$ for the *plane strain* case.

Similarly, when we consider the governing equation $\boldsymbol{\chi}$=-$\mathbf{L}^T\mathbf{U}$–$\mathbf{S}\ \boldsymbol{\lambda}$–$\mathbf{G}\ \boldsymbol{\lambda}$ and need to establish a relationship between the entries of $\boldsymbol{\chi}$ and a *dissipation* *function* $D(\boldsymbol{\lambda})$ concerned with the entire structure, we shall make use of the volumes $\tilde{V}_S$ defined above. For instance, in the case of Mises' yield conditions, we can start from *dissipation functions* per unit volume $d_S(\boldsymbol{\lambda}_S)=\hat{s}_{ij}^{\,S}\lambda_{ij}^{\,S}$, where $\boldsymbol{\lambda}_S$, $\hat{s}_{ij}^{\,S}$ and $\lambda_{ij}^{\,S}$ are referred to the *strain point* S. Similarly, it will be possible to define the *dissipation functions* $D_S(\boldsymbol{\lambda}_S)=\tilde{V}_S\ d_S(\boldsymbol{\lambda}_S)$. Eventually, by summing all the contributions $D_S(\boldsymbol{\lambda}_S)$, we will obtain a dissipation function $D(\boldsymbol{\lambda})$ such that $\boldsymbol{\chi}_S=\partial D/\partial\boldsymbol{\lambda}_S$ if $\boldsymbol{\lambda}_S\neq\mathbf{0}$ and $\boldsymbol{\chi}_S\in\partial D(\boldsymbol{\lambda})$ if $\boldsymbol{\lambda}_S=\mathbf{0}$. Of course, when $\boldsymbol{\lambda}_S\neq\mathbf{0}$, the components of the subvector $\boldsymbol{\chi}_S$ will represent convenient *generalized forces*, *i.e.*, convenient *average deviatoric stresses* $\hat{s}_{ij}^{\,S}$ multiplied by $\tilde{V}_S$.

IV Convergence Properties of an Algorithm Based on the *Backward-Difference* Concept

In the case of structural systems subjected to uniaxial stress states or multiaxial stress states combined with Mises' yield condition, the governing equations discussed in this paper allow us to prove that convergence toward the solution of the *elastic-plastic incremental problem* can be guaranteed, if we adopt a proper algorithm that makes use of the so-called *backward-difference* technique. Indeed, it was shown that these convergence properties exist both for quasi-static and dynamic discrete models [3,4].

Here, we will briefly review the proof, since it represents a necessary, preliminary step for the an application to *piecewise-linear yield surfaces*, which represent the main topic of this paper.

With reference to typical *incremental formulations*, we can rewrite eqns. (3b) and (4) in the form

$$\mathbf{Q} = \mathbf{K}\ \{\mathbf{U}^o + \Delta\mathbf{U}\} + \mathbf{L}\ \{\boldsymbol{\lambda}^o + \Delta\boldsymbol{\lambda}\} \qquad (11a)$$

$$-\boldsymbol{\chi} = \mathbf{L}^T\ \{\mathbf{U}^o + \Delta\mathbf{U}\} + \mathbf{S}\ \{\boldsymbol{\lambda}^o + \Delta\boldsymbol{\lambda}\} + \mathbf{G}\ \{\boldsymbol{\lambda}^o + \Delta\boldsymbol{\lambda}\} \qquad (11b)$$

where $\mathbf{U}^o$ and $\boldsymbol{\lambda}^o$ are referred to nodal displacements and plastic strains at the *strain points* at the beginning of a given time-step, while $\Delta\mathbf{U}$ and $\Delta\boldsymbol{\lambda}$ are the unknown quantities to be determined when the structural system is subjected to a given vector $\mathbf{Q}$ of equivalent nodal loads. Clearly, the solution of the *incremental elastic-plastic problem* corresponds to the minimum point of the *convex function*

$$\omega(\Delta\mathbf{U},\Delta\boldsymbol{\lambda}) = \tfrac{1}{2}\,\Delta\mathbf{U}^T\,\mathbf{K}\,\Delta\mathbf{U} + \tfrac{1}{2}\,\Delta\boldsymbol{\lambda}^T\,\mathbf{S}\,\Delta\boldsymbol{\lambda} + $$
$$+ \tfrac{1}{2}\,\Delta\boldsymbol{\lambda}^T\,\mathbf{G}\,\Delta\boldsymbol{\lambda} + D(\Delta\boldsymbol{\lambda}) + \Delta\mathbf{U}^T\,\mathbf{L}\,\Delta\boldsymbol{\lambda} - \Delta\mathbf{U}^T\,\mathbf{Q} + \qquad (12)$$
$$+ \Delta\mathbf{U}^T\,\{\mathbf{K}\,\mathbf{U}^o + \mathbf{L}\,\boldsymbol{\lambda}^o\} + \Delta\boldsymbol{\lambda}^T\,\{\mathbf{L}^T\,\mathbf{U}^o + \mathbf{S}\,\boldsymbol{\lambda}_o + \mathbf{G}\,\boldsymbol{\lambda}^o\}$$

Therefore, if a certain iterative algorithm steadily converges, it must force the function $\omega(\Delta\mathbf{U},\Delta\boldsymbol{\lambda})$ to decrease at each iteration. Here, we will show that this is what exactly happens in the case of a solution technique, which is based on the *backward-difference* concept and essentially consists of two phases.

First, at the *i-th* iteration, there is a *prediction phase* during which we need to compute the stresses or deviatoric stresses that would occur at all the *strain points* if the response to the incremental displacements $\Delta\mathbf{U}_i=\mathbf{K}^{-1}\{\mathbf{Q}-\mathbf{L}\{\boldsymbol{\lambda}^o+\Delta\boldsymbol{\lambda}_{i-1}\}\}-\mathbf{U}^o$ were fully elastic. Here, $\Delta\boldsymbol{\lambda}_{i-1}$ denotes the vector of the plastic strain increments computed at the end of the previous iteration. In consequence, it is usually set equal to zero when $i=1$.

Next, we need to proceed with a *correction phase*, which essentially consists in satisfying the constitutive law for given increments of the displacements and, hence, of the total strains or the corresponding deviatoric strains. For instance, in the case of Mises' criterion, at every strain point where the predicted deviatoric stress components $s_{ij}=s_{ij}^{\,o}+2G\Delta e_{ij}$ denote a point, which is outside the elastic domain, we shall determine plastic strain increments $\Delta e_{ij}^{\,p}$ that satisfy two conditions:

- the deviatoric stresses $\hat{s}_{ij}=s_{ij}^{\,o}+2G(\Delta e_{ij}-\Delta e_{ij}^{\,p})-2G'\,\Delta e_{ij}^{\,p}$ must correspond to a point located along the yield surface that characterizes the fictitious *slip device* (of course, with $G'=0$ in the case of *elastic perfectly-plastic* materials)
- the plastic strain increments $\Delta e_{ij}^{\,p}$ shall be normal to the yield surface at the point whose coordinates are $s_{ij}=s_{ij}^{\,o}+2G(\Delta e_{ij}-\Delta e_{ij}^{\,p})$ and, hence, denote the stress components that take into account the contribution of the plastic strain increments

This means that the *correction phase* implies a *radial return* when Mises' yield condition is applicable, as schematically shown in Fig. 8 for *elastic perfectly-plastic* and *hardening* materials. Note that, in the case of *isotropic hardening*, the modulus of the vector whose components are $\Delta s_{ij}=2G'\Delta e_{ij}^{\,p}$ corresponds to the increment of the radius of the yield surface, while, in the case of *isotropic hardening*, that vector denotes the displacement of the center of the yield surface.

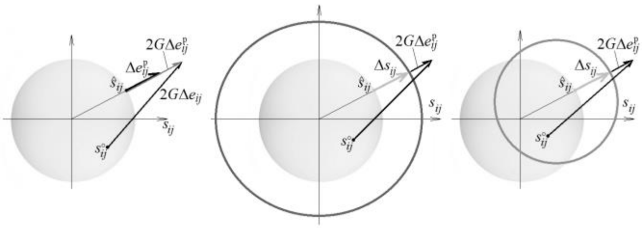

Fig. 8. *Mises' yield condition: correction phase.*

At the end of the *correction phase*, the plastic strain increments $\Delta e_{ij}^{\,p}$ will be collected in a vector $\Delta\boldsymbol{\lambda}_i$. Then, we can solve the system $\Delta\mathbf{U}_{i+1}=\mathbf{K}^{-1}\{\mathbf{Q}-\mathbf{L}\{\boldsymbol{\lambda}^o+\Delta\boldsymbol{\lambda}_i\}\}-\mathbf{U}^o$ and continue as before until a convenient Euclidean norm (*e.g.*, $\|\Delta\mathbf{U}_i-\Delta\mathbf{U}_{i-1}\|/\|\Delta\mathbf{U}_{i-1}\|$) is below a given threshold.

That said, we can consider the increment $\delta\omega=\omega(\Delta\mathbf{U}_i,\Delta\boldsymbol{\lambda}_i)-\omega(\Delta\mathbf{U}_{i-1},\Delta\boldsymbol{\lambda}_{i-1})$. If we set $\delta\mathbf{U}=\Delta\mathbf{U}_i-\Delta\mathbf{U}_{i-1}$, it turns out that the term $(\tfrac{1}{2}\Delta\mathbf{U}_i^T\mathbf{K}\Delta\mathbf{U}_i-\tfrac{1}{2}\Delta\mathbf{U}_{i-1}^T\mathbf{K}\Delta\mathbf{U}_{i-1})$ is equal to $(\tfrac{1}{2}\delta\mathbf{U}^T\mathbf{K}\delta\mathbf{U}+\delta\mathbf{U}^T\mathbf{K}\Delta\mathbf{U}_{i-1})$. Similarly, we obtain $(\tfrac{1}{2}\Delta\boldsymbol{\lambda}_i^T\mathbf{S}\Delta\boldsymbol{\lambda}_i-\tfrac{1}{2}\Delta\boldsymbol{\lambda}_{i-1}^T\mathbf{S}\Delta\boldsymbol{\lambda}_{i-1})=(\tfrac{1}{2}\delta\boldsymbol{\lambda}^T\mathbf{S}\delta\boldsymbol{\lambda}+\delta\boldsymbol{\lambda}^T\mathbf{S}\Delta\boldsymbol{\lambda}_{i-1})$, having set $\delta\boldsymbol{\lambda}=\Delta\boldsymbol{\lambda}_i-\Delta\boldsymbol{\lambda}_{i-1}$. Of course, we derive an analogous result when we deal with the term $\tfrac{1}{2}\Delta\boldsymbol{\lambda}^T\mathbf{G}\Delta\boldsymbol{\lambda}$ in eqn. (12). In addition, the binomial

$(\Delta\mathbf{U}_i^T\mathbf{L}\Delta\boldsymbol{\lambda}_i - \Delta\mathbf{U}_{i-1}^T\mathbf{L}\Delta\boldsymbol{\lambda}_{i-1})$ can be rewritten in a more convenient different way: $(\Delta\mathbf{U}_i^T\mathbf{L}\Delta\boldsymbol{\lambda}_i - \Delta\mathbf{U}_{i-1}^T\mathbf{L}\Delta\boldsymbol{\lambda}_{i-1} + \Delta\mathbf{U}_i^T\mathbf{L}\Delta\boldsymbol{\lambda}_{i-1} - \Delta\mathbf{U}_i^T\mathbf{L}\Delta\boldsymbol{\lambda}_{i-1})$. Hence we get $(\Delta\mathbf{U}_i^T\mathbf{L}\Delta\boldsymbol{\lambda}_i - \Delta\mathbf{U}_{i-1}^T\mathbf{L}\Delta\boldsymbol{\lambda}_{i-1}) = (\delta\mathbf{U}^T\mathbf{L}\Delta\boldsymbol{\lambda}_{i-1} + \delta\boldsymbol{\lambda}^T\mathbf{L}^T\Delta\mathbf{U}_i)$ in a straightforward way.

Now, by splitting $\delta\omega$ into the contributions $\delta\omega_1$ and $\delta\omega_2$, we can eventually set

$$\delta\omega_1 = \tfrac{1}{2}\,\delta\mathbf{U}^T\,\mathbf{K}\,\delta\mathbf{U} + \delta\mathbf{U}^T\,\mathbf{K}\,\Delta\mathbf{U}_{i-1} + \delta\mathbf{U}^T\,\mathbf{L}\,\Delta\boldsymbol{\lambda}_{i-1} - \delta\mathbf{U}^T\,\mathbf{Q} + \delta\mathbf{U}^T\,\{\mathbf{K}\,\mathbf{U}^o + \mathbf{L}\,\boldsymbol{\lambda}^o\} \tag{13a}$$

$$\delta\omega_2 = \tfrac{1}{2}\,\delta\boldsymbol{\lambda}^T\,\mathbf{S}\,\delta\boldsymbol{\lambda} + \delta\boldsymbol{\lambda}^T\,\mathbf{S}\,\Delta\boldsymbol{\lambda}_{i-1} + \tfrac{1}{2}\,\delta\boldsymbol{\lambda}^T\,\mathbf{G}\,\delta\boldsymbol{\lambda} + \delta\boldsymbol{\lambda}^T\,\mathbf{G}\,\Delta\boldsymbol{\lambda}_{i-1} + \delta\boldsymbol{\lambda}^T\,\mathbf{L}^T\,\Delta\mathbf{U}_i + D(\Delta\boldsymbol{\lambda}_i) - D(\Delta\boldsymbol{\lambda}_{i-1}) + \delta\boldsymbol{\lambda}^T\,\{\mathbf{L}^T\,\mathbf{U}^o + \mathbf{S}\,\boldsymbol{\lambda}^o + \mathbf{G}\,\boldsymbol{\lambda}^o\} \tag{13b}$$

Note that $\delta\mathbf{U}^T\{\mathbf{K}\ \mathbf{U}^o + \mathbf{L}\ \boldsymbol{\lambda}^o + \mathbf{L}\ \Delta\boldsymbol{\lambda}_{i-1} - \mathbf{Q}\}$ in eqn. (13a) is equal to $-\delta\mathbf{U}^T\mathbf{K}\ \Delta\mathbf{U}_i$. Thus, we end up with the result $\delta\omega_1 = -\tfrac{1}{2}\ \delta\mathbf{U}^T\mathbf{K}\delta\mathbf{U}$, which is obviously less than zero for any $\delta\mathbf{U}\neq\mathbf{0}$.

As for eqn. (13b), we can add and subtract the terms $\tfrac{1}{2}\ \delta\boldsymbol{\lambda}^T\ \mathbf{S}\ \delta\boldsymbol{\lambda}$ and $\tfrac{1}{2}\ \delta\boldsymbol{\lambda}^T\ \mathbf{G}\ \delta\boldsymbol{\lambda}$. In consequence, we obtain $(-\tfrac{1}{2}\delta\boldsymbol{\lambda}^T\mathbf{S}\delta\boldsymbol{\lambda} + \delta\boldsymbol{\lambda}^T\mathbf{S}\Delta\boldsymbol{\lambda}_i)$ instead of $(\tfrac{1}{2}\delta\boldsymbol{\lambda}^T\mathbf{S}\delta\boldsymbol{\lambda} + \delta\boldsymbol{\lambda}^T\mathbf{S}\Delta\boldsymbol{\lambda}_{i-1})$ and $(-\tfrac{1}{2}\delta\boldsymbol{\lambda}^T\mathbf{G}\delta\boldsymbol{\lambda} + \delta\boldsymbol{\lambda}^T\mathbf{G}\Delta\boldsymbol{\lambda}_i)$ instead of $(\tfrac{1}{2}\delta\boldsymbol{\lambda}^T\mathbf{G}\delta\boldsymbol{\lambda} + \delta\boldsymbol{\lambda}^T\mathbf{G}\Delta\boldsymbol{\lambda}_{i-1})$. Therefore, eqn. (13b) becomes

$$\delta\omega_2 = -\tfrac{1}{2}\,\delta\boldsymbol{\lambda}^T\,[\mathbf{S} + \mathbf{G}]\,\delta\boldsymbol{\lambda} + \delta\boldsymbol{\lambda}^T\,\{\mathbf{L}^T\,\{\mathbf{U}_o + \Delta\mathbf{U}_i\} + [\mathbf{S} + \mathbf{G}]\,\{\boldsymbol{\lambda}^o + \Delta\boldsymbol{\lambda}_i\}\} + D(\Delta\boldsymbol{\lambda}_i) - D(\Delta\boldsymbol{\lambda}_{i-1}) \tag{14}$$

where the second term on the right hand side represents the scalar product between $\delta\boldsymbol{\lambda}$ and $\boldsymbol{\chi}_i$, which (in turn) is the vector of *generalized forces* acting on the *slip devices* at the end of the *i-th* iteration. In view of this result, it is quite obvious that $\delta\omega_2$, too, can only be negative for any $\delta\boldsymbol{\lambda}\neq\mathbf{0}$, because $[\mathbf{S}+\mathbf{G}]$ is positive definite and $D(\Delta\boldsymbol{\lambda}_{i-1}) \geq D(\Delta\boldsymbol{\lambda}_i) + \boldsymbol{\chi}_i^T\{\Delta\boldsymbol{\lambda}_{i-1} - \Delta\boldsymbol{\lambda}_i\}$ since $D(\Delta\boldsymbol{\lambda})$ is convex.

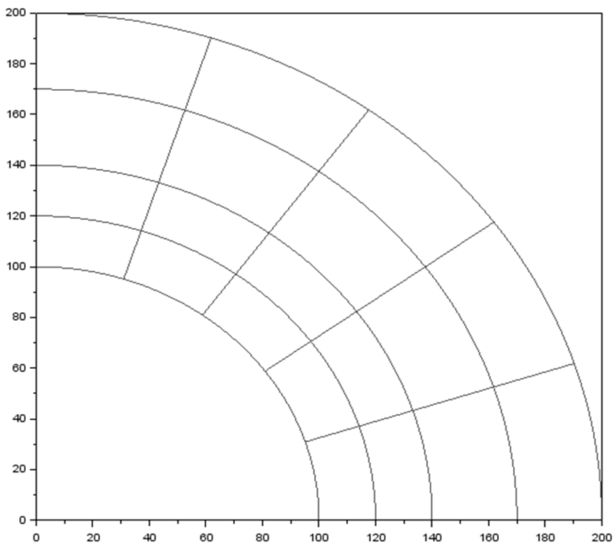

Fig. 9. *Discrete model (measures are given in mm).*

The trend of the function $\omega(\Delta\mathbf{U},\Delta\boldsymbol{\lambda})$ can be checked by solving any *incremental elastic-plastic problem*. For instance, we considered the classical case of an elastic perfectly-plastic tube subjected to internal pressure [10]. Assuming *Mises' yield condition* and imposing a *plane strain* state, we applied an increasing internal pressure to the discrete model in Fig. 9, which consists of twenty 8-node *isoparametric* elements characterized by four *strain points*.

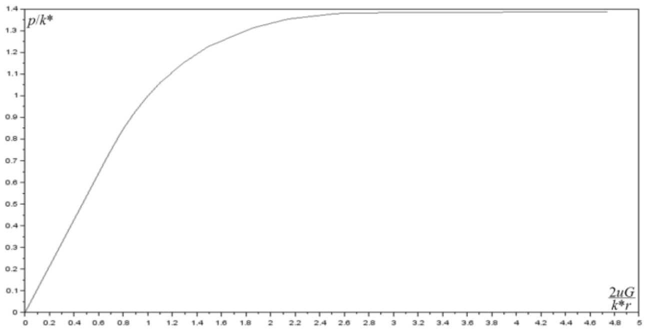

Fig. 10. *Internal pressure vs. external radial displacement in non-dimensional form.*

The relevant response is reported in Fig. 10 by using the non-dimensional quantities p/k^* and $2uG/(k^*r)$, where u, r and k^* denote the radial external displacement, the internal radius and the square root of the absolute value of the second invariant of the stress deviator, which can be used to define *Mises' yield surface* by setting $|s_{11}\ s_{22}+s_{22}\ s_{33}+s_{33}\ s_{11}-s_{12}^2-s_{23}^2-s_{31}^2|=k^{*2}$.

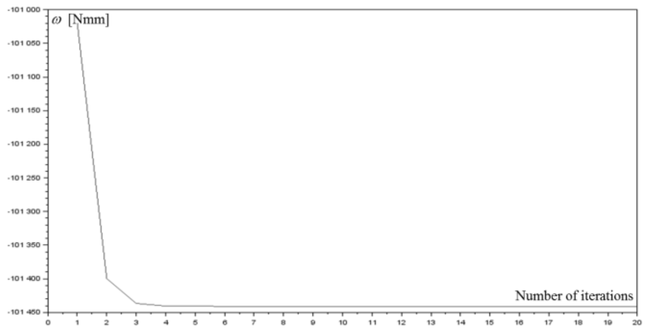

Fig. 11. *Typical trend of the function* $\omega(\Delta\mathbf{U},\Delta\boldsymbol{\lambda})$ *during a time-step.*

More precisely, the curve in Fig. 10 was obtained by subdividing the load history into fifty time-steps and Fig. 11 shows the trend of the function $\omega(\Delta\mathbf{U},\Delta\boldsymbol{\lambda})$ during the last step. As expected, a steady decrease was found.

V Applications to Piecewise-Linear Yield Surfaces

Even though this issue has never been explicitly studied before, the *internal variable* approach discussed here is applicable to every elastic-plastic material by introducing *piecewise-linear yield surfaces*, which are obviously able to approximate any surface with any required degree of accuracy.

In order to do so, it is necessary to express the vectors of the *incremental elastic-plastic strains* in the form $\Delta\boldsymbol{\varepsilon}^p = \Sigma_k\ \mathbf{n}_k\ \Delta\mu_k$, where $\Delta\mu_k$ represents a plastic multiplier, while $\mathbf{n}_k$ is the unit vector normal to the *k-th* plane, whose distance from the origin of the axes in the stress space is r_k (k=1,…,m, if m denotes the number of planes that define the yield surface).

Therefore, when we need to satisfy the constitutive law at a *strain point* for a given stress (say $\boldsymbol{\sigma}^{o}$) at the beginning of the current time-step and for any increment of the total strains $\Delta\boldsymbol{\varepsilon}$ (as typical of the *correction phase*), we shall solve the problem

$$\boldsymbol{\varphi} = \mathbf{n}^{T} \{\boldsymbol{\sigma}_{o} + \mathbf{D} \{\Delta\boldsymbol{\varepsilon} - \mathbf{n}\, \Delta\boldsymbol{\mu}\}\} - \{\mathbf{r} + \mathbf{H}\, \Delta\boldsymbol{\mu}\} \leq \mathbf{0} \qquad (15a)$$

$$\Delta\boldsymbol{\mu} \geq \mathbf{0} \quad , \quad \boldsymbol{\varphi}^{T} \Delta\boldsymbol{\mu} = 0 \qquad (15b,c)$$

where $\mathbf{n}$ is a matrix that collects the unit vectors $\mathbf{n}_k$, while $\mathbf{r}$ and $\Delta\boldsymbol{\mu}$ are vectors whose entries are the parameters r_k and $\Delta\mu_k$. As for $\boldsymbol{\varphi}$ and $\mathbf{H}$, they are, respectively, a vector of *yield functions* and a *hardening matrix*, which somehow plays the role of the matrix $\mathbf{G}$ considered above (of course, $\mathbf{H}=\mathbf{0}$ for *elastic perfectly-plastic* materials). A constant $\mathbf{H}$-matrix implies *linear hardening* and, in consequence, the conditions (15a,b,c) define a *linear complementarity problem*, for which well-known solution techniques do exist [7]. Instead, in the case of *nonlinear hardening*, we can introduce a convenient function $\psi(\Delta\boldsymbol{\mu})$ and consider the vector $\{\partial\psi/\partial\Delta\boldsymbol{\mu}\}$ instead of the product $\mathbf{H}\, \Delta\boldsymbol{\mu}$.

Clearly, we can also reason in terms of *dissipation functions* and preserve the framework discussed in the previous Sections. In fact, if we assume a number of *slip devices* equal to the number of planes and introduce the vector $\boldsymbol{\chi}'=\mathbf{n}^{T}\{\boldsymbol{\sigma}_{o}+ \mathbf{D}\{\Delta\boldsymbol{\varepsilon}-\mathbf{n}\, \Delta\boldsymbol{\mu}\}\}-\mathbf{H}\, \Delta\boldsymbol{\mu}$, the *dissipated energy* per unit volume can be expressed in the form $d(\Delta\boldsymbol{\mu})=\Sigma_k\, r_k\, \Delta\mu_k=\mathbf{r}^{T}\Delta\boldsymbol{\mu}$, in which the k-th plastic multiplier $\Delta\mu_k$ can be non-zero only if the k-th element of $\boldsymbol{\chi}'$ (*i.e.*, χ_k) is equal to r_k.

At this stage, it should be observed that *piecewise-linear yield functions* can be of practical use especially in the *Haigh-Westergaard space*, as suggested by classical Tresca or Mohr-Coulomb's yield surfaces or by a possible approximation of Drucker-Prager's yield surface by means of appropriate planes. Therefore, it is worth noting that the application of the *internal variable* approach (with its consequent convergence properties) is quite straightforward even when the constitutive law is enforced in the *Haigh-Westergaard space*, while the unknown vectors $\Delta\mathbf{U}$ and $\Delta\boldsymbol{\lambda}$ are defined in a generic x_1-x_2-x_3 space. This statement can be immediately checked.

For instance, it is possible to start from eqns. (11) under the hypothesis that the *generalized forces* are equivalent to stresses defined in any Euclidean space and that *Mises' yield condition* is applicable. The key point is that the matrix $\mathbf{G}$, as well as any other term in eqn. (11) and in the function $\omega(\Delta\mathbf{U},\Delta\boldsymbol{\lambda})$ introduced through eqn. (12), maintains its meaning, no matter if axis rotations are needed in order to satisfy the constitutive law in a different space (*e.g.*, in the *Haigh-Westergaard space*).

In actual fact, everything is fully analogous to the process to be followed in the space of *deviatoric stresses*. In the end, as shown below, it simply happens that the *incremental plastic strains* are determined at each *strain point* in the *Haigh-Westergaard space* (during the *correction phase*) and transformed into the correct components of the vector $\Delta\boldsymbol{\lambda}$ by using appropriate rotation matrices.

For the sake of example, let us consider a structural system consisting of an elastic-plastic material characterized by *linear isotropic hardening*, whose mechanical properties are E=200,000 Mpa, ν=0.3, G'=33,333 MPa and σ_Y=200 MPa (*yield stress*). In the *Haigh-Westergaard space*, *Mises' criterion* implies a cylindrical yield surface, which is defined by the function $\sigma_I^2+\sigma_{II}^2+\sigma_{III}^2-\sigma_I\sigma_{II}-\sigma_{II}\sigma_{III}-\sigma_I\sigma_{III}-\sigma_Y^2=0$, if σ_I, σ_{II}, σ_{III} denote *principal stresses*. The radius of its circular cross section is $\hat{r}$=163.299 MPa.

Next, just to simulate what might happen in the context of a numerical analysis, we can assume that the stress components σ_{11}*, σ_{22}*, σ_{33}*, σ_{12}*, σ_{23}*, σ_{31}* determined at the end of the *prediction phase* at a certain *strain point* are 100, 250, 130, 80, 120, 40 MPa. Thus, the corresponding principal stresses are σ_I*=52.3417 MPa, σ_{II}*=72.3821 MPa, σ_{III}*=355.2762 MPa, while the point whose coordinates are σ_I*, σ_{II}*, σ_{III}* is at a *distance* equal to 239.583 MPa from the axis of the cylindrical yield surface. In other words, plastic strain increments $\Delta\varepsilon_{ij}^{p}$ must be determined in order to find stress components in agreement with *Mises' yield condition*.

It turns out that the increments $\Delta\varepsilon_I^{p}$=-0.0001554500, $\Delta\varepsilon_{II}^{p}$=-0.0001265131, $\Delta\varepsilon_{III}^{p}$=0.0002819631 satisfy the constitutive law in the *Haigh-Westergaard space* in accordance with the *backward-difference* concept. In fact, we obtain the stress vector $\boldsymbol{\sigma}=\boldsymbol{\sigma}^{*}-\mathbf{D}\, \Delta\boldsymbol{\varepsilon}^{p}=[\sigma_I\ \sigma_{II}\ \sigma_{III}]^{T}$, whose entries are 76.2570, 91.8456 and 311.8974 MPa, respectively, if $\mathbf{D}$ denotes the stiffness matrix, while $\boldsymbol{\sigma}^{*}=[\sigma_I^{*}\ \sigma_{II}^{*}\ \sigma_{III}^{*}]^{T}$ and $\Delta\boldsymbol{\varepsilon}^{p}=[\Delta\varepsilon_I^{p}\ \Delta\varepsilon_{II}^{p}\ \Delta\varepsilon_{III}^{p}]^{T}$. Of course, the point whose coordinates are σ_I, σ_{II}, σ_{III} must belong to the updated yield surface and the new radius of the cross section turns out to be 186.362 MPa.

Note that the distance of any point whose coordinates are $\hat{\sigma}_I$, $\hat{\sigma}_{II}$, $\hat{\sigma}_{III}$ from the *hydrostatic axis* can be easily computed by setting $\sigma_I=t$, $\sigma_{II}=t$, $\sigma_{III}=t$ (*i.e.*, by writing the parametric equations of the *hydrostatic axis*) and by determining the value of the parameter t, which minimizes the square root of $((\hat{\sigma}_I-t)^2+(\hat{\sigma}_{II}-t)^2+(\hat{\sigma}_{III}-t)^2)$.

Since the *plastic incremental strains* are deviatoric, an alternative equation can be considered in order to derive the actual stress vector $\boldsymbol{\sigma}$ in the *Haigh-Westergaard space*. Indeed, we can set $\boldsymbol{\sigma}=\boldsymbol{\sigma}^{*}-2G\,\Delta\boldsymbol{\varepsilon}^{p}$. Similarly, when we consider the coordinates of the point that represents the stress acting on the *slip device* (and belongs to the *initial yield surface*), we obtain $\chi_I=\sigma_I-2G'\Delta\varepsilon_I^p=86.62$ MPa, $\chi_{II}=\sigma_{II}-2G'\Delta\varepsilon_{II}^p=100.28$ MPa, $\chi_{III}=\sigma_{III}-2G'\Delta\varepsilon_{III}^p=293.10$ MPa.

As for the energy per unit volume which is given by $G'((\Delta\varepsilon_I^p)^2+(\Delta\varepsilon_{II}^p)^2+(\Delta\varepsilon_{III}^p)^2)$ and corresponds to the energy stored in the torsion spring in Fig. 4, it is equal to $G'\Delta\varepsilon_{ij}^p\Delta\varepsilon_{ij}^p$. Instead, the dissipated energy per unit volume $\chi_I\Delta\varepsilon_I^p+\chi_{II}\Delta\varepsilon_{II}^p+\chi_{III}\Delta\varepsilon_{III}^p$ is equal to $\chi_{ij}\ \Delta\varepsilon_{ij}^p$ and, naturally, it is also equal to $\check{r}$ multiplied by the modulus of the vector whose components are $\Delta\varepsilon_I^p$, $\Delta\varepsilon_{II}^p$, $\Delta\varepsilon_{III}^p$ or $\Delta\varepsilon_{ij}^p$. Consequently, if we had a *piecewise-linear yield surface* in the *Haigh-Westergaard space* instead of the cylindrical one due to Mises, a product such as $\chi_{ij}\ \Delta\varepsilon_{ij}^p$ in the x_1-x_2-x_3 space would correspond to the dissipated energy $d(\Delta\boldsymbol{\mu})=\Sigma_k\ r_k\ \Delta\mu_k$ per unit volume introduced above. Thus, it can be stated that the convergence properties discussed in the previous Section still hold if we consider *piecewise-linear yield surfaces* in the *Haigh-Westergaard space* and satisfy the constitutive law in this space.

Incidentally, we can also point out that, in the case of *Mises' yield criterion*, the results obtained in the *Haigh-Westergaard space* are absolutely identical to what we could find in the space of the *deviatoric stresses*. As a matter of fact, the relevant (spherical) yield surface $s_{ij}\ s_{ij}-k^2=0$ would be characterized by the same radius $\check{r}=163.299$ Mpa and the *deviatoric stresses* $s_{ij}*$ corresponding to the stresses $\sigma_{ij}*$ would eventually lead to the same plastic strain increments $\Delta e_{ij}^p=\Delta\varepsilon_{ij}^p$. In addition, the dissipated energy per unit volume is again equal to $\check{r}$ multiplied by the modulus of the vector whose components are $\Delta\varepsilon_{ij}^p$.

VI Closing Remarks

In the first part of the paper, a non-traditional *internal variable approach* to the elastic-plastic analysis of structural systems, whose material is *stable in Drucker's sense*, has been revised. The main feature of the theoretical framework discussed here is that the constitutive law is enforced by focusing on *dissipation functions* rather than *yield functions*. This alternative formulation has interesting consequences when computational aspects come into play.

In fact, as already shown in the past with reference to the case of Mises' yield condition, the use of *dissipation functions* implies that the numerical solution of an *elastic-plastic incremental problem* corresponds to the minimum point of a convex unconstrained function. Instead, convenient constrains (to be introduced through *yield functions*) should be taken into account in order to derive similar properties by starting with a traditional formulation.

It has also been observed that convergence toward the correct solution can be guaranteed by making use of the *backward difference* concept. As explained in the paper, this implicit algorithm essentially consists of determining incremental plastic strains that satisfy the associated flow rule with respect to the stresses and the yield functions determined at the end of each time-step.

The innovative contribution of the present work is a detailed investigation of the possible use of *piecewise linear yield surfaces*, which practically allow one to consider any yield condition in any stress space (*e.g.*, Tresca's yield criterion in the *Haigh-Westergaard space*).

Hence, adequate *dissipation functions* can be introduced in order to define the convex unconstrained function, whose minimum point must correspond to the solution of the *elastic-plastic incremental problem*.

In the end, it was possible to prove that the *internal variable approach* presented here can be applied to any yield criterion (not only to the relatively simple case of Mises' yield condition). Therefore, the relevant properties in terms of convergence of the *backward-difference integration scheme* still hold, in the presence of *piecewise linear yield surfaces* and properly defined *dissipation functions*, under the condition that the material is *stable in Drucker's sense*.

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