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Thermodynamic Properties Of A Microcanonical Ensemble Of Mixture Of Helium-3 And Helium-4 Isotopes With Duo-Fermion Spin Degeneracy

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***Abstract*— A system of a mixture of Helium-3 and Helium-4 was considered in a weakly interacting system in this study. The partition function with duo-fermion spin was developed to bring out the superfluid properties of the system. The partition function was used to study and analyze the thermodynamic properties such as internal energy, specific heat and entropy of the system by singling out the duo-fermion spin component. It was found out that the internal energy and entropy increases with temperature. At high temperatures, the internal energy and entropy remains constant due to particle saturation. The specific heat increases exponentially with temperature and attains a peak value of $5.0x10^{19}$ $electron\, Volt\, per\, mole\, Kelvin$ at 37 Kelvins.**

Keywords—Micro canonical ensemble, Helium-3, Helium-4, Partition Function, Thermodynamic Properties and Duo-spin component

1. INTRODUCTION

A micro canonical ensemble represents a collection of configurations of isolated systems that have reached thermal equilibrium. A system is isolated from its environment if it does not exchange either particles or energy with its surroundings. The properties of such a system like volume, internal energy and number of particles are the same for all configurations. In this study a configuration of a mixture of Helium-3 and Helium-4 isotope is studied considering the role of the duo-fermion spin degeneracy. The partition function and internal energy were studied by obtaining the most probable distribution of boson (Helium-4) – fermion (Helium-3) mixture with duo- fermion spin degeneracy.

First attempt to generalize quantum Bose and Fermion statistics for a mixture of bosons and fermions was made by Gentile [1]. Gentile proposed statistics in which up to N particles ($N \gg 1$) were allowed to occupy a single quantum state instead of just one particle for Fermi case due to Pauli's exclusion principle. However, Gentile's approach was found to be too much generalization and violated the conventionally accepted Pauli's exclusion principle. Furthermore his model did not distinguish which particles were fermions and which ones were bosons. However, his work laid a foundation that statistical mechanics of a mixture of bosons and fermions can be worked out.

The next attempt was of Medvedev [2], in his work he proposed a new class of identical particles which may exhibit both Bose and Fermi statistics with respective probabilities of p_b' and p_f'. The model admits only Bose –Einstein and Fermi-Dirac statics as existing. He assumed that a particle is neither a pure boson nor a pure fermion. He let another particle which interacts with the first one, play the role of an external observer. During the interaction it performs a measurement at the first particle and identifies it either a boson or a fermion. According to the result of this measurement it interacts with the first particle as if the last particle is a fermion or a boson, respectively. The first particle is an observer for the second particle and so the process is symmetric. Note that $(p_b' + p_f')$ is not necessarily equal to one and if not, it means that second particle (observer) does not detect the first particle. The probability of this is $(1 - p_b' - p_f')$. The statistical uncertainty introduced here may be either the intrinsic property of a particle itself or the experimental uncertainty of measurement process.

Another attempt is the 'statistical independence model' of Landau and Lifshitz [3] in which two weakly interacting subsystems (bosons and fermions) are together regarded as one composite system and the subsystems are assumed to be quasi-closed. The statistical distribution for such a mixture is the product of the individual probabilities for two subsystems, one corresponding to bosons and the other to fermions. With these assumptions, the statistical independence model will only hold for and ideal gas assembly of bosons and fermions. In reality such an assembly does not exist and hence this model cannot be used for real mixtures of bosons and fermions.

Chan et.al. [4] Studied on disorder on superfluid 3_{He}- 4_{He} mixtures. Their studies related more of an ideal system rather than a real system.

Khanna et.al. [5] They developed a statistical distribution model for a mixture that was dominated by bosons. The partition function developed worked out well for liquid 3_{He}-4_{He} mixture.

Ayodo et.al. [6] Developed a statistical distribution model of a mixture that was dominated by fermions. They proposed that there exists a pair interaction between the bosons and fermions and that the concentrations of the bosons and fermions are different. Invoking the pair interaction in a given state of equilibrium, some fermions will be left unpaired. The value of occupation number of fermions in a given state will not exceed, rather will be much less than the degeneracy of the state so that Pauli's exclusion principle is not violated. A fermion concentration of 0.7 was used in this model and thermodynamic properties of 3_{He} - 4_{He} mixture were studied.

Lusamamba et.al. [7] Studied internal energy of grand canonical ensemble of a mixture of Helium isotopes with duo-fermion spin degeneracy. They found out that at low temperatures of below 5K the internal energy of the system approaches zero but it increases to a maximum value of 0.0025eV. This results agrees earlier study by Sakwa et.al. [8].

Lusamamba et.al. [10] Studied the effect of duo fermion spin degeneracy on the specific heat and entropy of a mixture of 3_{He} - 4_{He} isotopes and obtained a peak value of 0.008eV/mol.K at 35K for specific heat.

In this study we seek to study the effect of duo-fermion spin degeneracy component on a micro canonical ensemble and hence develop its partition function and study its internal energy, specific heat and entropy.

2.0 THEORETICAL DERIVATIONS

Consider a micro canonical assembly of N particle in which there are N_b bosons and N_f fermions such that:-

$$N = N_b + N_f \qquad (1)$$

Let $\varepsilon_1, \varepsilon_2, \varepsilon_3 . \varepsilon_j$. be the energy states of the assembly and in the statistical equilibrium, the number of particles assigned to this energy levels are $n_1, n_2, n_3, . n_j$,. respectively such that the number n_j must satisfy the conditions requiring the conservation of particles, N, and conservation of energy, E, such that:-

$$\sum_{j=1}^{\infty} n_j = N \text{ and } \sum_{j=1}^{\infty} n_j \varepsilon_j = E \qquad (2)$$

and that,

$$n_{jbf} = n_{jb} + n_{jf} \qquad (3)$$

Where n_{jb} is number of bosons in the j-th energy level and n_{jf} is number of fermions in the j-th energy level.

Let ω_j be the number of states in the j-th energy level, that is, ω_j is the degeneracy of the j-th level. Then the number of ways P_{jb} in which n_{jb} bosons can be assigned to ω_j state in the j-th level is given by equation (4)

$$P_{jb} = (\omega_j)^{n_{jb}} \qquad (4)$$

Similarly, if for fermions we allow only two particles to be put in a compartment, then the number of ways P_{jf} in which n_{jf} fermions can be assigned to ω_j states in the j-th energy level is given by equation (5)

$$P_{jf} = \frac{\omega_j!}{n_{jf}!(\omega_j - n_{jf})!} \qquad (5)$$

The probabilities from the above equations (4) and (5) confirms that bosons and fermion occupancy in the energy level is due to pair interaction. Occupancy of fermions for any energy level is independent of boson occupancy of same energy level i.e. $n_{jb} > n_{jf}$ hence the above probability are multiplicative. The combined number of ways of assigning n_{jb} bosons and n_{jf} fermions in an energy level is given by equation (6)

$$P_{jbf} = \frac{(2_j)^{n_{jb}}.(2_j)!}{n_{jf}!(2-n_{jf})!(n_{jb}-n_{jf})!} \qquad (6)$$

Where, 2 is the spin degeneracy of a binary system of helium isotopes particles. Dividing through by $n_{jf}!$, in equation (6), eliminates the permutations among identical pairs. On the other hand, dividing through equation (6) by $(n_{jb} - n_{jf})!$, eliminates all identical complexions of unpaired bosons. The number of ways of distributing these particles among the independent energy levels ($j = 1,2,3.$) is given by the product of such expressions in equation (7)

$$C_{bf} = \prod_{j=1}^{\infty} P_{jbf} = \prod_{j=1}^{\infty} \left[\frac{(2_j)^{n_{jb}}.(2_j)!}{n_{jf}!(2-n_{jf})!(n_{jb}-n_{jf})!}\right] \qquad (7)$$

Equation (7) is used to calculate the most probable distribution of particles in a boson – fermion mixture with duo-fermion spin degeneracy.

2.1 MOST PROBABLE DISTRIBUTION OF PARTICLES FOR THE ENSEMBLE

The aim is to calculate the values of to n_{jb} and n_{jf}, the statistical count C_{bf} is maximum if we allow C_{bf} to vary with n_{jb} and n_{jf} and set the result equal to zero while N and E are fixed. The distribution numbers and corresponding energies must satisfy the following conditions,

$$\sum_{j=1}^{\infty} n_{jb} = N_b \text{ and } \sum_{j=1}^{\infty} n_{jf} = N_f \qquad (8)$$

$$\sum_{j=1}^{\infty} n_{jb}\varepsilon_j = E_b \text{ and } \sum_{j=1}^{\infty} n_{jf}\varepsilon_j = E_f \qquad (9)$$

Where N_b and N_f are total number of bosons and fermions in the ensemble such that:-

$$N = N_b + N_f \qquad (10)$$

Similarly the total energy E of the ensemble is given by:-

$$E = E_b + E_f \qquad (11)$$

Where E_b and E_f are internal energies of the bosons and fermions respectively.

For maximum C_{bf} we get:-

$$\partial \ln C_{bf} = \sum_{j=1}^{\infty}\left[\frac{\partial}{\partial n_{jb}}\left(\ln C_{bf}\right)dn_{jb}\right] + \sum_{j=1}^{\infty}\left[\frac{\partial}{\partial n_{jf}}\left(\ln C_{bf}\right)dn_{jf}\right] = 0 \qquad (12)$$

Where ∂n_{jb} and ∂n_{jf} are allowable changes in the distribution number from the required distribution, they should continue to satisfy the equations (8) and (9).

Since N and E are constant, the variation in n_{jb} and n_{jf} must satisfy equations (12) and (13).

$$\sum_{j=1}^{\infty} dn_{jb} + \sum_{j=1}^{\infty} dn_{jf} = 0 \qquad (13)$$

$$\sum_{j=1}^{\infty} \varepsilon_j dn_{jb} + \sum_{j=1}^{\infty} \varepsilon_j dn_{jf} = 0 \qquad (14)$$

Combining equations (12), (13) and (14) by Lagrange's undetermined multipliers denoted by α and β. Multiplying 1st and 2nd terms of equation (13) by $-\alpha_b$ and $-\alpha_f$ respectively, equation (14) by $-\beta$ and adding to equation (11) gives equation (15),

$$\sum_{j=1}^{\infty} -\alpha_b dn_{jb} + \sum_{j=1}^{\infty} -\beta\varepsilon_j dn_{jb} + \sum_{j=1}^{\infty} \frac{\partial}{\partial n_{jb}}\left(\ln C_{bf}\right)dn_{jb} = 0 \qquad (15)$$

Rearranging equation (15) yields equation (16)

$$\sum_{j=1}^{\infty}\left\{\frac{\partial}{\partial n_{jb}}\left(\ln C_{bf}\right) - \left(\alpha_b + \beta\varepsilon_j\right)\right\}dn_{jb} = 0 \qquad (16)$$

In the same case, multiplying 1st and 2nd terms of equation (13) by $-\alpha_b$ and $-\alpha_f$ respectively, equation (14) by $-\beta$ and adding to equation (15), we get equation (17),

$$\sum_{j=1}^{\infty}\left\{\frac{\partial}{\partial n_{jf}}\left(\ln C_{bf}\right) - \left(\alpha_f + \beta\varepsilon_j\right)\right\}dn_{jf} = 0 \qquad (17)$$

Adding equation (16) to equation (17), yields equation (18),

$$\sum_{j=1}^{\infty}\left\{\frac{\partial}{\partial n_{jb}}\left(\ln C_{bf}\right) - \left(\alpha_b + \beta\varepsilon_j\right)\right\}dn_{jb} + \sum_{j=1}^{\infty}\left\{\frac{\partial}{\partial n_{jf}}\left(\ln C_{bf}\right) - \left(\alpha_f + \beta\varepsilon_j\right)\right\}dn_{jf} = 0 \qquad (18)$$

In equation (18), we let the coefficients of dn_{jb} and dn_{jf} to tend to zero; since dn_{jb} and dn_{jf} are allowable variables hence equation (18) can be written as,

$$\frac{\partial}{\partial n_{jb}}\left(\ln C_{bf}\right) - \left(\alpha_b + \beta\varepsilon_j\right) = 0 \quad \text{and} \quad \frac{\partial}{\partial n_{jf}}\left(\ln C_{bf}\right) - \left(\alpha_f + \beta\varepsilon_j\right) = 0 \qquad (19)$$

Since $\ln\left(\frac{2}{n_{jb}-n_{jf}}\right) = \left(\alpha_b + \beta\varepsilon_j\right)$ or $\frac{2}{n_{jb}+n_{jf}} = exp\left(\alpha_b + \beta\varepsilon_j\right)$ (20)

Then making n_{jb} the subject of equation (20) yields equation(21),

$$n_{jb} = \frac{2+n_{jf}exp\left(\alpha_b+\beta\varepsilon_j\right)}{exp\left(\alpha_b+\beta\varepsilon_j\right)} = \frac{2}{exp\left(\alpha_b+\beta\varepsilon_j\right)} + n_{jf} \qquad (21)$$

Substituting for $\beta = \frac{1}{KT}$ and $\alpha_b = \frac{\mu_b}{KT}$, in equation (21) we get equation (22),

$$n_{jb} = \frac{2}{exp\left(\frac{-\mu_b+\varepsilon_j}{KT}\right)} + n_{jf} = 2.exp\left[\frac{\mu_b-\varepsilon_j}{KT}\right] + n_{jf} \qquad (22)$$

Similarly

$$\frac{\partial}{\partial n_{if}}\left(\ln C_{bf}\right) - \left(\alpha_f + \beta\varepsilon_j\right) = \ln\left(\frac{\left(2-n_{jf}\right)\left(n_{jb}-n_{jf}\right)}{n_{jf}}\right) - \left(\alpha_f + \beta\varepsilon_j\right) = 0$$

Or

$$\left[\frac{(2-n_{jf})(n_{jb}-n_{jf})}{n_{jf}}\right] = exp\left(\alpha_f + \beta\varepsilon_j\right) \qquad (23)$$

Multiplying equation (23) on both sides by n_{jf}, we get equation (24),

$$\left(2-n_{jf}\right)\left(n_{jb}-n_{jf}\right) = n_{jf}.exp\left(\alpha_f + \beta\varepsilon_j\right) \qquad (24)$$

Substituting n_{jb} from equation (22) into equation (24), we get equation (25)

$$\left(2-n_{jf}\right)\left(\frac{2}{exp\left(\alpha_b+\beta\varepsilon_j\right)}\right) = n_{jf}.exp\left(\alpha_f + \beta\varepsilon_j\right) \qquad (25)$$

Multiplying both sides with $exp\left(\alpha_b + \beta\varepsilon_j\right)$ in equation (25), we obtain equation (26)

$$n_{jf} = \frac{4}{2+exp\left(\alpha_f+\beta\varepsilon_j\right).exp\left(\alpha_b+\beta\varepsilon_j\right)} \qquad (26)$$

Substituting $\alpha_f = \frac{\mu_f}{K_BT}$, $\beta = \frac{1}{K_BT}$ and $\alpha_b = -\frac{\mu_b}{K_BT}$ in equation (26) obtain equation (27);

$$n_{jf} = \frac{4}{2+exp\left[\frac{-\mu_f+\varepsilon_j-\mu_b+\varepsilon_j}{K_BT}\right]} = \frac{2}{1+\frac{1}{2}exp\left[\frac{2\varepsilon_j-\mu_f+-\mu_b}{K_BT}\right]}. \qquad (27)$$

Equation (27) gives the expression for the distribution particles for a fermion (helium-3 particles) which have a spin degeneracy of two.

From equation (27), letting the exponential term to be equal to zero, we obtain equation (28).

$$n_{jf} = \frac{2}{1+\frac{1}{2}X0} = 2 \qquad (28)$$

Equation (28) gives a result of 2, this tells us that the maximum number of fermions that occupy a given energy level is 2 as long as they have opposite spin. This satisfies the Pauli's exclusion principle.

Similarly, substitute n_{jf} in equation (22), we get equation (29);

$$n_{jb} = 2\left[exp\left(\frac{\mu_b-\varepsilon_j}{K_BT}\right)\right] + n_{jf} = 2.exp\left(\frac{\mu_b-\varepsilon_j}{K_BT}\right) + 4\left[2 + exp\left(\frac{2\varepsilon_j-\mu_b-\mu_f}{K_BT}\right)\right]^{-1} \qquad (29)$$

Equation (29) gives the most probable distribution particles for bosons (Helium-4) in the mixture. Where T is the absolute temperature, K_B is the Boltzmann constant, μ_f if the chemical potential of fermions and μ_b is the chemical potential of bosons

Substituting equation (27) and equation (29) into equation (3), we get equation (30);

$$n_{jbf} = 2.exp\left(\frac{\mu_b-\varepsilon_j}{K_BT}\right) + 4\left[2 + exp\left(\frac{2\varepsilon_j-\mu_b-\mu_f}{K_BT}\right)\right]^{-1} + 4\left[2 + exp\left[\frac{2\varepsilon_j-\mu_f+-\mu_b}{K_BT}\right]\right]^{-1} \text{ or}$$

$$n_{jbf} = 8\left[2 + exp\left(\frac{2\varepsilon_j-\mu_b-\mu_f}{K_BT}\right)\right]^{-1} + 2.exp\left(\frac{\mu_b-\varepsilon_j}{K_BT}\right) \quad (30)$$

Equation (30) gives an expression for the most probable distribution for a mixture of bosons and fermions with a duo-fermion spin degeneracy.

2.2 PARTITION FUNCTION

The partition function (Q) of a micro canonical ensemble where N_b and N_f are variables can be derived from the standard equation (31) [3, 9].

$$Q = Nexp\left(\frac{-\mu_f-\mu_b}{kT}\right) \quad (31)$$

Substituting equation (30) in the equation (31) and simplifying it we get equation (32);

$$Q = 2\left[4 + 2exp\left[\frac{(\mu_b-\mu_f)}{(k.T)}\right] + exp\left[2.\frac{(-\mu_f+\varepsilon_j)}{(k.T)}\right]\right] \times \frac{exp\left[\frac{-(\mu_f+\mu_b)}{(k.T)}\right]}{\left[2+exp\left[\frac{2\varepsilon_j-\mu_b-\mu_f}{k.T}\right]\right]} \quad (32)$$

Equation (32) gives the partition function of a micro canonical ensemble of a mixture of helium isotopes where helium-3 particles have a degeneracy of 2. From equation (32) it ought to be understood that partition function consists of two brackets; the first bracket consists of the difference of the chemical potential of bosons and fermions and a factor 2 which is due to interaction of bosons and fermions in the mixture. The second bracket comprises of μ_f which implies that fermions are not affected by the distribution of the bosons in the mixture because in whichever way the bosons are arranged or distributed, there should only be two fermions (spin up and spin down)

The partition function as a function of temperature is given by equation (33);

$$Q(T) = \left\{2\left[4 + 2exp\left[\frac{(\mu_b-\mu_f)}{(k.T)}\right] + exp\left[2.\frac{(-\mu_f+\varepsilon_j)}{(k.T)}\right]\right] \times \frac{exp\left[\frac{-(\mu_f+\mu_b)}{(k.T)}\right]}{\left[2+exp\left[\frac{2\varepsilon_j-\mu_b-\mu_f}{k.T}\right]\right]}\right\} \quad (33)$$

2.3 INTERNAL ENERGY

For a micro canonical ensemble, internal energy E is given by the standard equation (34)

$$E = NkT^2\left(\frac{\partial lnQ}{\partial T}\right)_V \quad (34)$$

Substituting equation (33) in equation (34) of internal energy and simplifying, it yields equation (35). Equation (35) is an expression of internal energy of a micro canonical ensemble as a function of temperature T.

$$E = NkT^2\Bigg[\frac{\left[-4\times\frac{(\mu_b-\mu_f)}{(kT^2)}.exp\left[\frac{(\mu_b-\mu_f)}{(kT)}\right]-8\frac{(-\mu_f+\varepsilon_j)}{(kT^2)}.exp\left[2.\frac{(-\mu_f+\varepsilon_j)}{(kT)}\right]\right]}{\left[8+4.exp\left[\frac{(\mu_b-\mu_f)}{(kT)}\right]+4exp\left[2.\frac{(-\mu_f+\varepsilon_j)}{(kT)}\right]\right]} \times \frac{exp\left[\frac{-(\mu_f+\mu_b)}{(k.T)}\right]}{\left[2+exp\left[\frac{2\varepsilon_j-\mu_b-\mu_f}{k.T}\right]\right]} - ln\left[8 + 4exp\left[\frac{(\mu_b-\mu_f)}{(kT)}\right] + 4exp\left[2.\frac{(-\mu_f+\varepsilon_j)}{(kT)}\right]\right].\frac{(\mu_f-\mu_b)}{(kT^2)}.\frac{exp\left[\frac{(-\mu_f-\mu_b)}{(kT)}\right]}{\left[2+exp\left[\frac{2\varepsilon_j-\mu_b-\mu_f}{kT}\right]\right]} + ln\left[8 + 4exp\left[\frac{(\mu_b-\mu_f)}{(kT)}\right] + 4exp\left[2.\frac{(-\mu_f+\varepsilon_j)}{(kT)}\right]\right].\frac{exp\left[\frac{(-\mu_f-\mu_b)}{(kT)}\right]}{\left[2+exp\left[\frac{(2\varepsilon_j-\mu_b-\mu_f)}{(kT)}\right]\right]}.\frac{(2\varepsilon_j-\mu_b-\mu_f)}{(kT^2)}exp\left[\frac{(2\varepsilon_j-\mu_b-\mu_f)}{(kT)}\right]\Bigg]$$

. (35)

2.4 SPECIFIC HEAT

For a micro canonical ensemble, the specific heat C_v is given by the standard equation (36) [3, 9]

$$C_V = 2NkT\left(\frac{\partial ln(Q)}{\partial T}\right)_V + NkT^2\left(\frac{\partial^2 lnQ}{\partial T^2}\right)_V \quad (36)$$

Substituting equation (33) in equation (36) of specific heat and simplifying, it yields equation (37). Equation (37) is an expression of specific heat of a micro canonical ensemble as a function of temperature T.

$$C_V(T) = 2NkT\left(\frac{\partial ln\left(2\left[4+2exp\left[\frac{(\mu_b-\mu_f)}{(k.T)}\right]+exp\left[2.\frac{(-\mu_f+\varepsilon_j)}{(k.T)}\right]\right]\times\frac{exp\left[\frac{-(\mu_f+\mu_b)}{(k.T)}\right]}{\left[2+exp\left[\frac{2\varepsilon_j-\mu_b-\mu_f}{k.T}\right]\right]}\right)}{\partial T}\right)_V + NkT^2\left(\frac{\partial^2 ln\left(2\left[4+2exp\left[\frac{(\mu_b-\mu_f)}{(k.T)}\right]+exp\left[2.\frac{(-\mu_f+\varepsilon_j)}{(k.T)}\right]\right]\times\frac{exp\left[\frac{-(\mu_f+\mu_b)}{(k.T)}\right]}{\left[2+exp\left[\frac{2\varepsilon_j-\mu_b-\mu_f}{k.T}\right]\right]}\right)}{\partial T^2}\right)_V$$

2.5 ENTROPY

For a micro canonical ensemble, the entropy S is given by the standard equation (38) [3, 9]

$$S = kN\left[ln\left(\frac{Q}{N}\right)+T\frac{\partial lnQ}{\partial T}\right] \quad (38)$$

Substituting equation (33) in equation (38) of entropy and simplifying, it yields equation (39). Equation (39) is an expression of entropy of a micro canonical ensemble as a function of temperature T.

$$\begin{aligned}S(T) &= K.N\left[ln\left[\left[8+4.\exp\left[\frac{(\mu_b-\mu_f)}{(k.T)}\right]+2.\exp\left[2.\frac{(-\mu_f+\varepsilon_j)}{(k.T)}\right]\right].\frac{\exp\left[\frac{(-\mu_f-\mu_b)}{(k.T)}\right]}{\left[\left[2+\exp\left[\frac{(2.\varepsilon_j-\mu_b-\mu_f)}{(k.T)}\right]\right].N\right]}\right]\right.\\ &+T.\frac{\left[\left[-4.\frac{(\mu_b-\mu_f)}{(k.T^2)}.\exp\left[\frac{(\mu_b-\mu_f)}{(k.T)}\right]-4.\frac{(-\mu_f+\varepsilon_j)}{(k.T^2)}.\exp\left[2.\frac{(-\mu_f+\varepsilon_j)}{(k.T)}\right]\right].\frac{\exp\left[\frac{(-\mu_f-\mu_b)}{(k.T)}\right]}{\left[2+\exp\left[\frac{(2.\varepsilon_j-\mu_b-\mu_f)}{(k.T)}\right]\right]}-\left[\mathbf{8}+4.\exp\left[\frac{(\mu_b-\mu_f)}{(k.T)}\right]+2.\exp\left[2.\frac{(-\mu_f+\varepsilon_j)}{(k.T)}\right]\right].\frac{(-\mu_f+\varepsilon_j)}{(k.T^2)}.\frac{\exp\left[\frac{(-\mu_f-\mu_b)}{(k.T)}\right]}{\left[2+\exp\left[\frac{(2.\varepsilon_j-\mu_b-\mu_f)}{(k.T)}\right]\right]}+\left[\mathbf{8}+4.\exp\left[\frac{(\mu_b-\mu_f)}{(k.T)}\right]+2.\exp\left[2.\frac{(-\mu_f+\varepsilon_j)}{(k.T)}\right]\right].\frac{\boldsymbol{exp}\left[\frac{(-\boldsymbol{\mu_f}-\boldsymbol{\mu_b})}{(\boldsymbol{k.T})}\right]}{\left[\mathbf{2}+\boldsymbol{exp}\left[\frac{(\mathbf{2}.\boldsymbol{\varepsilon_j}-\boldsymbol{\mu_b}-\boldsymbol{\mu_f})}{(\boldsymbol{k.T})}\right]\right]^2}.\frac{(2.\varepsilon_j-\mu_b-\mu_f)}{(k.T^2)}.\boldsymbol{exp}\left[\frac{(\mathbf{2}.\boldsymbol{\varepsilon_j}-\boldsymbol{\mu_b}-\boldsymbol{\mu_f})}{(\boldsymbol{k.T})}\right]\right]}{\left[\left[8+4.exp\left[\frac{(\mu_b-\mu_f)}{(k.T)}\right]+\mathbf{2}.\boldsymbol{exp}\left[\mathbf{2}.\frac{(-\boldsymbol{\mu_f}+\boldsymbol{\varepsilon_j})}{(\boldsymbol{k.T})}\right]\right].\exp\left[\frac{(-\mu_f-\mu_b)}{(k.T)}\right]\right]}.\left[\mathbf{2}\right.\\ &\left.\left.+\boldsymbol{exp}\left[\frac{(\mathbf{2}.\boldsymbol{\varepsilon_j}-\boldsymbol{\mu_b}-\boldsymbol{\mu_f})}{(\boldsymbol{k.T})}\right]\right]\right]\end{aligned}$$

. (39)

3. Results and Discussion

To perform calculations we use the experimental data in table 1 [4, 11, 12]

Parameter	Liquid Helium-3	Liquid Helium-4
Volume (cm^3)	40.00	28.00
Density (g/cm^3)	0.07	0.14
Mass (g)	2.80	3.92
Chemical Potential (eV)	$3.184x10^{-27}$	$6.215x10^{-28}$

Equation (35) is used to investigate the variation of internal energy of the system with temperature. Figure 1 shows the variation of internal energy and temperature of the system. The plot shows that as the temperature tends to 0 K, the internal energy also approaches zero. At low temperature the particles occupy the low quantum state. When the temperature is increased, the fermions shift quickly to the higher state where they possess greater kinetic energy usually manifested as internal energy. The increase in energy with temperature declines in higher temperatures is as a result of particle saturation. This plot agrees with [8, 10]

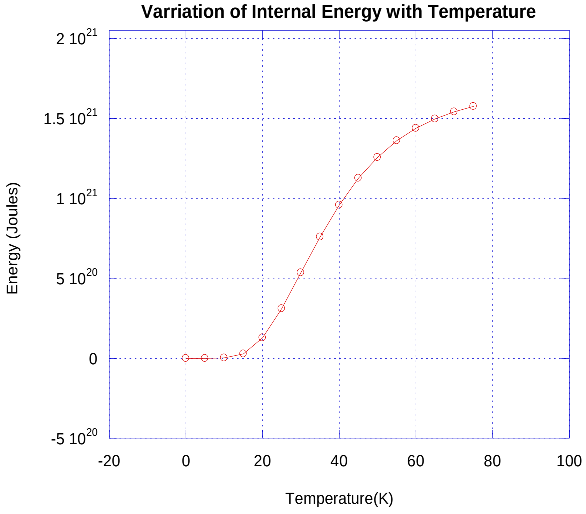

Figure 1: *The plot of variation of internal energy with Temperature*

Equation (37) was used to investigate the variation of specific heat and temperature. Figure 2 shows the variation of specific heat with temperature in the range 0K to 100K. At very low temperatures below 10K, the specific heat is zero. In this temperature range all the particles occupy the lower quantum states. When the temperature is increased, there is a shift of fermions from the lower states to higher states as a result of them gaining energy. Specific heat increasing exponentially with increase in temperature to a peak value of 5.0 x 10^{19} *eV/Mol/K* at a temperature of 37K and finally, it gradually reduces to very low values.

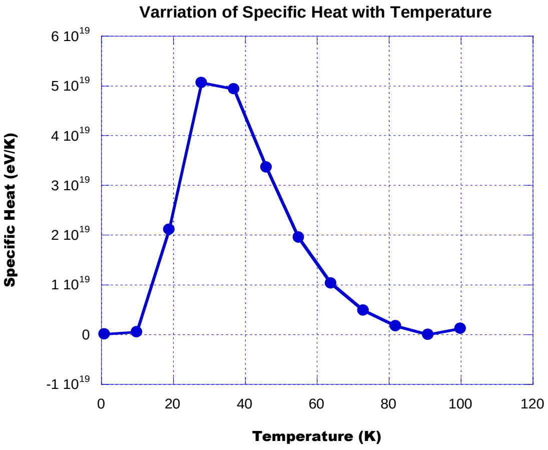

Figure 2: *Plot of variation of Specific Heat with Temperature.*

Equation (39) was used to investigate the variation of entropy with temperature as shown in figure 3. The results show that entropy varies logarithmically with temperature. At higher temperatures the entropy remains constant as temperature increases showing that it approaches saturation at higher temperatures. Figure 3 shows that the graph of entropy against temperature gives a sigmoid curve. The similar shapes of the graphs were noted by other researchers [8, 10] they noted that the entropy of a system increases with increase in temperature and also decreases with decrease in temperature. Hence the entropy is a temperature dependent property.

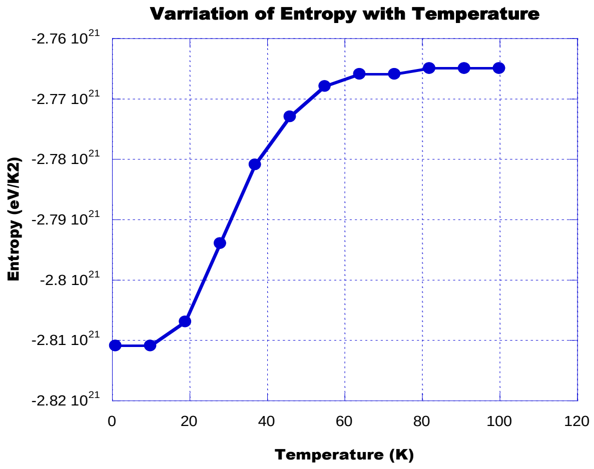

Figure 3: *Plot of variation of Entropy with Temperature*

4. Conclusion

It was found that the internal energy and entropy are very low at low temperatures of T<20K because at low temperatures there are very few states to be occupied by particles. In the temperature range of $20K < T < 60K$ the internal energy and entropy increases logarithmically with temperature. The internal energy and entropy becomes constant at temperatures $T > 60K$ this is due to particle saturation of the system. The specific heat at low temperatures $T < 10K$ is zero, in this temperature range all particles occupy the lowest quantum states. As temperatures increases particles shifts to higher energy levels and the specific heat increases exponentially to a peak value of $5.0x10^{19}eV/mol.K$ at 37K. At 37K a phase transition occurs and then the specific heat gradually reduces to very low values after the phase transition.

Conflicts of Interest

The authors declare no conflicts of interests regarding the publication of this paper.

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