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Modeling The Cable Equation Of Brain Neurons With The Finite Element Method

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***Abstract*—The brain, which acts as a central processing unit, is wired by neurons. Signals that are transmitted along the brain circuitry are amplified by action potentials generated inside the neurons. In this paper, the finite element method is used to simulate the propagation of the electric signals along the branches of the neuron. For this purpose, the method of weighted residual is applied to the cable equation of neurons. Two types of elements were tested and the corresponding finite element formulations were derived. The first type of element was a linear one-dimensional element, and the second was based on the cubic-Hermitian polynomials. The performances of the above two elements were compared to exact closed form solutions.**

The advantages/disadvantages of using these two elements, as well as the benefits of using the finite element method, are discussed.

Keywords— brain neurons; finite element method; acion potential

PLAN

I. Introduction

The brain, which acts as a central processing unit, is wired by neurons. Signals that are transmitted along the brain circuitry are amplified by action potentials generated inside the neurons. On a global scale, the neuron is composed of: the soma, which is the main body; the dendrites, which represent the signal receiving branches of the neuron; and the axon, the body transmitting the signal from one neuron to an associated one, as shown in Fig. 1. The signal is transmitted from one neuron to the next through the release of transmitters. The strength of the signal felt by the receiving neuron depends on the number of active channels and gates on its membrane. The propagation of the action potential along the dendrites/axon was determined to be mainly one-dimensional. The equation governing the propagation of the action potential is referred to as the cable equation.

Solution methods were devised for this cable equation, such as closed-form for particular properties and boundary conditions, finite difference method, and others. These methods relied on the similarities between the cable equation and equations in other fields of science.

In this paper, the finite element method is used to solve the cable equation with emphasize on its similarities with the field of continuum mechanics.

The propagation of the signals along the neuron is modeled and simulated using the method of weighted residuals. Two types of elements were tested and the corresponding finite element formulations were derived. The first type of element was a linear one-dimensional element, and the second was based on the cubic-Hermitian polynomials. The performances of the above two elements were compared to the exact closed form solutions. The advantages/disadvantages of using the above two elements, as well as the benefits of using the finite element method, are discussed.

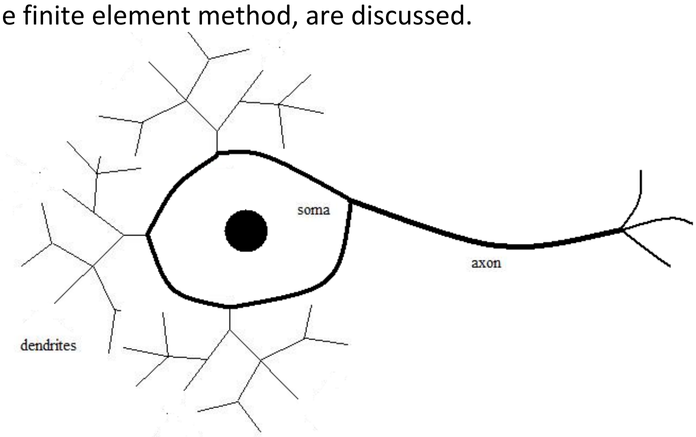

Fig. 1 Schematic representation of a brain nerve cell

II. THEORY

The brain nerve cell consists of the soma, as the main body, the dendrites and the axon. The dendrites represent the branches of the nerve cell that, in addition to the soma, receive the signals from other neurons. The action potential generated by the soma is transmitted to other neurons by the axon.

The membrane of the cell, which is mostly impermeable, allows ions to be exchanged between the extracellular and the intracellular media through channels that could be active/passive. For the active channels the flow of ions depends on the voltage across the membrane in a nonlinear fashion.

The modeling of systems could be attempted at the atomic level, molecular level, or micro-meter level [1-11]. For the present analysis, the membrane is modeled as an electric circuit made of a capacitor with capacitance C_m , resistance to the flow of ions represented by the membrane resistance R_m and resistance to the flow in the intracellular medium represented as R_c. The battery in the circuit is represented by the equilibrium potential E_m, as shown in Fig. 2.

Since the propagation of signals along the dendrites/axon was determined to be one-dimensional, the dendrites and axon are modeled as tubes with constant or variable diameters. One of the simplifications in deriving the cable equation is to assume the extracellular medium to be isopotential. The resulting equation that governs the variation of the potential ,V, along the dendrite/axon and in the presence of distributed external current, $I_e(s,t)$, is called the cable equation. For constant diameter "d" or a slightly tapered tube, the cable equation could be expressed as:

$$d.[\, C_m.\partial V/\partial t+(V-Em)/R_m \,] = \partial/\partial s\, [\, (d^2/4R_c).\partial V/\partial s \,] + d.I_e(s,t) \quad (1)$$

In this equation, "s" represents the position along the tube and "t" the time.

III. FINITE ELEMENT FORMULATION – Method of Weighted Residual

The finite element method has been used as a numerical simulation tool in almost every field of science. In this paper, the finite element method is used to solve the cable equation of brain neurons by emphasizing the similarities with the field continuum mechanics.

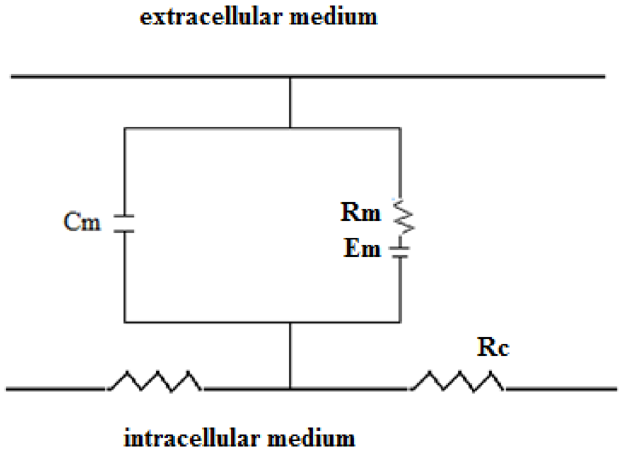

Fig. 2 Equivalent electrical circuit of cell membrane

For this purpose, the method of weighted residual is applied to the cable equation. Two types of elements were tested and the corresponding finite element equations were derived. The first is a linear one-dimensional element, sometimes used in one-dimensional heat transfer analysis. The second is based on the cubic-Hermitian polynomials, used mainly in modeling two-dimensional beam elements in solid mechanics. The performance of the above two elements is compared to exact closed form solutions. One of the advantages of using the finite element method is the possibility of embedding these neuron elements as substructures into models of higher dimensions. This will increase both the scope of the problems that could be solved as well as the complexity of the details that could be included in the analysis.

A. Linear Element

The finite element equations could be derived by using the method of weighted residuals [12].

We start by space discretization of the voltage along an element

$$V(s,t)=\Sigma_J N_J(s).V_J(t) \qquad (2)$$

where $N_J(s)$ is the shape function of node J , $V_J(t)$ the potential at node J and "s" the distance along the dendrite/axon.

The time derivative of the potential is expressed as:

$$\partial V(s,t)/\partial t=\Sigma_J N_J(s).dV_J(t)/dt \qquad (3)$$

Multiplying eq.(1) by $N_I(s)$ and integrating over the length of the tube, ds, we get the following equation in the absence of external distributed current:

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$$\int N_I(s).[\, C_m.\partial V/\partial t + (V-E_m)/R_m \,].(\pi d).ds = \int N_I(s).\partial/\partial s\, [\, (\pi d^2/4R_c).\partial V/\partial s \,].ds \quad (4)$$

Replacing eqs. (2) and (3) in eq. (4) and integrating by parts, leads to (summation is implied over repeated index J):

$$C_{IJ}.dV_J(t)/dt + K_{IJ}.V_J(t) = F_I \quad (5)$$

where

$$C_{IJ} = \int_0^L C_m.N_I(s).N_J(s).d.ds \quad (6)$$

$$K_{IJ} = \int_0^L [\, N_I(s).N_J(s).d/R_m \,]\, ds + \int_0^L [\, (d^2/4R_c).\partial N_I(s)/\partial s.\partial N_J(s)/\partial s\, ds$$

$$F_I = \int_0^L [\, N_I(s).E_m.d/R_m \,]\, ds + (1/\pi).[N_I(s).(\pi d^2/4R_c).\partial V/\partial s]_0^L$$
$$= \int_0^L [\, N_I(s).E_m.d/R_m \,]\, ds + (1/\pi).[N_I(s).(-I_{long}(s,t))]_0^L$$

where $I_{long}(s,t)$ is the longitudinal current $= -(\pi d^2/4R_c).\partial V/\partial s$.

The matrices C_{IJ}, K_{IJ} and F_I are be expressed as:

$$C_{IJ} = (Cm{*}L{*}d/6)\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (7)$$

$$K_{IJ} = (L.d/(6Rm))\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + (d.d/(4Rc{*}L))\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (8)$$

$$F_I = \begin{bmatrix} Em{*}L{*}d/(2Rm) + I(0)/\pi \\ Em{*}L{*}d/(2Rm) - I(L)/\pi \end{bmatrix} \quad (9)$$

B. Cubic-Hermitian polynomial

Eventhough this element is used in solid mechanics for fourth order differential equations, it is employed in this finite element formulation of the cable equation to increase the accuracy of the solution and reduce the number of elements to be used. However, in this case only the continuity of the voltage across elements is enforced.

We start by space discretization of the voltage along an element

$$V(s,t)=\Sigma^4_{J=1}N_J(s).V_J(t) \quad (10)$$

where $N_J(s)$ is the shape function of node J based on the cubic-Hermitian polynomials, $V_J(t)$ the degrees of freedom at node J and “s” the distance along the tube.

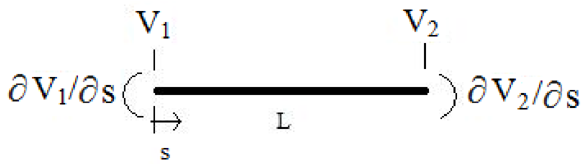


Fig. 3 Degrees of freedom of the cubic-Hermitian element

with

$$\mathbf{N} = [\, 1-3s^2/L^2+2s^3/L^3 ,\ s-2s^2/L+s^3/L^2 ,\ -s^2/L+s^3/L^2 ,\ 3s^2/L^2-2s^3/L^3 \,] \quad (11)$$

and

$$\partial\mathbf{N}/\partial s = [\, -6s/L^2+6s^2/L^3 ,\ 1-4s/L+3s^2/L^2 ,\ -2s/L+3s^2/L^2 ,\ 6s/L^2-6s^2/L^3 \,] \quad (12)$$

with

$$\mathbf{V} = [\ V_1 ,\ \partial V_1/\partial t ,\ \partial V_2/\partial t ,\ V_2 \,] \quad (13)$$

The time derivative of the potential is:

$$\partial V(s,t)/\partial t=\Sigma_J N_J(s).dV_J(t)/dt \quad (14)$$

Multiplying eq.(1) by $N_I(s)$ and integrating over the length of the tube, ds, we get:

$$\int N_I(s).[\, C_m.\partial V/\partial t + (V-E_m)/R_m \,].(\pi d).ds = \int N_I(s).\partial/\partial s\, [\, (\pi d^2/4R_c).\partial V/\partial s].ds + \int N_I(s).Ie(s).(\pi d).ds \quad (15)$$

Using the relative potential, we replace $V-E_m$ by V, and, then, integrate by parts:

$$\int N_I(s).[\, C_m.\partial V/\partial t + V/R_m].(\pi d).ds = [N_I(s).(\pi d^2/4R_c).\partial V/\partial s]_0^L - \int(\pi d^2/4R_c).\partial N_I(s)/\partial s\, .\partial V/\partial s.ds + \int N_I(s).Ie(s).(\pi d).ds \quad (16)$$

Replacing eqs. (11) and (12) in eq. (16) and integrating by parts, leads to:

$$C_{IJ}.dV_J(t)/dt + K_{IJ}.V_J(t) = F_I \quad (17)$$

where

$C_{IJ} = \int_0^L d.C_m.N_I(s).N_J(s).ds$ (18)

$K_{IJ} = \int_0^L [\ N_I(s).N_J(s).].d/R_m\ ]\ ds$
$+ \int_0^L [\ ((d^2)/4R_c).\partial N_I(s)/\partial s.\partial N_J(s)/\partial s\ ds$

$F_I = (1/\pi).[N_I(s).(\pi d^2/4R_c).\delta V/\delta s]_0^L + \int N_I(s).Ie(s).d.ds$
$F = [\ I_{node1}/\pi\ ,\ 0\ ,\ 0\ ,\ -I_{node2}/\pi]^T\ +\ \int \mathbf{N}^T(s).Ie(s).d.ds$

For a constant diameter "d" and constant material properties, the matrices C_{IJ} and K_{IJ} could be expressed as:

$$C_{IJ} = \frac{Cm.d.L}{420}\begin{bmatrix} 156 & 22L & -13L & 54 \\ 22L & 4L^2 & -3L^2 & 13L \\ -13L & -3L^2 & 4L^2 & -22L \\ 54 & 13L & -22L & 156 \end{bmatrix} \quad (19)$$

$$K_{IJ}^1 = (d^2)/(4^*Rc)\begin{bmatrix} 6/5L & 1/10 & 1/10 & -6/5L \\ 1/10 & 4L/30 & -L/30 & -1/10 \\ 1/10 & -L/30 & 4L/30 & -1/10 \\ -6/5L & -1/10 & -1/10 & 6/5L \end{bmatrix} \quad (20)$$

$$K_{IJ}^2 = (\ d.L\ )/(420^*Rm)\begin{bmatrix} 156 & 22L & -13L & 54 \\ 22L & 4L^2 & -3L^2 & 13L \\ -13L & -3L^2 & 4L^2 & -22L \\ 54 & 13L & -22L & 156 \end{bmatrix} \quad (21)$$

with eq. (17) expressed as:

$$C_{IJ}.\frac{dV_J}{dt} + (K_{IJ}^1 + K_{IJ}^2)V_J = \begin{bmatrix} I_{node1}/\pi \\ 0 \\ 0 \\ -I_{node2}/\pi \end{bmatrix} + \int N_I(s).Ie(s).d.ds \quad (22)$$

IV.TRANSIENT SOLUTION: TIME INTEGRATION

The transient cable equation can be expressed as:

$C_{IJ}.dV_J(t)/dt\ +\ K_{IJ}.V_J(t) = F_I(t)$ (23)

or, in matrix form,

$\mathbf{C}.d\mathbf{V}(t)/dt\ +\ \mathbf{K}.\mathbf{V}(t) = \mathbf{F}(t)$ (24)

Integrating over time, we get:

$$\int_t^{t+\Delta t} \mathbf{C}.d\mathbf{V}(t) + \int_t^{t+\Delta t} \mathbf{K}.\mathbf{V}(t)\ dt = \int_t^{t+\Delta t} \mathbf{F}(t)\ dt \quad (25)$$

For constant **C** and **K** matrices, the trapezoidal rule is used to integrate over time. This leads to:

$$\mathbf{C}.(\ \mathbf{V}(t+\Delta t) - \mathbf{V}(t)\ ) + \mathbf{K}.(\ \alpha\mathbf{V}(t+\Delta t) + (1-\alpha)\mathbf{V}(t)\ )\Delta t = (\ \alpha\mathbf{F}(t+\Delta t) + (1-\alpha)\mathbf{F}(t)\ )\Delta t \quad (26)$$

where **α** is a parameter in the interval [0,1]. Rearranging the terms,

$$[\ \mathbf{C} + \alpha\mathbf{K}\Delta t\ ]\mathbf{V}(t+\Delta t) = [\ \mathbf{C} - (1-\alpha)\mathbf{K}\Delta t\ ]\mathbf{V}(t) + [\ \alpha\mathbf{F}(t+\Delta t) + (1-\alpha)\mathbf{F}(t)\ ]\Delta t \quad (27)$$

Consequently,

$$\mathbf{V}(t+\Delta t) = [\ \mathbf{C} + \alpha\mathbf{K}\Delta t\ ]^{-1}.\ \{\ [\ \mathbf{C} - (1-\alpha)\mathbf{K}\Delta t\ ]\mathbf{V}(t) + [\ \alpha\mathbf{F}(t+\Delta t) + (1-\alpha)\mathbf{F}(t)\ ]\Delta t\ \} \quad (28)$$

For $\alpha=0$, we get the forward Euler integration scheme.

V. NUMERICAL SIMULATIONS

A. Numerical solution for one dendrite

The dendritic element is assumed to have a current injected at the left end and sealed at the right end, as shown in Fig. 4.

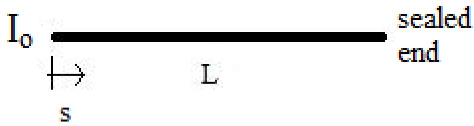


Fig 4. Dendritic cable with current injected at the left-end and sealed at the right-end

In the absence of external current, the cable equation can be expressed as:

$C_m.\partial V/\partial t+(V-E_m)/R_m=(d/4R_c).\partial^2 V/\partial s^2$ (29)

If the voltage is measured relative to E_m, then we can write the cable equation as:

$C_m.\partial V_R/\partial t + V_R/R_m\ = (d/4R_c).\partial^2 V_R/\partial s^2$ (30)

with $V_R=V-E_m$.

1) Steady-State solution

The steady-state solution of the cable equation with no external distributed current can be expressed as:

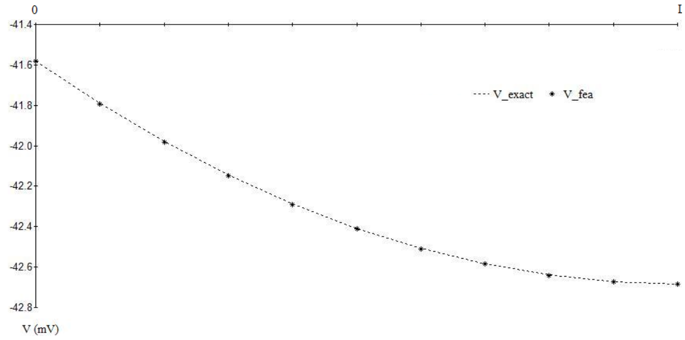
$\partial^2 V_R/\partial s^2 - \beta^2.V_R\ =0$ (31)

The solution is of the form $V_R = Ae^{-\beta s} + Be^{\beta s}$ where $\beta^2 = 2R_c/(a.R_m)$ and "a" the radius of the dendritic element.

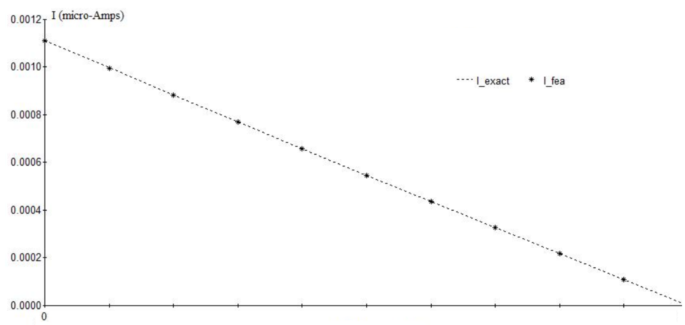
The values of the constants "A" and "B" are given by:

$A = -I_o.R_c.e^{2\beta L}/[\beta.\pi a^2.(1-e^{2\beta L})]$ and
$B = -I_o*R_c/[\beta.\pi a^2.(1-e^{2\beta L})]$

In this problem, the model parameters are as follows:
$L=70x10^{-3}$ cm , $d=2x10^{-3}$ cm , E_m=-60 mV ,
R_m=7. kΩ.cm^2 , and R_c=0.09 kΩ.cm [8].



(a)



(b)

Fig. 5. Steady-state solution of one dendrite with current injected at left and sealed at right: FEA vs. Exact solution
(a) Total voltage along length (b) Current along length

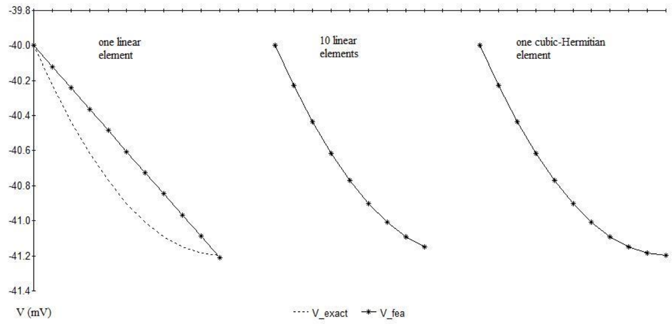
In Fig. 5, the closed-form solution is compared to the finite element solution using one cubic-Hermitian element.
The voltage and current from the finite element solution are cubic and quadratic, respectively. These solutions match well the closed form solution.

Another type of problem is solved that imposes a voltage Vo at the left-end and seals the right-end.
In this case, the solution is expressed as follows:

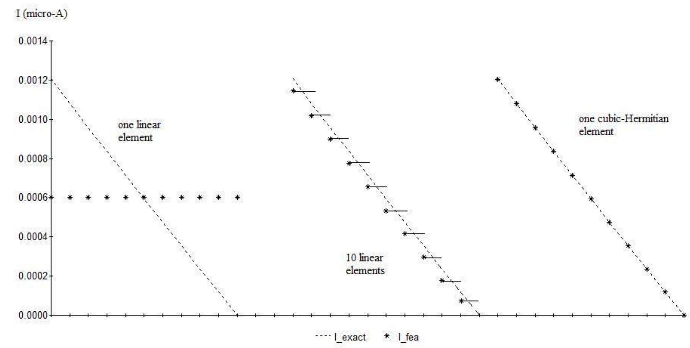
$$V_R(s) = Ae^{-\beta s} + Be^{\beta s} \quad \text{where} \quad \beta^2 = 2R_c/(a.R_m) \quad (32)$$

with

$$A = V_R(0).e^{2\beta L}/(1+e^{2\beta L}) \ , \ B = V_R(0)/(1+e^{2\beta L}) \quad (33)$$



(a)



(b)

Fig. 6. Steady-state solution of a dendrite with voltage specified at left and sealed at right: FEA vs. Exact solution
(a) Total voltage along length (b) Current along length

For the same parameters used in the previous solution, the finite element results for three types of elements are compared to the exact solution, as shown in Fig. 6. The cubic-Hermitian element matches well the closed-form solution as well as the 10 linear elements solution.

2) Transient Solution

Starting with the transient cable equation expressed relative to the equilibrium potential

$$C_m.\partial V/\partial t + V/R_m - (d/4R_c).\partial^2 V/\partial s^2 = 0 \quad (34)$$

The problem to be solved is of a dendrite element, with initial total voltage E_m , injected by a current at the left end and sealed at the right end.

Using the method of separation of variables with V(s,t)=X(s)T(t), the solution could be expressed as:

$$V(s,t)=V_s(s)+\Sigma^{\infty}_{n=0} \{ c_n.\cos(\alpha_n.s).\exp(-\lambda_n^2(t/\tau_m)) \} \quad (35)$$

where

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$V_s(s)$ = steady-state solution, $\alpha_n = n.\pi/L$,
$\lambda_n^2 = \alpha_n^2.(d.R_m/4R_c)+1$, and $\tau_m = R_m . C_m$

The finite element solution could be expressed as follows for the forward-Euler integration scheme (α=0):

$$\mathbf{V}(t+\Delta t) = \mathbf{C}^{-1} . \{ [\mathbf{C} - \mathbf{K}\Delta t]\mathbf{V}(t) + \mathbf{F}(t)\Delta t \} \quad (36)$$

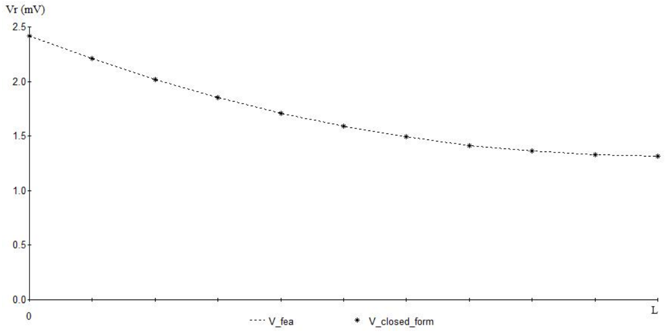


Fig. 7. Transient solution of the relative voltage at t=2.1ms : Closed-form vs. FEA solution

For an injected current at the left end of value I_o=1.1x10^{-3} μA and using one cubic-Hermitian element, the finite element solution matches well the closed-form solution at 2.1ms, as shown in Fig. 7.

B. Solution of Y-branched dendrites

1) Comparison with steady-state solution

A classical problem for dendrites, that has a closed form solution for steady-state, is the Y-branched dendrites problem. The geometry and mesh are shown in Fig. 8. A current $\mathbf{I_o}$ is injected at node "1" of dendrite #1 and the ends of dendrites #2 and #3 are sealed.

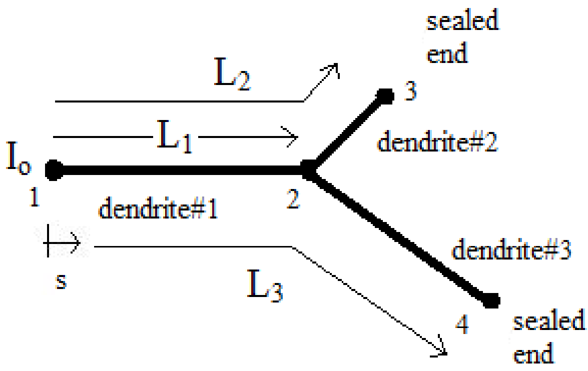


Fig. 8. Y-branched dendrites

Three cubic-Hermitian elements were used, one for each dendrite. The compatibility conditions are applied at node "2" and the currents at nodes "3" and "4" are set to zero.

A closed-form solution could be derived for this case, with the potentials of the dendrites expressed as:

$$V_1(s)=A_1.\sinh(\beta_1 s)+B_1\cosh(\beta_1 s) \quad (37)$$
$$V_2(s)=A_2.\sinh(\beta_2(s-L_2))+B_2\cosh(\beta_2(s-L_2))$$
$$V_3(s)=A_3.\sinh(\beta_3(s-L_3))+B_3\cosh(\beta_3(s-L_3))$$

For the properties of the dendrites used in [8], the values of constants A_i and B_i are as follows:

Table 1. Values of A_i and B_i

A1	B1	A21	B21	A22	B22
-21.74	22.36	0.0	1.39	0.0	0.071

Fig. 9 compares the finite element solution with the closed-form solution. The use of one cubic-Hermitian element gives results close to the exact/10 linear elements solutions.

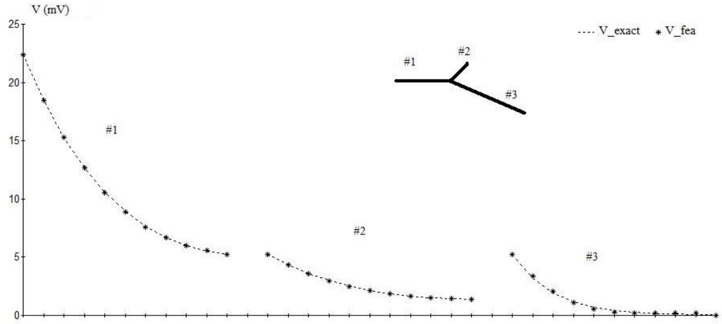
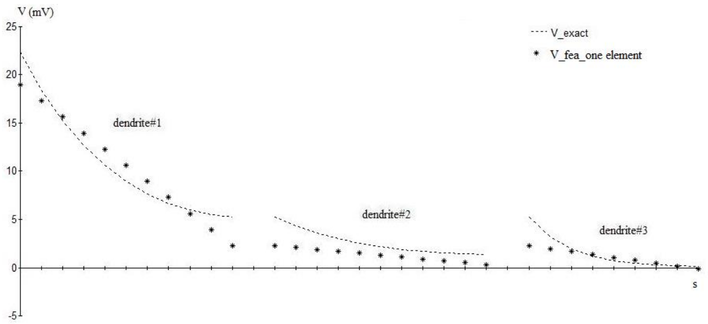
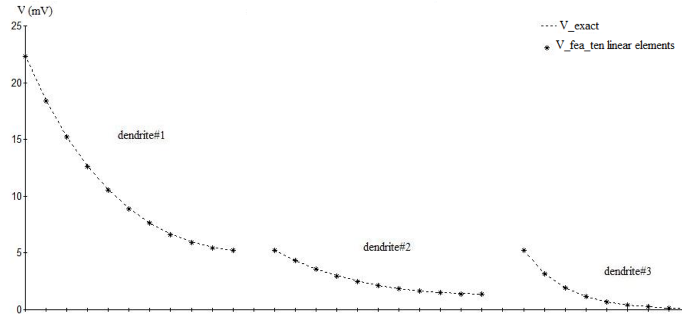


Fig. 9. Steady-state of the relative voltage of Y-branched dendrites: FEA vs. exact

However, the use of one linear element shows considerable variation from the exact solution. By using two cubic-Hermitian elements along dendrite#3, the finite element solution is made even closer to the exact solution at node "4".



(a)



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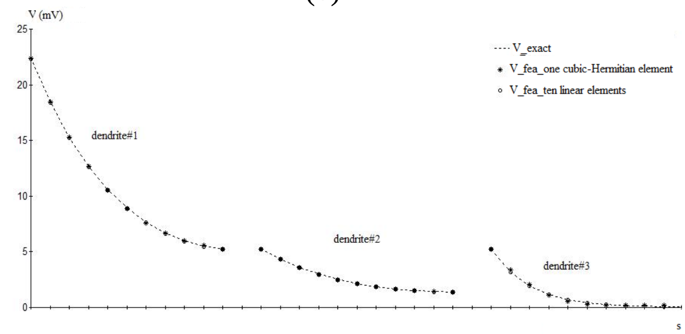

(c)

Fig. 10. Steady-state voltage of Y-branched dendrites: (a) One linear element (b) Ten linear elements (c) One cubic-Hermitian/Ten Linear elements/exact solution

VI. CONCLUSIONS

In this paper, the finite element method was used to simulate the propagation of potentials along the dendrites/axon of a brain neuron.

Two types of elements were tested, namely, linear and cubic elements. Numerical simulations using the above two elements were presented and compared to available closed-form solutions.

In general, for linear elements a considerable number of them is needed to achieve accurate results.

However, the cubic-Hermitian element showed improvement in the solution even with one element. For a constant diameter of dendrites/axon this element has the advantage of modeling the potential with a cubic polynomial and the current with a quadratic polynomial.

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