Global Regularity for Very Weak Solutions to Boundary Value Problem of Homogeneous A-Harmonic Equation

Shuo Yan
College of Science
North China University of Science and Technology
Hebei Tangshan, China

Wenrui Chang
College of Science
North China University of Science and Technology
Hebei Tangshan, China

Xiujuan Xu*
College of Science
North China University of Science and Technology
Hebei Tangshan, China
*Corresponding Author
xxjluck@126.com

***Abstract*—The very weak solution to elliptic boundary value problems is considered. A global regularity result is derived for very weak solutions under some controllable and coercivity conditions, by using the Hodge decomposition theorem and the methods in Sobolev spaces.**

Keywords—Hodge decomposition theorem; A-harmonic equation; global regularity

I. INTRODUCTION

Let Ω be a bounded regular open set of $\mathbf{R}^n (n \geq 2)$. We consider the boundary value problem for the second order degenerate elliptic equation

$$\begin{cases} \operatorname{div} A(x, \nabla u) = 0, & \text{in } \Omega \\ u = u_0, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $A(x,\xi): \Omega \times \mathbf{R}^n$ a $\mathbf{R}^n$ is a Carathéodory function satisfying the coercivity and growth conditions: for almost all $x \in \Omega$, all $\xi \in \mathbf{R}^n$,

(H1) $|A(x,\xi)| \leq \beta |\xi|^{p-1}$,

(H2) $\langle A(x,\xi), \xi \rangle \geq \alpha |\xi|^p$,

where $1 < p < \infty$, $0 < \alpha \leq \beta < \infty$, $u_0 \in W^{1,r}(\Omega)$ is a boundary value function.

Definition 1.1 A function $u \in u_0 + W_0^{1,r}(\Omega)$, $\max\{1, p-1\} \leq r < p$ is called a very weak solution to (1.1), if

$$\int_\Omega \langle A(x, \nabla u), \nabla \phi \rangle \mathrm{d}x = 0$$

holds true for any $\phi \in W_0^{1,\frac{r}{r-p+1}}(\Omega)$ with compact support in Ω.

A crucial fact is that r can be smaller than the natural exponent p. For variational extremals the global higher integrability of the derivative ∇u has been studied by Granlund S[1] in the case $p = n$. For this it seems necessary to impose a regularity condition for $\partial\Omega$.

We say that $\partial\Omega$ is r-Poincaré thick, if there is $0 < C < \infty$ such that for all open cube $Q_R \subset \mathbf{R}^n$ with side length $R > 0$, it holds

$$\left(\int_{Q_{2R}} |u|^r \, \mathrm{d}x \right)^{\frac{1}{r}} \leq C \left(\int_{Q_{2R}} |\nabla u|^{\frac{rn}{r+n}} \, \mathrm{d}x \right)^{\frac{r+n}{rn}} \tag{1.2}$$

whenever $u \in W^{1,r}(Q_{2R})$, $u = 0$ a.e. on $(\mathbf{R}^n \setminus \Omega) \cap Q_{2R}$, and $Q_{\frac{3R}{2}} \cap \Omega^C \neq \varnothing$. Here, and in the following, $Q(\lambda R)$, $\lambda > 0$, means a cube parallel to $Q(R)$ with the same center as $Q(R)$ and with side length λR. See [2].

The following is the main conclusion of this paper.

Theorem 1.2 Suppose that a bounded regular domain Ω has a r-Poincaré thick boundary and that $r \geq \frac{n}{n-1}$, operator A satisfy conditions (H1)-(H2). If $u_0 \in W^{1,r}(\Omega)$ is the boundary value function, $u \in W^{1,r}(\Omega)$ is the very weak solution of Dirichlet problem (1.1), then there exists $R_0 > 0$ and r_1, r_2, satisfying

$$r_1 = r_1(n, p, K, R_0, \alpha, \beta, \Omega) < p < r_2 = r_2(n, p, K, R_0, \alpha, \beta, \Omega),$$

such that $\forall r \in [r_1, p)$, $u \in W^{1,r_2}(\Omega)$, then u is the weak solution in the classical meaning.

II. PRELIMINARY LEMMAS

Let Ω be a bounded regular domain, $x_0 \in \Omega$, $0 < R < \operatorname{dist}(x_0, \partial\Omega)$, $Q_R(x_0) \subset \Omega$, here $Q_R(x_0)$ is a cube with side length of R and a center of x_0.

Lemma 2.1 [3] Let $1 < p < n$, $0 < q \leq \frac{np}{n-p}$ if $u \in W^{1,p}(B_R(x_0))$, then

$$\|u-u_R\|_{L^q(B_R(x_0))}\le CR^{n(\frac{1}{q}-\frac{1}{p})+1}\|\nabla u\|_{L^p(B_R(x_0))}, \quad (2.1)$$

here $u_R @ \fint_{B_R(x_0)} u dx = \frac{1}{|B_R|}\int_{B_R(x_0)} u dx$, C is a positive constant only depending on p, q, n.

Specially, if $u\in W_0^{1,p}(B_R(x_0))$, then

$$\|u\|_{L^q(B_R(x_0))}\le CR^{n\left(\frac{1}{q}-\frac{1}{p}\right)+1}\|\nabla u\|_{L^p(B_R(x_0))}. \quad (2.2)$$

This lemma gives the dependence of embedding theorem on region size. For the case $q=\frac{np}{n-p}$ see [4]. This lemma is a direct corollary of theorem 7.10 and the Hölder inequality in Gilbarg-Trudinger [4].

This lemma also applies to cubes.

Lemma 2.2 [3] (Hodge decomposition) Let $\Omega\subset\mathbf{R}^n$ be a regular domain and $u\in W_0^{1,r}(\Omega,\mathbf{R}^m)$,and let $0<\varepsilon<r-1$, $r=p-\varepsilon\ge\max\{1,p-1\}$ .Then there exist $\phi(x)\in W_0^{1,\frac{r}{1-\varepsilon}}(\Omega,\mathbf{R}^m)$ and a (divergence free) matrix-field $H(x)\in L^{\frac{r}{1-\varepsilon}}(\Omega,\mathbf{R}^{n\times m})$, such that

$$|\nabla u|^{-\varepsilon}\nabla u=\nabla\phi+H. \quad (2.3)$$

Moreover

$$\|H\|_{\frac{r}{1-\varepsilon}}\le C\varepsilon\|\nabla u\|_r^{1-\varepsilon} \quad (2.4)$$

where C is a constant that only depends on n , r and Ω.

Remark 2.3 It can be seen from (2.3) and (2.4), estimates of $\nabla\phi$ are similar to those of (2.4).

Lemma 2.4 [5] Suppose X and Y are vectors of an inner product space, $0\le\varepsilon<1$. Then

$$\| X|^{-\varepsilon}X-|Y|^{-\varepsilon}Y|\le\frac{2^{\varepsilon}(1+\varepsilon)}{1-\varepsilon}|X-Y|^{1-\varepsilon}.$$

Lemma 2.5 [6] (Reverse Hölder inequality) Let Q be an n-cube. Suppose

$$\fint_{Q_R(x_0)} g^q dx\le\theta\fint_{Q_{2R}(x_0)} g^q dx+c\left(\fint_{Q_{2R}(x_0)} g dx\right)^q+\fint_{Q_{2R}(x_0)} f^q dx$$

for each $x_0\in Q$ and each $R<\frac{1}{2}\text{dist}(x_0,\partial Q)\wedge R_0$, where R_0 , b , θ are constants with $b>1$, $R_0>0$, $0\le\theta<1$. Then $g\in L^p_{loc}(Q)$ for $p\in[q,q+\varepsilon)$ and

$$\left(\fint_{Q_R(x_0)} g^p dx\right)^{\frac{1}{p}}\le c\left\{\left(\fint_{Q_{2R}(x_0)} g^q dx\right)^{\frac{1}{q}}+\left(\fint_{Q_{2R}(x_0)} f^p dx\right)^{\frac{1}{p}}\right\}$$

for $Q_{2R}\subset Q$, $R<R_0$, where c and ε are positive constants only depending on b , θ , q , n .

III. Proof of Theorem 1.2

Proof. Let $x_0\in\Omega$, $Q_\rho=Q_\rho(x_0)$ is a cube with side length of ρ and a center of x_0 . ε is a sufficiently small positive number, $r=p-\varepsilon$. Since Ω is bounded, we can choose a cube $Q_0=Q_{2R_0}$ such that $\Omega\subset Q_{R_0}$. Next let $Q_{2R}\subset Q_0$. There are two possibilities: (1) $Q_{\frac{3}{2}R}\subset\Omega$; (2) $Q_{\frac{3}{2}R}\cap\Omega^C\ne\varnothing$.

In the case (1), for $Q_{\frac{3}{2}R}\subset\Omega$, fix a cutoff function $\eta\in C_0^\infty(Q_{\frac{3}{2}R})$ such that $0\le\eta\le1$, $|\nabla\eta|\le\frac{C}{R}$, and $\eta\equiv1$ on $x\in Q_R$. Let $u\in W^{1,r}(\Omega)$ be a very weak solution of problem(1.1). Consider the following Hodge decomposition

$$|\nabla(\eta u)|^{-\varepsilon}\nabla(\eta u)=\nabla\phi+H, \quad (3.1)$$

here $\phi\in W_0^{1,\frac{r}{1-\varepsilon}}(Q_{\frac{3}{2}R})$, $H\in L^{\frac{r}{1-\varepsilon}}(Q_{\frac{3}{2}R})$ is a (divergence free) matrix-field, satisfying

$$\|\nabla\phi\|_{\frac{r}{1-\varepsilon}}\le C\|\nabla(\eta u)\|_r^{1-\varepsilon}, \quad (3.2)$$

$$\|H\|_{\frac{r}{1-\varepsilon}}\le C\varepsilon\|\nabla(\eta u)\|_r^{1-\varepsilon}. \quad (3.3)$$

Let

$$E(\eta,u)=|\nabla(\eta u)|^{-\varepsilon}\nabla(\eta u)-|\eta\nabla u|^{-\varepsilon}\eta\nabla u, \quad (3.4)$$

by Lemma 2.4 we have

$$|E(\eta,u)|\le 2^{\varepsilon}\frac{1+\varepsilon}{1-\varepsilon}|u\nabla\eta|^{1-\varepsilon}. \quad (3.5)$$

A useful technique in the following calculation is to use ϕ in Hodge decomposition (3.1) as the test function in Definition 1.1. Then

$$\begin{aligned}&\int_{Q_{\frac{3}{2}R}}\langle A(x,\nabla u),E(\eta,u)\rangle dx\\&+\int_{Q_{\frac{3}{2}R}}\langle A(x,\nabla u),|\eta\nabla u|^{-\varepsilon}\eta\nabla u\rangle dx\\&=\int_{Q_{\frac{3}{2}R}}\langle A(x,\nabla u),H\rangle dx,\end{aligned} \quad (3.6)$$

that is

$$\int_{Q_{\frac{3}{2}R}}\langle A(x,\nabla u),|\eta\nabla u|^{-\varepsilon}\eta\nabla u\rangle dx$$

$$\begin{aligned}&=\int_{Q_{\frac{3}{2}R}}\langle A(x,\nabla u),H\rangle dx-\int_{Q_{\frac{3}{2}R}}\langle A(x,\nabla u),E(\eta,u)\rangle dx\\&=I_1+I_2.\end{aligned} \quad (3.7)$$

Let's first estimate the left side of formula (3.7). By the hypothesis condition (H2) and the definition of η,

$$\begin{aligned}
&\int_{Q_{\frac{3}{2}R}} \left\langle A(x,\nabla u), |\eta\nabla u|^{-\varepsilon}\eta\nabla u\right\rangle \mathrm{d}x \\
&= \int_{Q_{\frac{3}{2}R}} \eta^{1-\varepsilon}|\nabla u|^{-\varepsilon}\left\langle A(x,\nabla u),\nabla u\right\rangle \mathrm{d}x \\
&\geq \alpha\int_{Q_{\frac{3}{2}R}} \eta^{1-\varepsilon}|\nabla u|^{-\varepsilon}|\nabla u|^{p}\,\mathrm{d}x \\
&\geq \alpha\int_{Q_R} |\nabla u|^{r}\,\mathrm{d}x.
\end{aligned} \tag{3.8}$$

The estimate of I_1 is given below. By the hypothesis (H1), the Hölder inequality and (3.3), we can get the result

$$\begin{aligned}
|I_1| &= \left|\int_{Q_{\frac{3}{2}R}} \langle A(x,\nabla u), H\rangle \mathrm{d}x\right| \leq \int_{Q_{\frac{3}{2}R}} |A(x,\nabla u)||H|\mathrm{d}x \\
&\leq \beta\int_{Q_{\frac{3}{2}R}} |\nabla u|^{p-1}|H|\mathrm{d}x \\
&\leq \beta\|\nabla u\|_r^{p-1}\|H\|_{\frac{r}{1-\varepsilon}} \\
&\leq \beta C\varepsilon\|\nabla u\|_r^{p-1}\|\nabla(\eta u)\|_r^{1-\varepsilon}.
\end{aligned} \tag{3.9}$$

Notice that u plus a constant vector does not affect ∇u and the A-harmonic equation in (1.1) in our case, so let's assume that the average integral of u on $Q_{\frac{3}{2}R}$ is zero, and then by using Lemma 2.1,

$$\begin{aligned}
\|\nabla(\eta u)\|_r^{1-\varepsilon} &= \|u\nabla\eta + \eta\nabla u\|_r^{1-\varepsilon} \\
&\leq (\|u\nabla\eta\|_r + \|\eta\nabla u\|_r)^{1-\varepsilon} \\
&\leq (\frac{C}{R}\|u\|_r + \|\nabla u\|_r)^{1-\varepsilon} \\
&\leq (\frac{C}{R}CR\|\nabla u\|_r + \|\nabla u\|_r)^{1-\varepsilon} \\
&\leq C\|\nabla u\|_r^{1-\varepsilon},
\end{aligned} \tag{3.10}$$

then we have

$$|I_1| \leq C\beta\varepsilon\|\nabla u\|_r^{r}. \tag{3.11}$$

The estimate of I_2 is given below. By the hypothesis (H1), (3.5) and the definition of η, we have

$$\begin{aligned}
|I_2| &= \left|\int_{Q_{\frac{3}{2}R}} \langle A(x,\nabla u), E(\eta,u)\rangle \mathrm{d}x\right| \\
&\leq \beta\int_{Q_{\frac{3}{2}R}} |\nabla u|^{p-1}|E(\eta,u)|\mathrm{d}x \\
&\leq \beta 2^{\varepsilon}\frac{1+\varepsilon}{1-\varepsilon}\int_{Q_{\frac{3}{2}R}} |\nabla u|^{p-1}|u\nabla\eta|^{1-\varepsilon}\,\mathrm{d}x \\
&\leq \beta 2^{\varepsilon}\frac{1+\varepsilon}{1-\varepsilon}C^{1-\varepsilon}\int_{Q_{\frac{3}{2}R}} |\nabla u|^{p-1}\left|\frac{u}{R}\right|^{1-\varepsilon}\mathrm{d}x \\
&\leq \beta C\int_{Q_{\frac{3}{2}R}} |\nabla u|^{p-1}\left|\frac{u}{R}\right|^{1-\varepsilon}\mathrm{d}x,
\end{aligned} \tag{3.12}$$

By Young's inequality, for any $\theta_1 > 0$,

$$|I_2| \leq \beta C\theta_1\int_{Q_{\frac{3}{2}R}} |\nabla u|^{r}\,\mathrm{d}x + \beta C\int_{Q_{\frac{3}{2}R}} \left|\frac{u}{R}\right|^{r}\mathrm{d}x. \tag{3.13}$$

For the second integral formula at the right end of the upper formula, take t such that $\max\{1,\frac{nr}{n+r}\} \leq t < r$, then by Lemma 2.1,

$$\beta C\int_{Q_{\frac{3}{2}R}} \left|\frac{u}{R}\right|^{r}\mathrm{d}x \leq \beta CR^{n}\left(⨍_{Q_{\frac{3}{2}R}} |\nabla u|^{t}\,\mathrm{d}x\right)^{\frac{r}{t}}, \tag{3.14}$$

it doesn't effect ∇u and A-harmonic equation when u plus a constant, so assuming the integral average of u is zero in $Q_{\frac{3}{2}R}$, then we have

$$|I_2| \leq \beta C\theta_1\int_{Q_{\frac{3}{2}R}} |\nabla u|^{r}\,\mathrm{d}x + \beta CR^{n}\left(⨍_{Q_{\frac{3}{2}R}} |\nabla u|^{t}\,\mathrm{d}x\right)^{\frac{r}{t}}. \tag{3.15}$$

Combining the inequalities (3.7), (3.8), (3.11), (3.15), we obtain

$$\begin{aligned}
\alpha\int_{Q_R} |\nabla u|^{r}\,\mathrm{d}x &\leq C\beta\varepsilon\int_{Q_{\frac{3}{2}R}} |\nabla u|^{r}\,\mathrm{d}x \\
&+ C\beta\theta_1\int_{Q_{\frac{3}{2}R}} |\nabla u|^{r}\,\mathrm{d}x \\
&+ C\beta R^{n}\left(⨍_{Q_{\frac{3}{2}R}} |\nabla u|^{t}\,\mathrm{d}x\right)^{\frac{r}{t}}.
\end{aligned} \tag{3.16}$$

Divide the two sides of the formula above by $|Q_R| = \omega_n R^{n}$ (here ω_n is the volume of unit cube in $\mathbf{R}^{n}$), then

$$\begin{aligned}
\alpha⨍_{Q_R} |\nabla u|^{r}\,\mathrm{d}x &\leq C\beta\varepsilon⨍_{Q_{\frac{3}{2}R}} |\nabla u|^{r}\,\mathrm{d}x + C\beta\theta_1⨍_{Q_{\frac{3}{2}R}} |\nabla u|^{r}\,\mathrm{d}x \\
&+ C\beta\left(⨍_{Q_{\frac{3}{2}R}} |\nabla u|^{t}\,\mathrm{d}x\right)^{\frac{r}{t}}.
\end{aligned} \tag{3.17}$$

Since Ω is bounded, $\Omega \subset Q_{R_0}$, $R < R_0$, the formula above becomes

$$\fint_{Q_R} |\nabla u|^r \,dx \le C(\varepsilon+\theta_1)\fint_{Q_{\frac{3}{2}R}} |\nabla u|^r \,dx$$
$$+C\left(\fint_{Q_{\frac{3}{2}R}} |\nabla u|^t \,dx\right)^{\frac{r}{t}}. \tag{3.18}$$

Let ε, θ_1 be small enough such that $\theta = C(\varepsilon+\theta_1) < 1$. Then

$$\fint_{Q_R} |\nabla u|^r \,dx \le \theta \fint_{Q_{\frac{3}{2}R}} |\nabla u|^r \,dx + C\left(\fint_{Q_{\frac{3}{2}R}} |\nabla u|^t \,dx\right)^{\frac{r}{t}}, \tag{3.19}$$

where $C = C(n,p,r,\alpha,\beta,R_0,\Omega)$. Noticing that the case we considered is that r is close enough to p, then r can be removed from the parameter of C. For $1<t<r$, then (3.19) is a weak reverse Hölder inequality about ∇u.

Choosing $g = |\nabla u|^t$, $G=0$ in $Q_{\frac{3}{2}R}$ and $g=G=0$ in $Q_{2R}\setminus Q_{\frac{3}{2}R}$ with $q=\frac{r}{t}$. Then we arrive at the following inequality in $Q_{2R}\subset\Omega$, that is

$$\fint_{Q_R} g^{\frac{r}{t}}dx \le \theta\fint_{Q_{2R}} g^{\frac{r}{t}}dx + C(\fint_{Q_{2R}} g dx)^{\frac{r}{t}} + C\fint_{Q_{2R}} G^{\frac{r}{t}}dx. \tag{3.20}$$

In the case (2), let $w = -\eta^p(u-u_0)\in W_0^{1,r}(Q_{2R})$, where $\eta\in C_0^\infty(Q_{2R})$ is a cutoff function, $0\le\eta\le1$, $|\nabla\eta|\le\frac{C}{R}$, and $\eta\equiv1$ in Q_R.

Extending the function $u-u_0$ with zero to $\mathrm{R}^n\setminus\Omega$ continuously. Then by Lemma 2.2, there exist $\phi\in W_0^{1,\frac{r}{1-\varepsilon}}(Q_{2R})$ and $H(x)\in L^{\frac{r}{1-\varepsilon}}(Q_{2R})$, such that

$$\begin{aligned}|\nabla w|^{-\varepsilon}\nabla w &= \nabla\phi + H\\ &= -\left|\nabla[\eta^p(u-u_0)]\right|^{-\varepsilon}\nabla[\eta^p(u-u_0)],\end{aligned} \tag{3.21}$$

and

$$\|H\|_{\frac{r}{1-\varepsilon}} \le C\varepsilon\left\|\nabla[\eta^p(u-u_0)]\right\|_r^{1-\varepsilon}, \tag{3.22}$$

$$\|\nabla\phi\|_{\frac{r}{1-\varepsilon}} \le C\left\|\nabla[\eta^p(u-u_0)]\right\|_r^{1-\varepsilon}, \tag{3.23}$$

where C is a constant only depending on n, r and Ω. Note that for Hodge decomposition (3.21), (3.22) and (3.23), we have $u-u_0=0$, H and $\nabla\phi$ are equal to zero when $u-u_0\in\mathbf{R}^n\setminus\Omega$. By the Minkowski inequality and the selection of η, we have

$$\begin{aligned}\left\|\nabla[\eta^p(u-u_0)]\right\|_r^{1-\varepsilon} &\le C\Big[\|(u-u_0)\nabla\eta\|_r\\ &\quad+\|\nabla(u-u_0)\|_r\Big]^{1-\varepsilon}.\end{aligned} \tag{3.24}$$

Noting that the boundary $\partial\Omega$ is r-Poincaré thick. Since $u-u_0$ is continually zero to $\mathbf{R}^n\setminus\Omega$, then by (1.2),

$$\begin{aligned}\|(u-u_0)\nabla\eta\|_r &\le CR^{-1}\left(\int_{Q_{2R}\cap\Omega}|u-u_0|^r\,dx\right)^{\frac{1}{r}}\\ &= CR^{-1}\left(\int_{Q_{2R}}|u-u_0|^r\,dx\right)^{\frac{1}{r}}\\ &\le CR^{-1}\left(\int_{Q_{2R}}|\nabla(u-u_0)|^{\frac{nr}{n+r}}\,dx\right)^{\frac{n+r}{nr}}\\ &= CR^{-1}\left(\int_{Q_{2R}\cap\Omega}|\nabla(u-u_0)|^{\frac{nr}{n+r}}\,dx\right)^{\frac{n+r}{nr}},\end{aligned} \tag{3.25}$$

here we used $u-u_0=0$ in $\mathbf{R}^n\setminus\Omega$. Substitute the above formula into (3.24), we get

$$\begin{aligned}&\left\|\nabla[\eta^p(u-u_0)]\right\|_r^{1-\varepsilon}\\ &\le C\left[R^{-1}\left(\int_{Q_{2R}\cap\Omega}|\nabla(u-u_0)|^{\frac{nr}{n+r}}\,dx\right)^{\frac{n+r}{nr}}\right.\\ &\quad\left.+C\left(\int_{Q_{2R}\cap\Omega}|\nabla(u-u_0)|^r\,dx\right)^{\frac{1}{r}}\right]^{1-\varepsilon}.\end{aligned} \tag{3.26}$$

Then (3.22) and (3.23) are

$$\begin{aligned}&\left(\int_{Q_{2R}\cap\Omega}|H|^{\frac{r}{1-\varepsilon}}\,dx\right)^{\frac{1-\varepsilon}{r}}\\ &\le C\varepsilon\left[R^{-1}\left(\int_{Q_{2R}\cap\Omega}|\nabla(u-u_0)|^{\frac{nr}{n+r}}dx\right)^{\frac{n+r}{nr}}\right.\\ &\quad\left.+C\left(\int_{Q_{2R}\cap\Omega}|\nabla(u-u_0)|^r dx\right)^{\frac{1}{r}}\right]^{1-\varepsilon},\end{aligned} \tag{3.27}$$

$$\begin{aligned}&\left(\int_{Q_{2R}\cap\Omega}|\nabla\phi|^{\frac{r}{1-\varepsilon}}\,dx\right)^{\frac{1-\varepsilon}{r}}\\ &\le C\left[R^{-1}\left(\int_{Q_{2R}\cap\Omega}|\nabla(u-u_0)|^{\frac{nr}{n+r}}\,dx\right)^{\frac{n+r}{nr}}\right.\\ &\quad\left.+C\left(\int_{Q_{2R}\cap\Omega}|\nabla(u-u_0)|^r\,dx\right)^{\frac{1}{r}}\right]^{1-\varepsilon},\end{aligned} \tag{3.28}$$

By the conditions (H1), (H2), Lemma 2.4, Hodge decomposition (3.21), and the Definition1.1, we obtain

$$\begin{aligned}&\alpha\int_\Omega \eta^{p(1-\varepsilon)}|\nabla u|^r\,dx\\ &\le \int_\Omega\left\langle A(x,\nabla u), |\eta^p\nabla u|^{-\varepsilon}\eta^p\nabla u\right\rangle dx\\ &= \int_\Omega\left\langle A(x,\nabla u), |\eta^p\nabla u|^{-\varepsilon}\eta^p\nabla u\right.\\ &\quad\left.-|\eta^p\nabla(u-u_0)|^{-\varepsilon}\eta^p\nabla(u-u_0)\right\rangle dx\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega \left\langle A(x,\nabla u), \left|\eta^p \nabla(u-u_0)\right|^{-\varepsilon} \eta^p \nabla(u-u_0)\right. \\
&\left. -\left|\nabla[\eta^p(u-u_0)]\right|^{-\varepsilon} \nabla[\eta^p(u-u_0)]\right\rangle \mathrm{d}x \\
&+\int_\Omega \left\langle A(x,\nabla u), \left|\nabla[\eta^p(u-u_0)]\right|^{-\varepsilon} \nabla[\eta^p(u-u_0)]\right\rangle \mathrm{d}x \\
&\le C\int_\Omega \eta^{p(1-\varepsilon)} \left|A(x,\nabla u)\right| \left|\nabla u_0\right|^{1-\varepsilon} \mathrm{d}x \\
&+C\int_\Omega \left|A(x,\nabla u)\right| \left|\nabla \eta^p\right|^{1-\varepsilon} \left|u-u_0\right|^{1-\varepsilon} \mathrm{d}x \qquad (3.29)\\
&-\int_\Omega \left\langle A(x,\nabla u), \nabla\phi + H\right\rangle \mathrm{d}x \\
&\le C\beta[\int_\Omega \eta^{p(1-\varepsilon)} |\nabla u|^{p-1} |\nabla u_0|^{1-\varepsilon} \mathrm{d}x \\
&+\int_\Omega \eta^{(p-1)(1-\varepsilon)} |\nabla u|^{p-1} |\nabla \eta|^{1-\varepsilon} |u-u_0|^{1-\varepsilon} \mathrm{d}x \\
&+\int_\Omega |\nabla u|^{p-1} |H| \mathrm{d}x] \\
&@ C\beta[I_3 + I_4 + I_5].
\end{aligned}
$$

The estimate of I_3 is given below. By Young's inequality, for any $\theta_2 > 0$,

$$
\begin{aligned}
I_3 &= \int_\Omega \eta^{p(1-\varepsilon)} |\nabla u|^{p-1} |\nabla u_0|^{1-\varepsilon} \mathrm{d}x \\
&\le \theta_2 \int_\Omega \eta^{r(1-\varepsilon)} |\nabla u|^r \mathrm{d}x + C\int_\Omega \eta^r |\nabla u_0|^r \mathrm{d}x \qquad (3.30)\\
&\le \theta_2 \int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u|^r \mathrm{d}x + C\int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u_0|^r \mathrm{d}x.
\end{aligned}
$$

The estimate of I_4 is given below. By Young's inequality, for any $\theta_3 > 0$,

$$
\begin{aligned}
I_4 &= \int_\Omega \eta^{(p-1)(1-\varepsilon)} |\nabla u|^{p-1} |\nabla \eta|^{1-\varepsilon} |u-u_0|^{1-\varepsilon} \mathrm{d}x \\
&\le \theta_3 \int_\Omega \eta^{r(1-\varepsilon)} |\nabla u|^r \mathrm{d}x + C\int_\Omega |\nabla \eta|^r |u-u_0|^r \mathrm{d}x \qquad (3.31)\\
&\le \theta_3 \int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u|^r \mathrm{d}x + C\int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla \eta|^r |u-u_0|^r \mathrm{d}x
\end{aligned}
$$

For the second integral formula at the right end of the upper formula, noticing that $\partial\Omega$ is r-Poincaré thick. By (3.25) we get

$$
\begin{aligned}
&C\int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla \eta|^r |u-u_0|^r \mathrm{d}x \\
&\le CR^{-r} \left(\int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla(u-u_0)|^{\frac{nr}{n+r}} \mathrm{d}x \right)^{\frac{n+r}{n}}. \qquad (3.32)
\end{aligned}
$$

Then by Minkowski inequality and the Hölder inequality, we have

$$
\begin{aligned}
&\left(\int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla(u-u_0)|^{\frac{nr}{n+r}} \mathrm{d}x \right)^{\frac{n+r}{n}} \\
&\le \left[\left(\int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u|^{\frac{nr}{n+r}} \mathrm{d}x \right)^{\frac{n+r}{nr}} + \left(\int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u_0|^{\frac{nr}{n+r}} \mathrm{d}x \right)^{\frac{n+r}{nr}} \right]^r \\
&\le \left[\left(\int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u|^{\frac{nr}{n+r}} \mathrm{d}x \right)^{\frac{n+r}{nr}} + CR \left(\int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u_0|^r \mathrm{d}x \right)^{\frac{1}{r}} \right]^r \\
&\le 2^r \left[\left(\int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u|^{\frac{nr}{n+r}} \mathrm{d}x \right)^{\frac{n+r}{n}} + CR^r \int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u_0|^r \mathrm{d}x \right], \qquad (3.33)
\end{aligned}
$$

Then

$$
\begin{aligned}
I_4 &\le \theta_3 \int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u|^r \mathrm{d}x + CR^{-r} \left(\int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u|^{\frac{nr}{n+r}} \mathrm{d}x \right)^{\frac{n+r}{n}} \qquad (3.34)\\
&+ C\int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u_0|^r \mathrm{d}x.
\end{aligned}
$$

The estimate of I_5 is given below. By Young's inequality, (3.27) and the Hölder inequality, for any $\theta_4 > 0$, we have

$$
\begin{aligned}
I_5 &= \int_\Omega |\nabla u|^{p-1} |H| \mathrm{d}x \\
&\le \theta_4 \int_\Omega |\nabla u|^r \mathrm{d}x + C\int_\Omega |H|^{\frac{r}{1-\varepsilon}} \mathrm{d}x \\
&\le \theta_4 \int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u|^r \mathrm{d}x \\
&+ C\varepsilon \left[R^{-1} \left(\int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla(u-u_0)|^{\frac{nr}{n+r}} \mathrm{d}x \right)^{\frac{n+r}{nr}} \right. \\
&\left. + C\left(\int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla(u-u_0)|^r \mathrm{d}x \right)^{\frac{1}{r}} \right]^r \qquad (3.35)\\
&\le \theta_4 \int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u|^r \mathrm{d}x \\
&+ C\varepsilon \int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla(u-u_0)|^r \mathrm{d}x \\
&\le \theta_4 \int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u|^r \mathrm{d}x + C\varepsilon \int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u|^r \mathrm{d}x \\
&+ C\varepsilon \int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u_0|^r \mathrm{d}x
\end{aligned}
$$

Combining the inequalities (3.29),(3.30), (3.34), (3.35), we obtain

$$
\begin{aligned}
&\int_\Omega \eta^{p(1-\varepsilon)} |\nabla u|^r \mathrm{d}x \\
&\le C(\theta_2 + \theta_3 + \theta_4 + \varepsilon) \int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u|^r \mathrm{d}x \\
&+ CR^{-r} \left(\int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u|^{\frac{nr}{n+r}} \mathrm{d}x \right)^{\frac{n+r}{n}} \qquad (3.36)\\
&+ C\int_{Q_{2R} \mathrm{I}\, \Omega} |\nabla u_0|^r \mathrm{d}x,
\end{aligned}
$$

where $C = C(n, p, \alpha, \beta, K, \Omega)$.

Choosing θ_2, θ_3, θ_4 and $\varepsilon_0 > 0$ small enough, there exist $r_1 = p - \varepsilon_0 < p$, such that $\theta = C(\theta_2 + \theta_3 + \theta_4 + \varepsilon) < 1$ when $\varepsilon < \varepsilon_0$. By (3.36),

$$
\begin{aligned}
&⨍_{Q_R} |\nabla u|^r \mathrm{d}x \le \theta ⨍_{Q_{2R}} |\nabla u|^r \mathrm{d}x + C\left(⨍_{Q_{2R}} |\nabla u|^t \mathrm{d}x \right)^{\frac{r}{t}} \qquad (3.37)\\
&+ C ⨍_{Q_{2R}} |\nabla u_0|^r \mathrm{d}x,
\end{aligned}
$$

where $t = \frac{nr}{n+r} < r$. Let $g = |\nabla u|^t$, $G = 0$ .Then we arrive at the following inequality when $\varepsilon < \varepsilon_0$, that is

$$\begin{aligned} ⨍_{Q_R} g^{\frac{r}{t}} \mathrm{d}x \le \theta ⨍_{Q_{2R}} g^{\frac{r}{t}} \mathrm{d}x + C(⨍_{Q_{2R}} g \mathrm{d}x)^{\frac{r}{t}} \\ + C ⨍_{Q_{2R}} |G|^{\frac{r}{t}} \mathrm{d}x, \end{aligned} \tag{3.38}$$

where $C = C(n, p, \alpha, \beta, K, \Omega)$ .Then by (3.20),(3.38) and Lemma 2.5, there exists r' , and $r' > r$, such that $u \in W^{1,r'}(\Omega)$. For r' , repeating the above process, the integrability of ∇u is improved over and over again. In this way, there must be an integrable exponent r_1 and r_2 , satisfying $r_1 < p < r_2$, such that $u \in W^{1,\tau}(\Omega)$, $\forall \tau \in (r_1, r_2)$. The proof is complete.

REFERENCES

[1] Granlund S. An L^p-estimate for the gradient of extremals[J]. Math. Scand., 1982,50(1):66-72.

[2] Li Gongbao, Martio O. Local and global integrability of gradients in obstacle problems[J]. Ann Acad Sci Fenn Ser A I Math, 1994,19:25-34.

[3] Iwaniec T, Sbordone C. Weak minima of variational integrals[J]. J Reine Angew Math., 1994, 454: 143-161.

[4] Gilbarg D, Trudinger N S. Elliptic Partial Differential Equations of Second Order[M]. Berlin, Heidelberg: Springer-Verlag, 1977.

[5] Iwaniec T, Migliaccio L, Nania L, Sbordone C. Integrability and Removability Results for Quasiregular Mappings in High Dimensions[J]. MathScand,1994,75, 263-279.

[6] Giaquinta M. Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems[M]. Ann Math Stud 105, New Jersey: Princeton University Press, 1983.

[7] Elcrat A, Meyers N. Some results on regularity for non-linear elliptic systems and quasi-regular functions[J]. Duke Math J, 1975,42:121-136.

[8] Iwaniec T. p-Harmonic tensor and quasi-regular maps[J]. Ann Math, 1992,136(3):589-624.