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Proof Of Hardy Inequality

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***Abstract*—Starting from the discrete Hardy inequality, this paper derives the integral type Hardy inequality.In this paper, the two methods of balance coefficient method and error estimation method are used to prove the integral inequality, and the proof of the best coefficient is given.We give an example in the end ,**

***Keywords*—Hardy inequality : Balance factor method : Error estimation method**

Introduction

Hardy first proposed the following inequality in 1920:

$$\sum_{k=1}^{n}(\frac{x_1+x_2+\cdots+x_k}{k})^p \le \sum_{i=1}^{n} r_i x_i^p\ ,\quad r_i<(\frac{p}{p-1})^p\ .$$

among them, $\left(\dfrac{p}{p-1}\right)^p$ is the best coefficient [1].

The integral type of Hardy inequality is given in [2-4] :Assume $f(x)$ is non-negative in $[0,a]$, $p>1$,Define $(Tf)(x)=\dfrac{\int_0^x f(t)dt}{x}$, then $\|Tf\|_p \le \dfrac{p}{p-1}\|f\|_p$, among them, $\dfrac{p}{p-1}$ is the best coefficient.In this paper, the equilibrium coefficient method is used to prove the discrete Hardy inequality.For the integral type Hardy inequality, two methods of error estimation method and balance coefficient method are given.

Main content

1. Discrete Hardy inequality

Theorem 1：

Assume $x_1,x_2,\cdots,x_n\ge 0$, $p>1$,then $\sum_{k=1}^{n}\left(\dfrac{x_1+x_2+\cdots+x_k}{k}\right)^p \le (\dfrac{p}{p-1})^p\sum_{i=1}^{n}x_i^p$ .Inequality takes the equal sign if and only if $x_1=x_2=\cdots=x_n=0$.

Proof :

Step 1 Introduce balance factor

Let $q=\dfrac{p}{p-1}$ is the conjugate coefficient of p.Arbitrarily select positive numbers $\alpha_1,\alpha_2,\cdots,\alpha_n,\alpha$. According to

Hardy inequality,we can get：

$$x_1+x_2+\cdots+x_k=\frac{x_1}{a_1^{\alpha}}a_1^{\alpha}+\frac{x_2}{a_2^{\alpha}}a_2^{\alpha}+\cdots+\frac{x_k}{a_k^{\alpha}}a_k^{\alpha}\le(\frac{x_1^p}{a_1^{\alpha p}}+\frac{x_2^p}{a_2^{\alpha p}}+\cdots+\frac{x_k^p}{a_k^{\alpha p}})^{\frac{1}{p}}(\alpha_1^{\alpha q}+\alpha_2^{\alpha q}+\cdots+\alpha_k^{\alpha q})^{\frac{1}{q}}$$ Let

$\alpha q=1$, then $\alpha=\frac{1}{q}=1-\frac{1}{p}$,

then,

$$x_1+x_2+\cdots+x_k\le(\frac{x_1^p}{a_1^{(1-\frac{1}{p})p}}+\frac{x_2^p}{a_2^{(1-\frac{1}{p})p}}+\cdots+\frac{x_k^p}{a_k^{(1-\frac{1}{p})p}})^{\frac{1}{p}}(\alpha_1+\alpha_2+\cdots+\alpha_k)^{1-\frac{1}{p}},$$

$$=(\frac{x_1^p}{a_1^{p-1}}+\frac{x_2^p}{a_2^{p-1}}+\cdots+\frac{x_k^p}{a_k^{p-1}})^{\frac{1}{p}}(\alpha_1+\alpha_2+\cdots+\alpha_k)^{1-\frac{1}{p}}$$

then,

$$(x_1+x_2+\cdots+x_k)^p\le(\alpha_1+\alpha_2+\cdots+\alpha_k)^{p-1}(\frac{x_1^p}{a_1^{p-1}}+\frac{x_2^p}{a_2^{p-1}}+\cdots+\frac{x_k^p}{a_k^{p-1}})$$

then,

$$(\frac{x_1+x_2+\cdots+x_k}{k})^p\le\frac{(\alpha_1+\alpha_2+\cdots+\alpha_k)^{p-1}}{k^p}(\frac{x_1^p}{a_1^{p-1}}+\frac{x_2^p}{a_2^{p-1}}+\cdots+\frac{x_k^p}{a_k^{p-1}}),$$

then,

$$\sum_{k=1}^{n}(\frac{x_1+x_2+\cdots+x_k}{k})^p\le\sum_{k=1}^{n}\frac{(\alpha_1+\alpha_2+\cdots+\alpha_k)^{p-1}}{k^p}\sum_{i=1}^{k}\frac{x_i^p}{a_i^{p-1}}$$

$$=\sum_{i=1}^{n}\left[\frac{1}{a_i^{p-1}}\sum_{k=i}^{n}\frac{(\alpha_1+\alpha_2+\cdots+\alpha_k)^{p-1}}{k^p}\right]x_i^p$$

So the coefficient of x_i^p is $r_i=\frac{1}{a_i^{p-1}}\sum_{k=i}^{n}\frac{(\alpha_1+\alpha_2+\cdots+\alpha_k)^{p-1}}{k^p}$.

Step 2 Determine the appropriate balance factor

In order to make the form of $\alpha_1+\alpha_2+\cdots+\alpha_k$ as simple as possible，we can assume

$$\alpha_1+\alpha_2+\cdots+\alpha_k=k^{\beta},$$

Among them，$\beta>0$，then $\alpha_k=\begin{cases}1,k=1\\ k^{\beta}-(k-1)^{\beta},k>1\end{cases}=k^{\beta}-(k-1)^{\beta},k=1,2,\cdots,n,$

$$r_i=\frac{1}{a_i^{p-1}}\sum_{k=i}^{n}\frac{(\alpha_1+\alpha_2+\cdots+\alpha_k)^{p-1}}{k^p}=\frac{1}{a_i^{p-1}}\sum_{k=i}^{n}\frac{(k^{\beta})^{p-1}}{k^p}=\frac{1}{a_i^{p-1}}\sum_{k=i}^{n}k^{\beta p-\beta-p}$$

Choose β to make $\beta p-\beta-p<0, \beta p-\beta-p\neq -1$, then

$$(x^{\beta p-\beta-p})''=(\beta p-\beta-p)(\beta p-\beta-p-1)x^{\beta p-\beta-p-2}>0, x\in(0,+\infty),$$

So $x^{\beta p-\beta-p}$ is a strict lower convex function in $(0,+\infty)$,

then,

$$k^{\beta p-\beta-p}<\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} x^{\beta p-\beta-p}dx=\frac{(k-\frac{1}{2})^{(1-\beta)(p-1)}-(k+\frac{1}{2})^{(1-\beta)(p-1)}}{(1-\beta)(p-1)},$$

Then,

$$r_i=\frac{1}{a_i^{p-1}}\sum_{k=i}^{n}k^{\beta p-\beta-p}<\frac{1}{a_i^{p-1}}\sum_{k=i}^{n}\frac{(k-\frac{1}{2})^{(1-\beta)(p-1)}-(k+\frac{1}{2})^{(1-\beta)(p-1)}}{(1-\beta)(p-1)}$$
$$=\frac{(i-\frac{1}{2})^{(1-\beta)(p-1)}-(n+\frac{1}{2})^{(1-\beta)(p-1)}}{a_i^{p-1}(1-\beta)(p-1)}$$

As long as $(1-\beta)(p-1)>0$, that is $0<\beta<1$.Now, we can continue to scale this inequality

$$r_i=\frac{(i-\frac{1}{2})^{(1-\beta)(p-1)}-(n+\frac{1}{2})^{(1-\beta)(p-1)}}{a_i^{p-1}(1-\beta)(p-1)}<\frac{(i-\frac{1}{2})^{(1-\beta)(p-1)}}{(1-\beta)(p-1)a_i^{p-1}}$$

$$\left[\frac{(i-\frac{1}{2})^{(1-\beta)(p-1)}}{a_i^{p-1}}\right]^{\frac{1}{p-1}}=\frac{(i-\frac{1}{2})^{1-\beta}}{a_i}$$

$0<\beta<1$, So $x^{\beta-1}$ is a strict lower convex function in $(0,+\infty)$,

then,

$$\alpha_i=i^{\beta}-(i-1)^{\beta}=\beta\int_{i-1}^{i}x^{\beta-1}dx>\beta(i-\frac{1}{2})^{\beta-1}, i=1,2,\cdots,$$

then,

$$\frac{(i-\frac{1}{2})^{1-\beta}}{a_i}<\frac{(i-\frac{1}{2})^{1-\beta}}{\beta(i-\frac{1}{2})^{\beta-1}}=\frac{1}{\beta},$$

then,

$$r_i < \frac{\frac{1}{\beta^{p-1}}}{(1-\beta)(p-1)} = \frac{1}{(p-1)(1-\beta)\beta^{p-1}}$$

then,

$$\sum_{k=1}^{n}(\frac{x_1+x_2+\cdots+x_k}{k})^p \le \sum_{i=1}^{n}\frac{1}{(p-1)(1-\beta)\beta^{p-1}}x_i^p$$

Step 3 Determine the optimal balance factor

Consider the maximum value of the function $f(x)=(1-x)x^{p-1}=x^{p-1}-x^p$ in $[0,1]$.

$f(x)$ is non-negative and continuous in $[0,1]$.Derivative,

$$f'(x)=(p-1)x^{p-2}-px^{p-1}=px^{p-2}(\frac{p-1}{p}-x),$$

When $0<x<\frac{p-1}{p}$, $f'(x)>0$. When $\frac{p-1}{p}<x<1$, $f'(x)<0$,

So, $f(x)$ gets the maximum when $x=\frac{p-1}{p}$.

Choose $\beta=\frac{p-1}{p}$, then

$$\sum_{k=1}^{n}(\frac{x_1+x_2+\cdots+x_k}{k})^p \le \sum_{i=1}^{n}\frac{1}{(p-1)(1-\frac{p-1}{p})(\frac{p-1}{p})^{p-1}}x_i^p = (\frac{p}{p-1})^p\sum_{i=1}^{n}x_i^p$$

□

2.Integral Hardy Inequality

Theorem 2：

Assume that $f(x)$ is non-negative and continuous in $[0,a]$, $p>1$ and $(Tf)(x)=\frac{\int_0^x f(t)dt}{x}$, then

$$\|Tf\|_p \le \frac{p}{p-1}\|f\|_p$$

Proof :

Method 1

Step 1 Continuity of $(Tf)(x)$

$(Tf)(x)=\frac{\int_0^x f(t)dt}{x}$ is continuous in $(0,a]$,and

$$\lim_{x\to 0^+}(Tf)(x)=\lim_{x\to 0^+}\frac{\int_0^x f(t)dt}{x}=\lim_{x\to 0^+} f(x)=f(0),$$

We make a supplementary rule that $(Tf)(0)=f(0)$, then $(Tf)(x)$ is continuous in $[0,a]$.

Then $(Tf)(x)$ is uniform continuity in $[0,a]$.In other words, $\forall\varepsilon>0$, $\exists\delta>0$, s.t $\forall x,y\in[0,a]$, as long as $|x-y|<\delta$, $|f(x)-f(y)|<\varepsilon$. $\exists N>0$, s.t $\forall n>N$, $\frac{a}{n}<\delta$ 。

Step 2 Approximation of $(Tf)(x)$

$\int_0^{\frac{ia}{n}} f(t)dt$ can approximate approximation with $\frac{a}{n}\left[f(\frac{a}{n})+f(\frac{2a}{n})+\cdots+f(\frac{ia}{n})\right]$, then, $(Tf)(\frac{ia}{n})$ can approximate approximation with

$$\frac{\frac{a}{n}\left[f(\frac{a}{n})+f(\frac{2a}{n})+\cdots+f(\frac{ia}{n})\right]}{\frac{ia}{n}}=\frac{f(\frac{a}{n})+f(\frac{2a}{n})+\cdots+f(\frac{ia}{n})}{i}$$

According to Hardy inequality of discrete forms, we can get

$$\sum_{i=1}^{n}\left[\frac{f(\frac{a}{n})+f(\frac{2a}{n})+\cdots+f(\frac{ia}{n})}{i}\right]^p\le(\frac{p}{p-1})^p\sum_{i=1}^{n}\left[f(\frac{ia}{n})\right]^p,$$

then,

$$\sum_{i=1}^{n}\left[\frac{f(\frac{a}{n})+f(\frac{2a}{n})+\cdots+f(\frac{ia}{n})}{i}\right]^{p}\frac{a}{n}$$

$$\le(\frac{p}{p-1})^{p}\sum_{i=1}^{n}\left[f(\frac{ia}{n})\right]^{p}\frac{a}{n}\to(\frac{p}{p-1})^{p}\int_{0}^{a}f^{p}(x)dx, n\to\infty$$

Step 3 Error estimate

Estimate the difference between $\frac{\frac{a}{n}\left[f(\frac{a}{n})+f(\frac{2a}{n})+\cdots+f(\frac{ia}{n})\right]}{\frac{ia}{n}}$ (approximation value) and $(Tf)(\frac{ia}{n})=\frac{\int_{0}^{\frac{ia}{n}}f(t)dt}{\frac{ia}{n}}$ (real value).

$\forall n>N$, there is $\frac{a}{n}<\delta$, then, $\forall i\in\{1,2,\cdots,n\}$ and $x,y\in\left[\frac{(i-1)a}{n},\frac{ia}{n}\right]$,

there is $|f(x)-f(y)|<\varepsilon$.

$$\frac{a}{n}\left[f(\frac{a}{n})+f(\frac{2a}{n})+\cdots+f(\frac{ia}{n})\right]-\int_{0}^{\frac{ia}{n}}f(x)dx$$
$$=\frac{a}{n}\sum_{j=1}^{i}f(\frac{ja}{n})-\sum_{j=1}^{i}\int_{\frac{(j-1)a}{n}}^{\frac{ja}{n}}f(x)dx=\sum_{j=1}^{i}\int_{\frac{(j-1)a}{n}}^{\frac{ja}{n}}\left[f(\frac{ja}{n})-f(x)\right]dx,$$

$$\left|\frac{a}{n}\left[f(\frac{a}{n})+f(\frac{2a}{n})+\cdots+f(\frac{ia}{n})\right]-\int_{0}^{\frac{ia}{n}}f(x)dx\right|=\left|\sum_{j=1}^{i}\int_{\frac{(j-1)a}{n}}^{\frac{ja}{n}}\left[f(\frac{ja}{n})-f(x)\right]dx\right|$$
$$\le\sum_{j=1}^{i}\int_{\frac{(j-1)a}{n}}^{\frac{ja}{n}}\left|f(\frac{ja}{n})-f(x)\right|dx<\sum_{j=1}^{i}\int_{\frac{(j-1)a}{n}}^{\frac{ja}{n}}\varepsilon dx=\sum_{j=1}^{i}\frac{a\varepsilon}{n}=\frac{ia\varepsilon}{n},$$

then,

$$\left|\frac{\frac{a}{n}\left[f(\frac{a}{n})+f(\frac{2a}{n})+\cdots+f(\frac{ia}{n})\right]}{\frac{ia}{n}}-(Tf)(\frac{ia}{n})\right|=\left|\frac{\int_{0}^{\frac{ia}{n}}f(t)dt}{\frac{ia}{n}}-\frac{\frac{a}{n}\left[f(\frac{a}{n})+f(\frac{2a}{n})+\cdots+f(\frac{ia}{n})\right]}{\frac{ia}{n}}\right|\le\frac{\frac{ia\varepsilon}{n}}{\frac{ia}{n}}=\varepsilon$$

$$0 \le \frac{\frac{a}{n}\left[f(\frac{a}{n}) + f(\frac{2a}{n}) + \cdots + f(\frac{ia}{n})\right]}{\frac{ia}{n}} = \frac{f(\frac{a}{n}) + f(\frac{2a}{n}) + \cdots + f(\frac{ia}{n})}{i} \le M \ ,$$

$$0 \le (Tf)(x) = \frac{\int_0^x f(t)dt}{x} \le \frac{Mx}{x} = M \ ,$$

Among them, $M = \max_{x\in[0,a]} f(x)$.

Then,

$$\left|\sum_{i=1}^{n}\left[\frac{f(\frac{a}{n}) + f(\frac{2a}{n}) + \cdots + f(\frac{ia}{n})}{i}\right]^p \frac{a}{n} - \sum_{i=1}^{n}\left[(Tf)(\frac{ia}{n})\right]^p \frac{a}{n}\right|$$

$$= \left|\sum_{i=1}^{n}\left\{\left[\frac{f(\frac{a}{n}) + f(\frac{2a}{n}) + \cdots + f(\frac{ia}{n})}{i}\right]^p - \left[(Tf)(\frac{ia}{n})\right]^p\right\}\frac{a}{n}\right|$$

$$\le \sum_{i=1}^{n}\left|\left[\frac{f(\frac{a}{n}) + f(\frac{2a}{n}) + \cdots + f(\frac{ia}{n})}{i}\right]^p - \left[(Tf)(\frac{ia}{n})\right]^p\right|\frac{a}{n}$$

$$\overset{\text{Lagrange mean value theorem}}{=} \sum_{i=1}^{n} p\xi_i^{p-1}\left|\frac{f(\frac{a}{n}) + f(\frac{2a}{n}) + \cdots + f(\frac{ia}{n})}{i} - (Tf)(\frac{ia}{n})\right|\frac{a}{n}$$

$$\le \sum_{i=1}^{n} pM^{p-1}\varepsilon\frac{a}{n} = pM^{p-1}a\varepsilon$$

Here, ξ_i is between $\frac{f(\frac{a}{n}) + f(\frac{2a}{n}) + \cdots + f(\frac{ia}{n})}{i}$ and $(Tf)(\frac{ia}{n})$ .Then,

$$\lim_{n\to\infty}\left\{\sum_{i=1}^{n}\left[\frac{f(\frac{a}{n}) + f(\frac{2a}{n}) + \cdots + f(\frac{ia}{n})}{i}\right]^p \frac{a}{n} - \sum_{i=1}^{n}\left[(Tf)(\frac{ia}{n})\right]^p \frac{a}{n}\right\} = 0 \, ,$$

then,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{f(\frac{a}{n})+f(\frac{2a}{n})+\cdots+f(\frac{ia}{n})}{i}\right]^p\frac{a}{n}=\lim_{n\to\infty}\sum_{i=1}^{n}\left[(Tf)(\frac{ia}{n})\right]^p\frac{a}{n}=\int_0^a[(Tf)(x)]^p dx\,,$$

then,

$$\int_0^a[(Tf)(x)]^p dx\le(\frac{p}{p-1})^p\int_0^a f^p(x)dx\,,$$

then,

$$\left(\int_0^a[(Tf)(x)]^p dx\right)^{\frac{1}{p}}\le\frac{p}{p-1}\left(\int_0^a f^p(x)dx\right)^{\frac{1}{p}},$$

then,

$$\|Tf\|_p\le\frac{p}{p-1}\|f\|_p$$

□

Method 2

Let q is the conjugate number of p, $q=\dfrac{p}{p-1}$.Assume that $g(x)$ is a non-negative continuous function with at most one zero point in $[0,+\infty)$. $\forall x>0$, According to Holder inequality

$$(Tf)(x)=\frac{\int_0^x f(t)dt}{x}=\frac{\int_0^x f(t)g(t)\frac{1}{g(t)}dt}{x}\le\frac{\left(\int_0^x f^p(t)g^p(t)dt\right)^{\frac{1}{p}}\left(\int_0^x\frac{1}{g^q(t)}\right)^{1-\frac{1}{p}}}{x},$$

Here， $g(x)$ needs to choose properly to make $\int_0^x f^p(t)g^p(t)dt$ and $\int_0^x\dfrac{1}{g^q(t)}$ are significant.Then,

$$[(Tf)(x)]^p\le\frac{\left(\int_0^x\frac{1}{g^q(t)}\right)^{p-1}\int_0^x f^p(t)g^p(t)dt}{x^p}$$

To make $\int_0^x\dfrac{1}{g^q(t)}$ have a simple for,let $\int_0^x\dfrac{1}{g^q(t)}=x^\beta$, $\beta>0$.Deriving on both sides of the equation，then $\dfrac{1}{g^q(x)}=\beta x^{\beta-1}$,

$g(x)=\dfrac{1}{\beta^{\frac{1}{q}}}x^{\frac{1-\beta}{q}}$.

Then,

$$g^p(x)=\left[\frac{1}{\beta^{1-\frac{1}{p}}}x^{(1-\beta)(1-\frac{1}{p})}\right]^p=\frac{1}{\beta^{p-1}}x^{(1-\beta)(p-1)}$$

This $g(x)$ guarantees that $\int_0^a \frac{1}{g^q(t)}$ is significant.In order to ensure $\int_0^x f^p(t)g^p(t)dt$ is significant,we limit $\frac{1-\beta}{q}>0$ .then, $0<\beta<1$,

then,

$$[(Tf)(x)]^p\le\frac{x^{\beta(p-1)}\int_0^x f^p(t)g^p(t)dt}{x^p}=x^{\beta p-\beta-p}\int_0^x f^p(t)g^p(t)dt\ ,$$

then,

$$\int_0^a[(Tf)(x)]^p dx\le\int_0^a\left[x^{\beta p-\beta-p}\int_0^x f^p(t)g^p(t)dt\right]dx$$

Notice that when $x\in(0,a]$, $x^{\beta p-\beta-p}f^p(t)g^p(t)$ is continuous.But when $x=0$,we can't judge.In order to exchange the order of points smoothly,choose $\varepsilon\in(0,a)$,

then,

$$\int_\varepsilon^a\left[x^{\beta p-\beta-p}\int_0^x f^p(t)g^p(t)dt\right]dx$$

$$=\int_0^\varepsilon f^p(t)g^p(t)dt\int_\varepsilon^a x^{\beta p-\beta-p}dx+\int_\varepsilon^a f^p(t)g^p(t)dt\int_t^a x^{\beta p-\beta-p}dx$$

$$0\le\int_0^\varepsilon f^p(t)g^p(t)dt\int_\varepsilon^a x^{\beta p-\beta-p}dx=\int_0^\varepsilon f^p(t)\frac{1}{\beta^{p-1}}t^{(1-\beta)(p-1)}\frac{\varepsilon^{(\beta-1)(p-1)}-a^{(\beta-1)(p-1)}}{(1-\beta)(p-1)}dt$$

$$\le\frac{1}{(p-1)(1-\beta)\beta^{p-1}}\int_0^\varepsilon f^p(t)\varepsilon^{(1-\beta)(p-1)}\left[\varepsilon^{(\beta-1)(p-1)}-a^{(\beta-1)(p-1)}\right]dt$$

$$=\frac{1}{(p-1)(1-\beta)\beta^{p-1}}\int_0^\varepsilon f^p(t)\left[1-a^{(\beta-1)(p-1)}\varepsilon^{(1-\beta)(p-1)}\right]dt\to 0,\varepsilon\to 0^+$$

Then, $\lim\limits_{\varepsilon\to 0^+}\int_0^\varepsilon f^p(t)g^p(t)dt\int_\varepsilon^a x^{\beta p-\beta-p}dx=0$。

$$\int_\varepsilon^a f^p(t)g^p(t)dt\int_t^a x^{\beta p-\beta-p}dx = \int_\varepsilon^a f^p(t)\frac{1}{\beta^{p-1}}t^{(1-\beta)(p-1)}\frac{t^{(\beta-1)(p-1)}-a^{(\beta-1)(p-1)}}{(1-\beta)(p-1)}dt$$
$$=\frac{1}{(p-1)(1-\beta)\beta^{p-1}}\int_\varepsilon^a f^p(t)\left[1-a^{(\beta-1)(p-1)}t^{(1-\beta)(p-1)}\right]dt$$
$$\to\frac{1}{(p-1)(1-\beta)\beta^{p-1}}\int_0^a f^p(t)\left[1-a^{(\beta-1)(p-1)}t^{(1-\beta)(p-1)}\right]dt, \varepsilon\to 0^+$$

then,

$$\int_0^a\left[x^{\beta p-\beta-p}\int_0^x f^p(t)g^p(t)dt\right]dx = \lim_{\varepsilon\to 0^+}\int_\varepsilon^a\left[x^{\beta p-\beta-p}\int_0^x f^p(t)g^p(t)dt\right]dx$$
$$=\lim_{\varepsilon\to 0^+}\left[\int_0^\varepsilon f^p(t)g^p(t)dt\int_\varepsilon^a x^{\beta p-\beta-p}dx+\int_\varepsilon^a f^p(t)g^p(t)dt\int_t^a x^{\beta p-\beta-p}dx\right]$$
$$=\frac{1}{(p-1)(1-\beta)\beta^{p-1}}\int_0^a f^p(t)\left[1-a^{(\beta-1)(p-1)}t^{(1-\beta)(p-1)}\right]dt$$
$$\le\frac{1}{(p-1)(1-\beta)\beta^{p-1}}\int_0^a f^p(t)dt$$

then,

$$\int_0^a[(Tf)(x)]^p dx\le\frac{1}{(p-1)(1-\beta)\beta^{p-1}}\int_0^a f^p(t)dt$$

When $\beta\in[0,1]$, $(p-1)(1-\beta)\beta^{p-1}$ is non-negative continuous, and we get the maximum value at $\beta=\frac{p-1}{p}$.

$$(p-1)(1-\frac{p-1}{p})(\frac{p-1}{p})^{p-1}=(\frac{p-1}{p})^p,$$

then,

$$\int_0^a[(Tf)(x)]^p dx\le(\frac{p}{p-1})^p\int_0^a f^p(t)dt,$$

then,

$$\left(\int_0^a[(Tf)(x)]^p dx\right)^{\frac{1}{p}}\le\frac{p}{p-1}\left(\int_0^a f^p(t)dt\right)^{\frac{1}{p}},$$

then,

$$\|Tf\|_p\le\frac{p}{p-1}\|f\|_p \#$$

□

3.An example of integral type Hardy inequality

Theorem3： Assume $q \ge 1, r > 0$， $g(x)$ is a non-negative continuous function in $(0,+\infty)$.Prove that

$$\left(\int_0^{+\infty}\left[\int_0^t g(u)du\right]^q t^{-r-1}dt\right)^{\frac{1}{q}} \le \frac{q}{r}\left(\int_0^{+\infty}[ug(u)]^q u^{-r-1}du\right)^{\frac{1}{q}}$$

Proof:

Let $q > 1$ and $p = \dfrac{q}{q-1}$ are conjugated. $\forall a > 0$ and $\beta < \dfrac{1}{p}$， $\forall t > 0$，

According to Holder inequality，

$$\int_0^t g(u)du = \int_0^t g(u)u^{\beta}\frac{1}{u^{\beta}}du \le \left(\int_0^t g^q(u)u^{q\beta}du\right)^{\frac{1}{q}}\left(\int_0^t \frac{1}{u^{p\beta}}du\right)^{\frac{1}{p}},$$

then,

$$\left(\int_0^t g(u)du\right)^q \le \left(\int_0^t g^q(u)u^{q\beta}du\right)\left(\int_0^t \frac{1}{u^{p\beta}}du\right)^{(1-\frac{1}{q})q}$$

$$= \left(\frac{t^{1-p\beta}}{1-p\beta}\right)^{q-1}\left(\int_0^t g^q(u)u^{q\beta}du\right) = \frac{t^{q-1-q\beta}}{(1-p\beta)^{q-1}}\int_0^t g^q(u)u^{q\beta}du$$

If $q - q\beta - r - 2 \ne -1$，then

$$\int_0^a\left[\int_0^t g(u)du\right]^q t^{-r-1}dt \le \int_0^a \frac{t^{q-q\beta-r-2}}{(1-p\beta)^{q-1}}\int_0^t g^q(u)u^{q\beta}dudt = \frac{1}{(1-p\beta)^{q-1}}\int_0^a g^q(u)u^{q\beta}du\int_u^a t^{q-q\beta-r-2}dt$$

$$= \frac{1}{(1-p\beta)^{q-1}}\int_0^a g^q(u)u^{q\beta}\frac{u^{q-q\beta-r-1}-a^{q-q\beta-r-1}}{q\beta+r+1-q}du$$

Let $q\beta + r + 1 - q > 0$， then,

$$\int_0^a\left[\int_0^t g(u)du\right]^q t^{-r-1}dt \le \frac{1}{(1-p\beta)^{q-1}}\int_0^a g^q(u)u^{q\beta}\frac{u^{q-q\beta-r-1}}{q\beta+r+1-q}du$$

$$= \frac{1}{(1-p\beta)^{q-1}(q\beta+r+1-q)}\int_0^a [ug(u)]u^{-r-1}du$$

Let $1 - p\beta = q\beta + r + 1 - q$， then

$$\beta = \frac{q-r}{p+q} = \frac{q-r}{\frac{q}{q-1}+q} = \frac{(q-r)(q-1)}{q^2},$$

Such β can satisfy :

$$\beta=\frac{(q-r)(q-1)}{q^2}<\frac{q(q-1)}{q^2}=\frac{q-1}{q}=\frac{1}{p},$$

then,

$$q\beta+r+1-q=1-p\beta>0,$$

then,

$$(1-p\beta)^{q-1}(q\beta+r+1-q)=(1-p\beta)^q=(1-p\frac{q-r}{p+q})^q$$

$$=(\frac{p+q-pq+pr}{p+q})^q=(\frac{\frac{q}{q-1}+q-\frac{q}{q-1}q+\frac{q}{q-1}r}{\frac{q}{q-1}+q})^q=(\frac{r}{q})^q$$

then,

$$\int_0^a\left[\int_0^t g(u)du\right]^q t^{-r-1}dt\le\frac{1}{(\frac{r}{q})^q}\int_0^a[ug(u)]u^{-r-1}du=(\frac{q}{r})^q\int_0^a[ug(u)]u^{-r-1}du,$$

then,

$$\left(\int_0^a\left[\int_0^t g(u)du\right]^q t^{-r-1}dt\right)^{\frac{1}{q}}\le\frac{q}{r}\left(\int_0^a[ug(u)]u^{-r-1}du\right)^{\frac{1}{q}},$$

Take $a\to+\infty$, we can get

$$\left(\int_0^{+\infty}\left[\int_0^t g(u)du\right]^q t^{-r-1}dt\right)^{\frac{1}{q}}\le\frac{q}{r}\left(\int_0^{+\infty}[ug(u)]^q u^{-r-1}du\right)^{\frac{1}{q}}$$

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