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Proof And Promotion Of Hardy Inequality

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***Abstract*—Based on the study of classical Hardy's inequality, this paper uses the integral transformation method to refine and improve the original inequality, and proposes and proves two derived Hardy's inequalities.**

Keywords—Hardy inequality;integral transform method

I. INTRODUCTION

Since Hardy first proved this inequality relationship in 1920, there has been a lot of application and promotion work in Hardy inequality .First ,we get the basic form of Hardy inequality: we assume $f(x)$ is a Non-negative continuous function in $[0,a]$ $[0,a]$, $p>1$,definition $(Tf)(x)=\frac{\int_0^x f(t)dt}{x}$, Then $\|Tf\|_p \le \frac{p}{p-1}\|f\|_p$, Which the $\frac{p}{p-1}$ is the best coefficient of the Hardy inequality[1-4].in references [5-6], The author gives thehigh dimensional form of Hardy's inequality: Assume $f(x_1,x_2,\cdots,x_n)$ is a Non-negative integrablefunctionin $[0,a_1]\times[0,a_2]\times\cdots\times[0,a_n]$, $p>1$, $q=\frac{p}{p-1}$, is the Conjugate number of p 。Assume $g(x_1,x_2,\cdots,x_n)=x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}$, $k_1,k_2,\cdots,k_n$ is a positive number to be determined, $k_j<\frac{1}{q}$, $j=1,2,\cdots,n$。definition

$(Tf)(x_1,x_2,\cdots,x_n)=\frac{\int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} f(u_1,u_2,\cdots,u_n)du_1du_2\cdots du_n}{x_1x_2\cdots x_n}$,, Then we can get $\|Tf\|_p \le (\frac{p}{p-1})^n\|f\|_p$, Which the $\left(\frac{p}{p-1}\right)^n$ is the best coefficient of the Hardy inequality.In this paper, We refine the Hardy's inequality and give an error estimate. Based on this error estimate ,we propose and prove two refined Hardy inequalities.

II. MAIN CONTENT

A. Theorem 1 : Assume $f(x)\in C^1[0,a]$, $f(0)=0$,

$$\begin{cases}\frac{\int_0^x f'(t)dt}{x}, x\in(0,a]=\frac{f(x)-f(0)}{x}=\frac{f(x)}{x}, x\in(0,a] \\ f'(0), x=a\end{cases}$$

Then we can get

$$f^4(x)\le x^2\int_0^a\left\{4[f'(t)]^2-[Tf'(t)]^2\right\}dt\int_0^a[Tf'(t)]^2dt$$

Proof.

By Hardy's inequality, we can get

$$\int_0^a [Tf'(x)]^2 dx \le (\frac{2}{2-1})^2 \int_0^a [f'(x)]^2 dx = 4\int_0^a [f'(x)]^2 dx$$,

Then

$$\int_0^a \frac{f^2(x)}{x^2} dx \le 4\int_0^a [f'(x)]^2 dx$$,

Then

$$\int_0^a \left\{ [f'(x)]^2 - \frac{f^2(x)}{4x^2} \right\} dx \ge 0$$

We make the following integral transformation $f(x) = g(x)h(x)$, $g(x), h(x)$ is differentiable,then

$$f'(x) = g'(x)h(x) + g(x)h'(x)$$,

Then

$$\begin{aligned} &[f'(x)]^2 - \frac{f^2(x)}{4x^2} \\ &= [g'(x)]^2 h^2(x) + g^2(x)[h'(x)]^2 + 2g'(x)h(x)g(x)h'(x) - \frac{g^2(x)h^2(x)}{4x^2} \\ &= [g'(x)]^2 h^2(x) + 2g'(x)h(x)g(x)h'(x) + \left\{ [h'(x)]^2 - \frac{h^2(x)}{4x^2} \right\} g^2(x) \end{aligned}$$

We assume $[h'(x)]^2 - \frac{h^2(x)}{4x^2} = 0$,take $h'(x) = \frac{h(x)}{2x}$,We can get

$$h(x) = Ce^{\int \frac{dx}{2x}} = C\sqrt{x}$$ 。

take $h(x) = \sqrt{x}$, $g(x) = \frac{f(x)}{\sqrt{x}}, x \in (0, a]$ then $h(x), g(x)$ is differentiable in $(0, a]$,then

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \frac{f(x)}{\sqrt{x}} = \lim_{x \to 0^+} \sqrt{x} \lim_{x \to 0^+} \frac{f(x)}{x} = 0 \cdot f'(0) = 0$$,

Now,we can assume $g(0) = 0$ .For any $x \in (0, a]$,we can get

$$\begin{aligned} &[f'(x)]^2 - \frac{f^2(x)}{4x^2} = [g'(x)]^2 h^2(x) + 2g'(x)h(x)g(x)h'(x) \\ &= x[g'(x)]^2 + 2g(x)g'(x)\sqrt{x}\frac{1}{2\sqrt{x}} = x[g'(x)]^2 + g(x)g'(x) \end{aligned}$$,

Then

$$\int_0^a \left\{[f'(x)]^2 - \frac{f^2(x)}{4x^2}\right\} dx = \int_0^a \left\{x[g'(x)]^2 + g(x)g'(x)\right\} dx = \int_0^a x[g'(x)]^2 dx + \frac{1}{2} g^2(x)\Big|_0^a$$

$$= \int_0^a x[g'(x)]^2 dx + \frac{1}{2} g^2(a)$$

For any $x \in [0,a]$,we can get

$$g^2(x) = g^2(x) - g^2(0) = 2\int_0^x g(t)g'(t)dt \quad ,$$

By Cauchy inequality

$$g^2(x) = 2\int_0^x g(t)g'(t)dt = 2\int_0^x g'(t)\sqrt{t}\frac{g(t)}{\sqrt{t}}dt \le 2\left(\int_0^x t[g'(t)]^2 dt\right)^{\frac{1}{2}} \left(\int_0^x \frac{g^2(t)}{t}dt\right)^{\frac{1}{2}}$$

$$\le 2\left(\int_0^a t[g'(t)]^2 dt\right)^{\frac{1}{2}} \left(\int_0^a \frac{g^2(t)}{t}dt\right)^{\frac{1}{2}}$$

Then

$$g^4(x) \le 4\int_0^a t[g'(t)]^2 dt \int_0^a \frac{g^2(t)}{t}dt \le 4\left\{\int_0^a t[g'(t)]^2 dt + \frac{1}{2}g^2(a)\right\}\int_0^a \frac{g^2(t)}{t}dt$$

$$= 4\int_0^a \left\{[f'(t)]^2 - \frac{f^2(t)}{4t^2}\right\} dt \int_0^a \frac{f^2(t)}{t^2}dt = \int_0^a \left\{4[f'(t)]^2 - \frac{f^2(t)}{t^2}\right\} dx \int_0^a \frac{f^2(t)}{t^2}dt$$

$$= \int_0^a \left\{4[f'(t)]^2 - [Tf'(t)]^2\right\} dt \int_0^a [Tf'(t)]^2 dt$$

Then

$$\frac{f^4(x)}{x^2} \le \int_0^a \left\{4[f'(t)]^2 - [Tf'(t)]^2\right\} dt \int_0^a [Tf'(t)]^2 dt \quad ,$$

Then

$$f^4(x) \le x^2 \int_0^a \left\{4[f'(t)]^2 - [Tf'(t)]^2\right\} dt \int_0^a [Tf'(t)]^2 dt$$

B. Theorem 2 : We assume $f(x) \in C^1[0,a]$, $f(0) = 0$*,then*

$$f^4(x) \le \left(\frac{4}{\sqrt{3}} + 2\right)^2 \int_0^a \left[|f'(t)|^2 - \frac{f^2(t)}{4t^2}\right] dt \int_0^a f^2(t)dt, x \in [0,a]$$

Proof.

We take $g(x) = \frac{f(x)}{\sqrt{x}}$,then we can get

$$\int_0^a \left\{ [f'(t)]^2 - \frac{f^2(t)}{4t^2} \right\} dt = \int_0^a t\left[g'(t)\right]^2 dt + \frac{1}{2} g^2(a)$$

For any $x \in (0, a]$,we can get

$$g^2(x) - g^2(a) = 2\int_a^x g(t)g'(t)dt \le 2\int_x^a |g(t)||g'(t)|dt = 2\int_x^a |g(t)||g'(t)|\frac{x}{x}dt$$

$$\le 2\int_x^a |g(t)||g'(t)|\frac{t}{x}dt = \frac{2}{x}\int_x^a \sqrt{t}|g'(t)|\sqrt{t}|g(t)|dt \le \frac{2}{x}\left[\int_x^a t|g'(t)|^2 dt\right]^{\frac{1}{2}}\left[\int_x^a tg^2(t)dt\right]^{\frac{1}{2}}$$

$$\le \frac{2}{x}\left[\int_0^a t|g'(t)|^2 dt\right]^{\frac{1}{2}}\left[\int_0^a tg^2(t)dt\right]^{\frac{1}{2}} \le \frac{2}{x}\left[\int_0^a t|g'(t)|^2 dt + \frac{1}{2}g^2(a)\right]^{\frac{1}{2}}\left[\int_0^a tg^2(t)dt\right]^{\frac{1}{2}}$$

Then

$$g^2(x) \le g^2(a) + \frac{2}{x}\left[\int_0^a t|g'(t)|^2 dt + \frac{1}{2}g^2(a)\right]^{\frac{1}{2}}\left[\int_0^a tg^2(t)dt\right]^{\frac{1}{2}}$$

We are now giving an estimate by $g^2(a)$,take $s > 0$, $h(x) = x^s g^2(x)$

Then $h(x)$ is a continuousfunctionin $[0, a]$, $h(0) = 0$,we can get

$$a^s g^2(a) = h(a) = h(a) - h(0) = \int_0^a h'(t)dt = \int_0^a \left[st^{s-1}g^2(t) + 2t^s g(t)g'(t)\right]dt$$

$$= \int_0^a t^{\frac{1}{2}} g(t)\left[st^{s-\frac{3}{2}}g(t) + 2t^{s-\frac{1}{2}}g'(t)\right]dt \le \left[\int_0^a tg^2(t)dt\right]^{\frac{1}{2}}\left\{\int_0^a \left[st^{s-\frac{3}{2}}g(t) + 2t^{s-\frac{1}{2}}g'(t)\right]^2 dt\right\}^{\frac{1}{2}}$$

$$= T^{\frac{1}{2}}\left[\int_0^a tg^2(t)dt\right]^{\frac{1}{2}}$$

with

$$T = \int_0^a \left[st^{s-\frac{3}{2}}g(t) + 2t^{s-\frac{1}{2}}g'(t)\right]^2 dt = \int_0^a \left[s^2t^{2s-3}g^2(t) + 4st^{2s-2}g(t)g'(t) + 4t^{2s-1}|g'(t)|^2\right]dt$$

If $s \ne 1$,then

$$T = \int_0^a \left\{ \frac{s^2}{2s-2}(t^{2s-2})'g^2(t) + 2st^{2s-2}\left[g^2(t)\right]' + 4t^{2s-1}|g'(t)|^2 \right\} dt$$

Assume $\frac{s^2}{2s-2} = 2s$,which leads to $s = \frac{4}{3}$,Now take $s = \frac{4}{3}$,we can get

$$T=\int_0^a\left\{\frac{8}{3}\left[t^{\frac{2}{3}}g^2(t)\right]'+4t^{\frac{5}{3}}|g'(t)|^2\right\}dt=4\int_0^a t^{\frac{5}{3}}|g'(t)|^2\,dt+\frac{8}{3}\left[t^{\frac{2}{3}}g^2(t)\right]\Big|_0^a$$

$$=4\int_0^a t^{\frac{5}{3}}|g'(t)|^2\,dt+\frac{8}{3}a^{\frac{2}{3}}g^2(a)=4a^{\frac{5}{3}}\int_0^a(\frac{t}{a})^{\frac{5}{3}}|g'(t)|^2\,dt+\frac{8}{3}a^{\frac{2}{3}}g^2(a)$$

$$\le 4a^{\frac{5}{3}}\int_0^a\frac{t}{a}|g'(t)|^2\,dt+\frac{8}{3}a^{\frac{2}{3}}g^2(a)=4a^{\frac{2}{3}}\int_0^a t|g'(t)|^2\,dt+\frac{8}{3}a^{\frac{2}{3}}g^2(a)$$

$$=4a^{\frac{2}{3}}\left[\int_0^a t|g'(t)|^2\,dt+\frac{2}{3}g^2(a)\right]\le 4a^{\frac{2}{3}}\bullet\frac{4}{3}\left[\int_0^a t|g'(t)|^2\,dt+\frac{1}{2}g^2(a)\right]$$

$$=\frac{16}{3}a^{\frac{2}{3}}\left[\int_0^a t|g'(t)|^2\,dt+\frac{1}{2}g^2(a)\right]$$

Then

$$a^{\frac{4}{3}}g^2(a)\le T^{\frac{1}{2}}\left[\int_0^a tg^2(t)dt\right]^{\frac{1}{2}}\le\left\{\frac{16}{3}a^{\frac{2}{3}}\left[\int_0^a t|g'(t)|^2\,dt+\frac{1}{2}g^2(a)\right]\right\}^{\frac{1}{2}}\left[\int_0^a tg^2(t)dt\right]^{\frac{1}{2}}$$

$$=\frac{4}{\sqrt{3}}a^{\frac{1}{3}}\left[\int_0^a t|g'(t)|^2\,dt+\frac{1}{2}g^2(a)\right]^{\frac{1}{2}}\left[\int_0^a tg^2(t)dt\right]^{\frac{1}{2}}$$

Then

$$g^2(a)\le\frac{4}{\sqrt{3a}}\left[\int_0^a t|g'(t)|^2\,dt+\frac{1}{2}g^2(a)\right]^{\frac{1}{2}}\left[\int_0^a tg^2(t)dt\right]^{\frac{1}{2}}$$

Then

$$g^2(x)\le g^2(a)+\frac{2}{x}\left[\int_0^a t|g'(t)|^2dt+\frac{1}{2}g^2(a)\right]^{\frac{1}{2}}\left[\int_0^a tg^2(t)dt\right]^{\frac{1}{2}}$$

$$\le\frac{4}{\sqrt{3a}}\left[\int_0^a t|g'(t)|^2\,dt+\frac{1}{2}g^2(a)\right]^{\frac{1}{2}}\left[\int_0^a tg^2(t)dt\right]^{\frac{1}{2}}+\frac{2}{x}\left[\int_0^a t|g'(t)|^2dt+\frac{1}{2}g^2(a)\right]^{\frac{1}{2}}\left[\int_0^a tg^2(t)dt\right]^{\frac{1}{2}}$$ We have

$$=(\frac{4}{\sqrt{3a}}+\frac{2}{x})\left[\int_0^a t|g'(t)|^2\,dt+\frac{1}{2}g^2(a)\right]^{\frac{1}{2}}\left[\int_0^a tg^2(t)dt\right]^{\frac{1}{2}}$$

completed the certificate

$$f^4(x)\le(\frac{4}{\sqrt{3}}+2)^2\int_0^a\left[|f'(t)|^2-\frac{f^2(t)}{4t^2}\right]dt\int_0^a f^2(t)dt, x\in(0,a]$$

C. Theorem3 : Assume $f(x)\in C^1[0,a]$, $f(0)=0, q>1$,*we can get*

$$|f(x)|^{2q}\le\left(\sqrt{\frac{2q^3}{2q-1}}+q\right)^2\int_0^a\left[|f'(t)|^2-\frac{f^2(t)}{4t^2}\right]dt\int_0^a|f(t)|^{2q-2}dt$$

In order to prove the theorem3 , we first introduce the following lemma

*Lemma 1.*Assume $f(x)\in C^1[a,b]$, $q>1$,we can get $|f(x)|^q \in C^1[a,b]$.

Proof :

Take $x_0 \in [a,b]$,

*(one)*If $f(x_0)>0$,then there exist a neighborhood $U(x_0)$ by x_0 ,then in $U(x_0)\cap[a,b]$

we can get $f(x)>0$

$$\lim_{x\to x_0}\frac{|f(x)|^q-|f(x_0)|^q}{x-x_0}=\lim_{x\to x_0}\frac{[f(x)]^q-[f(x_0)]^q}{x-x_0}=q[f(x_0)]^{q-1}f'(x_0)=q|f(x_0)|^{q-1}f'(x_0)\operatorname{sgn}(f(x_0))$$

we can get $|f(x)|^q$ is differentiable in x_0, then

$$\frac{d}{dx}|f(x)|^q\Big|_{x=x_0}=q|f(x_0)|^{q-1}f'(x_0)\operatorname{sgn}(f(x_0))$$

(two) If $f(x_0)<0$,then there exist a neighborhood $U(x_0)$ by x_0 ,then in $U(x_0)\cap[a,b]$

we can get $f(x)<0$

$$\lim_{x\to x_0}\frac{|f(x)|^q-|f(x_0)|^q}{x-x_0}=\lim_{x\to x_0}\frac{[-f(x)]^q-[-f(x_0)]^q}{x-x_0}$$
$$=q[-f(x_0)]^{q-1}[-f'(x_0)]=q|f(x_0)|^{q-1}f'(x_0)\operatorname{sgn}(f(x_0))$$

Now we can get $|f(x)|^q$ is differentiable in x_0, then

$$\frac{d}{dx}|f(x)|^q\Big|_{x=x_0}=q|f(x_0)|^{q-1}f'(x_0)\operatorname{sgn}(f(x_0))$$

(there) If $f(x_0)=0$,then

$$\lim_{x\to x_0}\frac{f(x)}{x-x_0}=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0),$$

Then $\lim_{x\to x_0}\left|\frac{f(x)}{x-x_0}\right|=|f'(x_0)|$,then

$$\lim_{x\to x_0}\left|\frac{|f(x)|^q-|f(x_0)|^q}{x-x_0}\right|=\lim_{x\to x_0}\left|\frac{|f(x)|^q}{x-x_0}\right|=\lim_{x\to x_0}|f(x)|^{q-1}\lim_{x\to x_0}\left|\frac{f(x)}{x-x_0}\right|=0,$$

$$\lim_{x\to x_0}\frac{|f(x)|^q-|f(x_0)|^q}{x-x_0}=0=q|f(x_0)|^{q-1}f'(x_0)\operatorname{sgn}(f(x_0))$$

we can get $\left|f(x)\right|^q$ is differentiable in x_0 ,

$$\frac{d}{dx}\left|f(x)\right|^q\Big|_{x=x_0}=0=q\left|f(x_0)\right|^{q-1}f'(x_0)\operatorname{sgn}\left(f(x_0)\right)$$

It follows from the arbitrariness of x_0 ,we can get $\left|f(x)\right|^q$ is differentiable in $[a,b]$,

$$\frac{d}{dx}\left|f(x)\right|^q=q\left|f(x)\right|^{q-1}f'(x)\operatorname{sgn}\left(f(x)\right)$$

Assume $f(x)\in C^1[a,b]$, $f'(x)$ is a continous function in $[a,b]$

Take any $x_0\in[a,b]$, If $f(x_0)>0$,then there exist a neighborhood $U(x_0)$ by x_0 ,then in $U(x_0)\bigcap[a,b]$,we can get $f(x)>0$

$$\lim_{x\to x_0}\frac{d}{dx}\left|f(x)\right|^q=\lim_{x\to x_0}\left[q\left|f(x)\right|^{q-1}f'(x)\operatorname{sgn}\left(f(x)\right)\right]=\lim_{x\to x_0}\left[q\left|f(x)\right|^{q-1}f'(x)\right]$$
$$=q\left|f(x_0)\right|^{q-1}f'(x_0)=q\left|f(x_0)\right|^{q-1}f'(x_0)\operatorname{sgn}\left(f(x_0)\right)=\frac{d}{dx}\left|f(x)\right|^q\Big|_{x=x_0}$$

If $f(x_0)<0$,then there exist a neighborhood $U(x_0)$ by x_0 ,then in $U(x_0)\bigcap[a,b]$

we can get $f(x)<0$

$$\lim_{x\to x_0}\frac{d}{dx}\left|f(x)\right|^q=\lim_{x\to x_0}\left[q\left|f(x)\right|^{q-1}f'(x)\operatorname{sgn}\left(f(x)\right)\right]=-\lim_{x\to x_0}\left[q\left|f(x)\right|^{q-1}f'(x)\right]$$
$$=-q\left|f(x_0)\right|^{q-1}f'(x_0)=q\left|f(x_0)\right|^{q-1}f'(x_0)\operatorname{sgn}\left(f(x_0)\right)=\frac{d}{dx}\left|f(x)\right|^q\Big|_{x=x_0}$$

If $f(x_0)=0$,then

$$\left|\frac{d}{dx}\left|f(x)\right|^q\right|=\left|q\left|f(x)\right|^{q-1}f'(x)\operatorname{sgn}\left(f(x)\right)\right|\le q\left|f(x)\right|^{q-1}\left|f'(x)\right|\to 0, x\to x_0$$,

Now ,we can get $\lim_{x\to x_0}\frac{d}{dx}\left|f(x)\right|^q=0=\frac{d}{dx}\left|f(x)\right|^q\Big|_{x=x_0}$

Therefore, $\frac{d}{dx}\left|f(x)\right|^q$ is continuousin x_0 .It follows from the arbitrariness of x_0

$\frac{d}{dx}\left|f(x)\right|^q$ is a continousfunction in $[a,b]$,then we can get $\left|f(x)\right|^q\in C^1[a,b]$

With this lemma, Now we can prove the theorem 3

Assume $f(x)\in C^1[0,a]$, $f(0)=0$, $q>1$, $g(x)=\frac{f(x)}{\sqrt{x}}$,for any $g(x)=\frac{f(x)}{\sqrt{x}}$,

$$|g(x)|^q-|g(a)|^q=\int_a^x \frac{d}{dt}|g(t)|^q\,dt=\int_a^x q|g(t)|^{q-1}g'(t)\operatorname{sgn}(g(t))dt$$

$$\le q\int_x^a |g(t)|^{q-1}|g'(t)||\operatorname{sgn}(g(t))|dt\le q\int_x^a |g(t)|^{q-1}|g'(t)|dt$$

$$=\frac{q}{x^{\frac{q}{2}}}\int_x^a |g(t)|^{q-1}|g'(t)|x^{\frac{q}{2}}dt\le\frac{q}{x^{\frac{q}{2}}}\int_x^a |g(t)|^{q-1}|g'(t)|t^{\frac{q}{2}}dt$$

$$=\frac{q}{x^{\frac{q}{2}}}\int_x^a |g(t)|^{q-1}t^{\frac{q-1}{2}}|g'(t)|\sqrt{t}dt\le\frac{q}{x^{\frac{q}{2}}}\left[\int_x^a t|g'(t)|^2\,dt\right]^{\frac{1}{2}}\left[\int_x^a t^{q-1}|g(t)|^{2q-2}dt\right]^{\frac{1}{2}},$$

Then

$$|g(x)|^q\le|g(a)|^q+\frac{q}{x^{\frac{q}{2}}}\left[\int_x^a t|g'(t)|^2\,dt\right]^{\frac{1}{2}}\left[\int_x^a t^{q-1}|g(t)|^{2q-2}dt\right]^{\frac{1}{2}}$$

$$\le|g(a)|^q+\frac{q}{x^{\frac{q}{2}}}\left[\int_0^a t|g'(t)|^2\,dt+\frac{1}{2}g^2(a)\right]^{\frac{1}{2}}\left[\int_0^a t^{q-1}|g(t)|^{2q-2}dt\right]^{\frac{1}{2}}$$

We are now giving an estimate by $|g(a)|^q$ .take $s>0$, $h(x)=x^s|g(x)|^q$,Then $h(x)$ is a continous function in $[0,a]$, $h(0)=0$,we can get

$$a^s|g(a)|^q=h(a)=h(a)-h(0)=\int_0^a h'(t)dt=\int_0^a\left[st^{s-1}|g(t)|^q+qt^s|g(t)|^{q-1}g'(t)\operatorname{sgn}(g(t))\right]dt$$

$$=\int_0^a|g(t)|^{q-1}t^{\frac{q-1}{2}}\left[st^{s-1-\frac{q-1}{2}}|g(t)|+qt^{s-\frac{q-1}{2}}g'(t)\operatorname{sgn}(g(t))\right]dt$$

$$\le\left[\int_0^a|g(t)|^{2q-2}t^{q-1}dt\right]^{\frac{1}{2}}\left\{\int_0^a\left[st^{s-1-\frac{q-1}{2}}|g(t)|+qt^{s-\frac{q-1}{2}}g'(t)\operatorname{sgn}(g(t))\right]^2dt\right\}^{\frac{1}{2}}$$

with

$$T=\int_0^a\left[st^{s-1-\frac{q-1}{2}}|g(t)|+qt^{s-\frac{q-1}{2}}g'(t)\operatorname{sgn}(g(t))\right]^2dt$$

$$T=\int_0^a\left[st^{s-1-\frac{q-1}{2}}|g(t)|+qt^{s-\frac{q-1}{2}}g'(t)\operatorname{sgn}(g(t))\right]^2dt$$

$$=\int_0^a\left[s^2t^{2s-q-1}g^2(t)+2sqt^{2s-q}g(t)g'(t)+q^2t^{2s-q+1}|g'(t)|^2|\operatorname{sgn}(g(t))|^2\right]dt$$

$$=\int_0^a\left[\frac{s^2}{2s-q}(t^{2s-q})'g^2(t)+sqt^{2s-q}\frac{d}{dt}g^2(t)+q^2t^{2s-q+1}|g'(t)|^2|\operatorname{sgn}(g(t))|^2\right]dt$$

Assume $\frac{s^2}{2s-q}=sq$,which leads to $s=\frac{q^2}{2q-1}$,then $2s-q=2\frac{q^2}{2q-1}-q=\frac{q}{2q-1}$,

$$T=\int_0^a\left[\frac{s^2}{2s-q}(t^{2s-q})'g^2(t)+sqt^{2s-q}\frac{d}{dt}g^2(t)+q^2t^{2s-q+1}\left|g'(t)\right|^2\left|\operatorname{sgn}\left(g(t)\right)\right|^2\right]dt$$

$$=\frac{q^3}{2q-1}\left[t^{\frac{q}{2q-1}}g^2(t)\right]\Big|_0^a+q^2\int_0^a t^{\frac{3q-1}{2q-1}}\left|g'(t)\right|^2\left|\operatorname{sgn}\left(g(t)\right)\right|^2dt$$

$$=\frac{q^3}{2q-1}a^{\frac{q}{2q-1}}g^2(a)+q^2\int_0^a t^{\frac{3q-1}{2q-1}}\left|g'(t)\right|^2\left|\operatorname{sgn}\left(g(t)\right)\right|^2dt$$

$$\le\frac{q^3}{2q-1}a^{\frac{q}{2q-1}}g^2(a)+q^2a^{\frac{3q-1}{2q-1}}\int_0^a(\frac{t}{a})^{\frac{3q-1}{2q-1}}\left|g'(t)\right|^2dt\le\frac{q^3}{2q-1}a^{\frac{q}{2q-1}}g^2(a)+q^2a^{\frac{3q-1}{2q-1}}\int_0^a\frac{t}{a}\left|g'(t)\right|^2dt$$

$$=\frac{q^3}{2q-1}a^{\frac{q}{2q-1}}g^2(a)+q^2a^{\frac{q}{2q-1}}\int_0^a t\left|g'(t)\right|^2dt=q^2a^{\frac{q}{2q-1}}\left[\int_0^a t\left|g'(t)\right|^2dt+\frac{q}{2q-1}g^2(a)\right]$$

$$\le\frac{2q}{2q-1}\cdot q^2a^{\frac{q}{2q-1}}\left[\int_0^a t\left|g'(t)\right|^2dt+\frac{1}{2}g^2(a)\right]=\frac{2q^3}{2q-1}a^{\frac{q}{2q-1}}\left[\int_0^a t\left|g'(t)\right|^2dt+\frac{1}{2}g^2(a)\right]$$

Then

$$a^{\frac{q^2}{2q-1}}\left|g(a)\right|^q\le T^{\frac{1}{2}}\left[\int_0^a\left|g(t)\right|^{2q-2}t^{q-1}dt\right]^{\frac{1}{2}}$$

$$\le\left\{\frac{2q^3}{2q-1}a^{\frac{q}{2q-1}}\left[\int_0^a t\left|g'(t)\right|^2dt+\frac{1}{2}g^2(a)\right]\right\}^{\frac{1}{2}}\left[\int_0^a\left|g(t)\right|^{2q-2}t^{q-1}dt\right]^{\frac{1}{2}}$$

$$=\sqrt{\frac{2q^3}{2q-1}}a^{\frac{q}{2(2q-1)}}\left[\int_0^a t\left|g'(t)\right|^2dt+\frac{1}{2}g^2(a)\right]^{\frac{1}{2}}\left[\int_0^a\left|g(t)\right|^{2q-2}t^{q-1}dt\right]^{\frac{1}{2}},$$

Then

$$\left|g(a)\right|^q\le\sqrt{\frac{2q^3}{2q-1}}a^{\frac{q}{2(2q-1)}-\frac{q^2}{2q-1}}\left[\int_0^a t\left|g'(t)\right|^2dt+\frac{1}{2}g^2(a)\right]^{\frac{1}{2}}\left[\int_0^a\left|g(t)\right|^{2q-2}t^{q-1}dt\right]^{\frac{1}{2}}$$

$$=\sqrt{\frac{2q^3}{2q-1}}a^{-\frac{q}{2}}\left[\int_0^a t\left|g'(t)\right|^2dt+\frac{1}{2}g^2(a)\right]^{\frac{1}{2}}\left[\int_0^a\left|g(t)\right|^{2q-2}t^{q-1}dt\right]^{\frac{1}{2}}$$

Then

$$\left|g(x)\right|^q\le\left|g(a)\right|^q+\frac{q}{x^{\frac{q}{2}}}\left[\int_0^a t\left|g'(t)\right|^2dt+\frac{1}{2}g^2(a)\right]^{\frac{1}{2}}\left[\int_0^a t^{q-1}\left|g(t)\right|^{2q-2}dt\right]^{\frac{1}{2}}$$

$$\le\sqrt{\frac{2q^3}{2q-1}}a^{-\frac{q}{2}}\left[\int_0^a t\left|g'(t)\right|^2dt+\frac{1}{2}g^2(a)\right]^{\frac{1}{2}}\left[\int_0^a\left|g(t)\right|^{2q-2}t^{q-1}dt\right]^{\frac{1}{2}}$$

$$+\frac{q}{x^{\frac{q}{2}}}\left[\int_0^a t\left|g'(t)\right|^2dt+\frac{1}{2}g^2(a)\right]^{\frac{1}{2}}\left[\int_0^a t^{q-1}\left|g(t)\right|^{2q-2}dt\right]^{\frac{1}{2}}$$

$$=\left(\sqrt{\frac{2q^3}{2q-1}}a^{-\frac{q}{2}}+\frac{q}{x^{\frac{q}{2}}}\right)\left[\int_0^a t\left|g'(t)\right|^2dt+\frac{1}{2}g^2(a)\right]^{\frac{1}{2}}\left[\int_0^a t^{q-1}\left|g(t)\right|^{2q-2}dt\right]^{\frac{1}{2}}$$

Then

$$|f(x)|^q \le \left(\sqrt{\frac{2q^3}{2q-1}}\left(\frac{x}{a}\right)^{\frac{q}{2}}+q\right)\left\{\int_0^a\left[|f'(t)|^2-\frac{f^2(t)}{4t^2}\right]dt\right\}^{\frac{1}{2}}\left[\int_0^a|f(t)|^{2q-2}dt\right]^{\frac{1}{2}}$$

$$\le\left(\sqrt{\frac{2q^3}{2q-1}}+q\right)\left\{\int_0^a\left[|f'(t)|^2-\frac{f^2(t)}{4t^2}\right]dt\right\}^{\frac{1}{2}}\left[\int_0^a|f(t)|^{2q-2}dt\right]^{\frac{1}{2}},$$

We have completed the certificate

$$|f(x)|^{2q} \le \left(\sqrt{\frac{2q^3}{2q-1}}+q\right)^2\int_0^a\left[|f'(t)|^2-\frac{f^2(t)}{4t^2}\right]dt\int_0^a|f(t)|^{2q-2}dt$$

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