Journal of Multidisciplinary Engineering Science and Technology (JMEST)
ISSN: 2458-9403
Vol. 6 Issue 12, December - 2019, Special Issue

# Approximation of Improper Integral Based on Haar Wavelets

**Mohammad Hasan Abdul Sathar**
Centre of Foundation Studies for Agricultural Science, University Putra Malaysia,
43400 Serdang, Selangor, Malaysia
mohdhasan@upm.edu.my

**Ahmad Fadly Nurullah Rasedee**
Faculty of Economics and Muamalat, Universiti Sains Islam Malaysia,
78100 Nilai, Negeri Sembilan, Malaysia
fadlynurullah@usim.edu.my

**Anvarjon A. Ahmedov**
Faculty of Industrial Science & Techonology, Universiti Malaysia Pahang, 26300, Gambang, Pahang, Malaysia
anvarjon@ump.edu.my

**Nor Fadzillah Mokhtar**
Centre of Foundation Studies for Agricultural Science, University Putra Malaysia,
43400 Serdang, Selangor, Malaysia
nor_fadzillah@upm.edu.my

***Abstract*— Previous numerical methods for solving definite integrals based on Haar wavelets was introduced. In this research we extend the previous works by approximate the solution of improper integral based on Haar wavelets functions. Error analysis of the approximation by Haar wavelets are given. Numerical examples are conducted to show the accuracy of the method.**

***Keywords—numerical integration; Haar wavelets; improper integral***

## I. Introduction

Many engineering application deal to evaluate improper integral which the integral is unbounded domain (limits of integration is infinite) or the function has an infinite discontinuity. One of the most important application is the probability density function that are studies in Probability Theory. There are two types of improper integrals, first case

$$\int_a^{\infty} f(t)\,dt = \lim_{x\to\infty}\int_a^x f(t)\,dt$$

$$\int_{-\infty}^{a} f(t)\,dt = \lim_{x\to-\infty}\int_x^a f(t)\,dt$$

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{a} f(t)\,dt + \int_a^{\infty} f(t)\,dt,$$ if both integral convergent.

The second case is the integrand has singularity or infinite discontinuity at some points in the range of integrations.

$$\int_a^b f(t)\,dt = \lim_{x\to b}\int_a^x f(t)\,dt,$$

if discontinuous at b,

$$\int_a^b f(t)\,dt = \lim_{x\to a}\int_x^b f(t)\,dt,$$

if discontinuous at a,

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt,$$

c is the discontinuous in $(a,b)$ and if both integral convergent.

In both of this case the normal procedure which is finding the antiderivative and find its limits. But if we couldn't find the antiderivative then, numerical methods are used to find the approximate solutions. Approximate an improper integral by numerical methods are even more challenging. First, we can't even approximate an improper integral with infinite domain, because we don't know at what point or interval to calculate the integral. Therefore, we must change to finite domain before applying numerical methods.

Many researches have been done in terms of quadrature methods see [6-8]. Recently a new approach in finding definite integral and improper integral of the second case has been develop by using wavelets functions [1-5]. In this result we will consider the improper integral of the first and second case by using Haar wavelets in approximate the integrals.

## II. Haar Wavelets

The formula for the Haar wavelets (HW) can be represented as follows:

$$H\left(2^j\frac{x-a}{b-a}-k\right)=\begin{cases}1 & \text{for } x\in\left[a+k\frac{(b-a)}{2^j},\, a+\left(k+\frac{1}{2}\right)\frac{(b-a)}{2^j}\right),\\ -1 & \text{for } x\in\left[a+\left(k+\frac{1}{2}\right)\frac{(b-a)}{2^j},\, a+(k+1)\frac{(b-a)}{2^j}\right),\\ 0 & \text{elsewhere,}\end{cases}$$

where $h_n(t) = 2^{\frac{j}{2}} H\left(2^j \frac{t-a}{b-a} - k\right)$, $n = 2^j + k$, $j = 0,1,2,\ldots, k = 0,1,\ldots,2^j - 1$. All functions $\{h_n(t)\}, n = 1,2,3,\ldots$ are denote on subintervals of $[a,b)$ and are produce from the function $H\left(2^j \frac{t-a}{b-a} - k\right)$ by operations of dilation and translation. The integer $j$ indicates the level dilation of the wavelet and $k$ is the translation parameter.

The HW functions form the orthonormal functions to each on $[a,b)$

$$\int_0^1 h_n(t) h_\ell(t) dx = \begin{cases} 1, & n = \ell = 2^j + k, \\ 0, & n \neq \ell. \end{cases}$$

and the scaling function is defined as

$$h_0(t) = \begin{cases} 1, & \text{for } \quad t \in [0,1), \\ 0, & \text{elsewhere.} \end{cases}$$

Any function $f(t) \in L^2(\mathbb{R})$ in the interval $[a,b)$ may be expanded as:

$$f(t) = \sum_{n=0}^{\infty} c_n h_n(t),$$

but in practice, only the first $2M - 1$ term of the sum is considered

$$f(x) \approx f_M(t) = \sum_{k=0}^{2M-1} c_n h_n(t) = C^T \Psi(t),$$

where $C$ and $\Psi(t)$ are matrices given by $C = [c_0, c_1, \ldots, c_{2M-1}]^T$ and $\Psi(t) = [h_0, h_1 \ldots, h_{2M-1}]^T$. $M = 2^j$ is the of maximum level of dilation for HW functions.

## III. Numerical Integration by HW

Consider the integral

$$\int_a^b f(t) dt.$$

if the function $f(t) \in L^2(\mathbb{R})$ then the integral can be approximate as

$$\int_a^b f(t)\, dt \approx \int_a^b C^T \Psi(t)\, dt.$$

The integral only involved one coefficient due to the properties of HW function. Therefore, the integral can be evaluated by only finding one coefficient $c_0$. To calculate the HW coefficient $c_0$, consider collocation points

$$f(t_i) \approx \sum_{n=0}^{2M-1} c_n h_n(t_i), \qquad (1)$$

where

$$t_i = a + (b-a)\frac{i+0.5}{2M}, \quad i = 0,1,\ldots,2M-1.$$

The solution of the system (1) for $c_0$ has a form as follows:

$$c_0 = \frac{1}{2M} \sum_{i=0}^{2M-1} f(t_i),$$

Proof see [4]

In this paper the same formula and notation are used in [4] to derive the numerical integration for single integrals with constant limit as

$$\int_a^b f(t) dt \approx \sum_{n=0}^{2M-1} c_n \int_a^b h_n(t)\, dt = \frac{(b-a)}{2M} \sum_{i=0}^{2M-1} f\left(a + (b-a)\frac{i+0.5}{2M}\right). \qquad (2)$$

## IV. Error Analysis in Holder Classes $H^s[0,1]$

Set of all continuous functions on [0,1], that satisfies the inequality as bellow:

$$|f(t) - f(y)| \leq c|t-y|^s, \forall t,y \in [0,1]$$

are defined as Holder classes $H^s[0,1]$ of order $0 < s < 1$.

The Holder classes are nested between $C[0,1]$ and $C^1[0,1]$ such that:

$$C^1[0,1] \subset H^s[0,1] \subset C[0,1], 0 < s < 1.$$

**Theorem 1:** Let $f(t) \in H^s[0,1], 0 < s < 1$, then

$$\| f - f_M \|_{L_2[0,1]} \leq \frac{L^2}{4(4^s - 1)M^{2s}},$$

**Prof:** see [4]

The error analysis of the approximation by HW is worked out, to show the convergence of the method, therefore the error bound will approach $0$ if the value of $M$ is increase.

## V. Numerical Example

In this section we consider three problems of an improper integrals and we wish to show the accuracy of HW in terms of absolute error. The numerical results for Haar wavelets with the different level of dilation
($j$ is the dilation for Haar wavelets) to validate the error estimation.

The improper integrals problem with the exact solution are given in Table 1.

TABLE I.

| Example | Problem | Exact solution (ES) |
|---|---|---|
| 1. | $\int_0^{\infty} e^{-t^2} dt$ | $\frac{1}{2}\pi$ |
| 2. | $\int_2^{10} \frac{1}{\sqrt{t-2}} dt$ | $4\sqrt{2}$ |
| 3. | $\int_0^{\infty} \frac{\sin(2t)}{t} dt$ | $\frac{\pi}{2}$ |

## VI. RESULTS AND DISCUSSION

In Tables 1-3 approximation HW in terms of absolute error is obtained. For example, 1 we must consider an upper limit to approximate the problem by HW. Let U be the upper limit and $\varepsilon$ is the error for the numerical solution.

$$ES - \int_0^U e^{-t^2} dt = \varepsilon,$$
$$\frac{1}{2}\pi - \frac{b-a}{2M} \sum_{i=0}^{2M-1} f\left(a + (b-a)\frac{i+0.5}{2M}\right) \leq \varepsilon,$$
$$\frac{1}{2}\pi - \varepsilon \leq \frac{U}{2M} \sum_{i=0}^{2M-1} f\left(U \frac{i+0.5}{2M}\right) \quad (3)$$

In equation (3) we could evaluate the upper limit for the integration by fixed the value of $\varepsilon$ and $M = 2^j$. Here we could also use the divide and conquer method in guessing the upper limit of the integral. For the Problem 1 let the upper limit as U. If the error is small, then we have increased the upper limit $U_1 = U + U$. Now check whether error estimation is smaller than the previous error. If the error is smaller, we need to increase U. Vice versa, the error is not decreasing or similar as the previous then we need to reduce the upper limit as $U_2 = \frac{U}{2}$. In example 1 the sufficient upper limit for example is 4.5 if we fixed the left side of equation (3) as 1.00000E-10. The absolute error for the example 1 is given in table 2.

In example 2 the improper integral will also cause a problem in numerical methods; this is due to which the function and first derivative of the function is undefined at the endpoint. The functions is discontinuous and not a smooth function even so the antiderivative of function can be evaluate easily. Therefore, need to increase the number of dilations $j$ (see table 3) in order to perform a better approximation. If the function is not defined at the endpoints of the domain of integration, hence methods such Trapezoidal and Simpson's methods are failed to evaluate the improper integrals.

In the Example 3 the function is an oscillatory and the upper limit of the integral are performed using the same technique as in example 1. We will consider the upper limit of integral as 150, because the error estimation are almost zero.

Table 2

| Upper limits Integral | Haar | Absolute Errors |
|---|---|---|
| 3 | $j=4$ | 1.93080E-05 |
| 3 | $j=6$ | 1.95605E-05 |
| 3 | $j=8$ | 1.95765E-05 |
| 4.5 | $j=4$ | 1.00000E-10 |
| 4.5 | $j=6$ | 1.00000E-10 |

**Absolute errors of Example 1**

Table 3

| Haar | Absolute Errors |
|---|---|
| $j=8$ | 7.56121E-02 |
| $j=14$ | 9.45151E-03 |
| $j=22$ | 5.83413E-04 |

**Absolute errors of Example 2**

Table 4

| Upper limits Integral | Haar | Absolute Errors |
|---|---|---|
| 145 | $j=5$ | 5.66454E-03 |
| 145 | $j=7$ | 2.05633E-03 |
| 145 | $j=9$ | 1.95566E-03 |
| 145 | $j=11$ | 1.94959E-03 |
| 150 | $j=5$ | 1.58019E-04 |
| 150 | $j=7$ | 8.8434 E-05 |
| 150 | $j=9$ | 8.4983E-05 |
| 150 | $j=11$ | 8.4771E-05 |

**Absolute errors of Example 3**

## VII. CONCLUSSION

In these results, the HW is presented to solve the improper integrals. The error analysis for the approximate solution of the functions belong in the Holder classes based on HW are given. Its cleat that, the accuracy of HW in approximating the improper integrals are better if increase the number of dilations $j$. Numerical results were obtained by Maple 2015 software.

## Acknowledgment

This paper has been supported by University Putra Malaysia (UPM) under its short term grant scheme (GP-IPM/9589600)

## References

[1] Aziz, I., & Haq, F. (2010). A comparative study of numerical integration based on Haar wavelets and hybrid functions. *Computers & Mathematics with Applications*, *59*(6), 2026-2036

[2] Ahmedov, A. A., & bin Abd Sathar, M. H. (2013). Numerical integration of the integrals based on Haar wavelets. In *Journal of Physics: Conference Series* (Vol. 435, No. 1, p. 012042). IOP Publishing.

[3] Alpert, B. K. (1999). Hybrid Gauss-trapezoidal quadrature rules. *SIAM Journal on Scientific Computing*, *20*(5), 1551-1584.

[4] Ahmedov, A. A., Sathar, M. H. A., Rasedee, A. F. N., & Mokhtar, N. F. B. (2017, September). Approximating of functions from Holder classes Hα [0,1] by Haar wavelets. In *Journal of Physics: Conference Series* (Vol. 890, No. 1, p. 012073). IOP Publishing

[5] Abd Sathar M. H. (2018). Numerical Integration Based on Linear Legendre Multi Wavelets *3rd International Conference on Mathematical Sciences and Statistics*

[6] Place, J., & Stach, J. (1999). Efficient numerical integration using Gaussian quadrature. *Simulation*, *73*(4), 232-237

[7] Rathod, H. T., Nagaraja, K. V., & Venkatesudu, B. (2007). Symmetric Gauss Legendre quadrature formulas for composite numerical integration over a triangular surface. *Applied Mathematics and Computation*, *188*(1), 865-876

[8] Rokhlin, V. (1990). End-point corrected trapezoidal quadrature rules for singular functions. *Computers & Mathematics with Applications*, *20*(7), 51-