On the Exponential Diophantine Equations of Degree Three

Nur Hidayah Amalul Hair
Institute for Mathematical Research
University Putra Malaysia
Serdang, Selangor
Nurhidayah_amalulhair@yahoo.com

Siti Hasana Sapar
Institute for Mathematical Research
University Putra Malaysia
Serdang, Selangor
sitihas@upm.edu.my

Abstract—Diophantine equation is known as a polynomial equation with two or more unknowns which only integral solutions are sought. This paper concentrates on finding an integral solution to the exponential Diophantine equation of degree three. By looking at the pattern of the solution to the equation under consideration, we will construct theorems and lemmas. From this study, there exist two solutions to the exponential Diophantine equation of degree three that we consider.

Keywords— Diophantine equation, exponential Diophantine equation, parity.

I. INTRODUCTION

The history of Diophantine equation $x^2 + C = y^n$ that has been studied by since 1850's where x, y are positive integer and proved that there is no integer solution when $C = 1$ as mention in [4]. For $C = 2$ has been solved by [5] as given in [1]. Then, for value of $C = 3,4,5$ solved by [7] as in stated in [6] then followed by [1] for seventy seven values of *C*, between 1 - 100.

In [2], the authors consider the Diophantine equation of the form $x^2 + q^m = c^{2n}$ by consider for $c > 1$ and t are positive integers from equation $q^t + 1 = c^n$ and *q* is an odd prime. The solution to this equation is $(x, m, n) = (c^2 - 1;\ t, 2,)$ by considering some cases where $m = t$. Meanwhile, [3] find all solutions to the Diophantine equation $x^2 + 2^a \cdot 3^b \cdot 11^c = y^n$, for the values of $n \geq 3$. Then, [6] studied the Diophantine equation in the form of $x^2 + 5^a.p^b = y^n$ for certain value of $p = 29, 41$ and the solution to this equation are $(x;\ y;\ a;\ b) = (2;\ 9;\ 2;\ 1)$ and $(2;\ 3;\ 2;\ 1)$ for *p*=29 and $n = 3$ and 6 respectively.

In this paper, we focused on finding an integral solution to the exponential Diophantine equations of degree three in the form of $x^2 + 2^a.7^b = y^n$ for *n*=3.

II. MAIN RESULT

In this section, we will discuss on finding an integral solutions to the Diophantine equation $x^2 + 2^a \cdot 7^b = y^n$ for *n*=3. In order to solve this equation, we will consider two cases depend on the parity of x and y in equation. By looking at the pattern of the solution and considering some cases, we obtain the following result:

Firstly, we consider for the parity of *x* and *y* both are even integers.

Theorem 1 Let be a, b, x, y, n be positive integers, then an integral solution to Diophantine equation $x^2 + 2^a.7^b = y^n$ for $(a, b, n) = (2,7,3)$ are $(x, y) = (2058, 196), (7546, 392)$.

Proof:

Consider the equation

$$x^2 + 2^a 7^b = y^n\ . \qquad (1)$$

From the hypothesis, (1) become

$$x^2 + 2^2 \cdot 7^7 = y^3. \qquad (2)$$

In order to solve this equation, we will consider a few cases depend on the possibility of the parity of *x* and *y*.

Now, we consider the first case where both *x, y* are even.
Suppose $x = 2^\alpha s$ and $y = 2^\beta r$, where $(2,s) = (2,r) = 1$, $\alpha, \beta \geq 1$ and $r, s \in \mathbb{N}$. By substituting these values into (2), we obtain,

$$2^{3\beta} r^3 - 2^{2\alpha} s^2 = 2^2 \cdot 7^7. \qquad (3)$$

From (3), we consider six possibilities for the case α and β as in the table below:

Table 1: Possible cases for α and β when $\alpha, \beta > 0$

(1)	$\beta > \alpha$	$\beta = 2$
(2)	$\beta > \alpha$	$\beta = 3$
(3)	$\beta > \alpha\ , \alpha = 1$	$\beta > 3$
(4)	$\beta > \alpha\ , \alpha > 1$	$\beta > 3$
(5)	$\alpha > \beta$	$\beta = 1$
(6)	$\alpha > \beta$	$\beta > 1$

Case (1) : Consider $\beta>\alpha$. Suppose $\beta=2$. From equation (3), it becomes,

$$2^6r^3 - 2^{2\alpha}s^2 = 2^2\cdot 7^7 . \quad (4)$$

Since $\beta > \alpha$, the smallest value of α is 1. By substituting these values into (4), we obtain

$$2^4r^3 - s^2 = 7^7 . \quad (5)$$

Since (2, s) = (2, r) = 1 and right hand side (RHS) has factor of 7, then left hand side (LHS) also has factor of 7.

Suppose, $s = 7^{\sigma_1}w_1$ and $s = 7^{\sigma_2}w_2$ with $(7, w_1) = (7, w_2) = 1$. Substitute these values in equation (5), we have

$$2^4 7^{3\sigma_2}{w_2}^3 - 7^{2\sigma_1}{w_1}^2 = 7^7 \quad (6)$$

By considering all possibilities of factor of 7 for both sides, we have the following:

i. $2^4 7^{3\sigma_2}{w_2}^3 - 7^{2\sigma_1}{w_1}^2 = 7\cdot 7^6$ (7)
ii. $2^4 7^{3\sigma_2}{w_2}^3 - 7^{2\sigma_1}{w_1}^2 = 7^2\cdot 7^5$ (8)
iii. $2^4 7^{3\sigma_2}{w_2}^3 - 7^{2\sigma_1}{w_1}^2 = 7^3\cdot 7^4$ (9)

Now, we consider the following three cases:

Case (1(i)):. From (7) it can be written as,

$$7^{2\sigma_1}(2^4 7^{3\sigma_2-2\sigma_1}{w_2}^3 - {w_1}^2) = 7^6\cdot 7.$$

By comparing both sides, the possible combination is $2\sigma_1 = 6$, imply that $\sigma_1 = 3$. Then, we have

$$2^4 7^{3\sigma_2-6}{w_2}^3 - {w_1}^2 = 7. \quad (10)$$

Equation (10) holds if $7^{3\sigma_2-6} = 7^0$. Thus, we have $\sigma_2 = 2$. Therefore, (10) become,

$$16{w_2}^3 - {w_1}^2 = 7.$$

It can be written as

$${w_1}^2 \equiv -7 \ (\mathrm{mod}16) \quad (11)$$

That is,

$${w_1}^2 \equiv 9 \ (\mathrm{mod}16)$$

By solving the equation above, we have

$$w_1 = 3 \text{ or } w_1 = -3,$$

in the least residue modulo 16.

Suppose $w_1 = 3$ and substitute in (10) we obtain

$$w_2 = 1.$$

By back substitution for all values of α, β, σ_1 and σ_2, we have

$$x = 2058 \text{ and } y = 196.$$

Suppose $w_1 = -3$. By using the same argument, we will obtain the same answer for *x* and *y*.

Case (1(ii)): From (6), we consider possibility of

$$2^4 7^{3\sigma_2}{w_2}^3 - 7^{2\sigma_1}{w_1}^2 = 7^2\cdot 7^5.$$

Firstly, consider the case of $\sigma_1 = \sigma_2 = 1$. The equation become,

$$2^4 7^3{w_2}^3 - 7^2{w_1}^2 = 7^2\cdot 7^5.$$

That is,

$$2^4 7{w_2}^3 - {w_1}^2 = 7^5. \quad (12)$$

Since RHS is odd then w_1 must be odd. We let both odd, that is $w_1 = 2g + 1$ and $w_2 = 2h + 1$. Then substitute these values into (12), we have

$$2^4 7(2g + 1)^3 - (2h + 1)^2 = 7^5.$$

By expanding and simplifying the equation above, we obtain

$$896g\left[\left(g-\tfrac{3}{4}\right)^2 - \tfrac{3}{16}\right] = 4\left[\left(h+\tfrac{1}{2}\right)^2 - \tfrac{4175}{4}\right].$$

Contradiction occurs since *h* and *g* are integers. We will obtain the same result for the second case $\sigma_2 > \sigma_1$.

Now, we consider the third case. That is

$$2^4 7^{3\sigma_2}{w_2}^3 - 7^{2\sigma_1}{w_1}^2 = 7^3\cdot 7^4 .$$

By using the same method and argument as in case (ii), contradiction occurs for all cases of σ_1 and σ_2.

Case (2). Now from Table 1, we consider for $\beta > \alpha$ and suppose $\beta = 3$. Then, substitute in (3), we have

$$2^{2\alpha}(2^{9-2\alpha}r^3 - s^2) = 2^2\cdot 7^7. \quad (13)$$

By comparing both sides and since LHS=RHS, we obtain α =1 and

$$2^7r^3 - s^2 = 7^7. \quad (14)$$

Since $(2, s) = (2, r) = 1$ and RHS have factor of 7, then LHS also has factor of 7. Therefore, equation (14) have a solution in the form of $s = 7^{\sigma_1}w_1$ and $r = 7^{\sigma_2}w_2$ with $(7, w_1) = (7, w_2) = 1$. Substitute these value into (14), we have

$$2^7.7^{3\sigma_2}{w_2}^3 - 7^{2\sigma_1}{w_1}^2 = 7^7 \quad (15)$$

By considering all possibilities of factor of 7 for both side, we have the following cases:

i. $2^7 7^{3\sigma_2}{w_2}^3 - 7^{2\sigma_1}{w_1}^2 = 7.7^6$ (16)
ii. $2^7 7^{3\sigma_2}{w_2}^3 - 7^{2\sigma_1}{w_1}^2 = 7^2 7^5$ (17)
iii. $2^7 7^{3\sigma_2}{w_2}^3 - 7^{2\sigma_1}{w_1}^2 = 7^3 7^4$ (18)

Case (2(i)): From equation (16), firstly we consider for the case $\sigma_1 > \sigma_2$. It can be written as,

$$7^{2\sigma_1}(2^7 7^{3\sigma_2-2\sigma_1}{w_2}^3 - {w_1}^2) = 7.7^6. \quad (19)$$

In order to solve (19), we will consider three different possibilities of σ_1 and σ_2 as follows:

Firstly, we consider if $\sigma_1 > \sigma_2$ .By comparing both sides and considering all possible combination of an expressions, equation (19) holds if $2\sigma_1 = 6$, that is $\sigma_1 = 3$. Thus, we have equation,

$$2^7 7^{3\sigma_2-6}{w_2}^3 - {w_1}^2 = 7. \quad (20)$$

Since $\sigma_1 > \sigma_2$, then equation (20) holds if $7^{3\sigma_2-6} = 7^0$. Then, we have $\sigma_2 = 2$. That is, equation (20) become

$$128{w_2}^3 - {w_1}^2 = 7 \quad (21)$$

It can be written as,

$${w_1}^2 = -7(\mathrm{mod}\ 128).$$

That is, ${w_1}^2 = 121(\mathrm{mod}\ 128)$.

By solving the equation above we obtain $w_1 = 11$ in the least residue modulo 128.
Suppose $w_1 = 11$ and substitute in (21) we obtain

$$w_2 = 1.$$

By back substitution for all values of $\alpha, \beta, \sigma_1, \sigma_2$, we obtain

$$x = 7546 \quad y = 392.$$

Secondly, for the case $\sigma_1 = \sigma_2 = 1$. From equation (20), we have,

$$7^2(2^7.7{w_2}^3 - {w_1}^2) = 7^6.7$$

Since LHS=RHS, the equation above become,

$$7(2^7{w_2}^3 - 7^4) = {w_1}^2 \qquad (22)$$

It is contradiction since $(7, w_1)$ =1. That is, LHS have a factor of 7 and RHS does not have factor of 7.

Now, we consider the third case. That is, we consider for the case $\sigma_1 < \sigma_2$. From (20), we have

$$7^{2\sigma_1}\,(2^7 7^{3\sigma_2 - 2\sigma_1}{w_2}^3 - {w_1}^2) = 7.7^6 \qquad (23)$$

By comparing both side, one of the possible combination is $2\sigma_1 = 6$, that is $\sigma_1 = 3$. Then, the smallest integer for $\sigma_2 > 3$, is $\sigma_2 = 4$. Thus, equation (23) become

$$2^7.7^6{w_2}^3 - {w_1}^2 = 7.$$

That is, $7(\,2^7.7^5{w_2}^3 - 1) = {w_1}^2$.
It is contradiction since $(7, w_1)$ =1. That is, LHS have a factor of 7 and RHS does not have factor of 7.

Case (2(ii)): From equation (17), we will consider the different possibilities of σ_1 and σ_2 for

$$2^7 7^{3\sigma_2}{w_2}^3 - 7^{2\sigma_1}{w_1}^2 = 7^2 7^5.$$

Firstly, suppose $\sigma_1 = \sigma_2 = 1$. The above equation become,

$$2^7.7^3{w_2}^3 - 7^2{w_1}^2 = 7^2.7^5.$$

It can be written as,

$$7(2^7{w_2}^3 - 7^4) = {w_1}^2.$$

The contradiction occurs since $(7, w_1)$ =1. That is, LHS have a factor of 7 and RHS does not have factor of 7. We will obtain the same result for the case when $\sigma_2 > \sigma_1$ and $\sigma_1 > \sigma_2$ by using the same argument.

Case (2(iii)): From (18), we consider for the different possibilities of σ_1 and σ_2.

Now, consider the first case. Suppose $\sigma_1 = \sigma_2 = 1$, equation (18) become,

$$2^7.7^{3\sigma_2}{w_2}^3 - 7^{2\sigma_1}{w_1}^2 = 7^4.7^3.$$

It can be written as,

$$7(2^7{w_2}^3 - 7^4) = {w_1}^2$$

By using the same argument as in the case 2(iii), contradiction occurs since $(7, w_1)$ =1. That is, LHS have a factor of 7 and RHS does not have factor of 7. We will obtain the same result for the case when $\sigma_2 > \sigma_1$ and $\sigma_1 > \sigma_2$.

Now, from Table 1 we will consider Case 3.
Case 3: Consider $\beta > \alpha$, $\alpha = 1$ for $\beta > 3$. Then, equation (3) becomes,

$$2^{3\beta}r^3 - 2^2s^2 = 2^2.7^7$$

It can be written as

$$2^{3\beta}r^3 - s^2 = 7^7$$

By using the same argument as in Case (1) and equation (5), the contradiction occurs.

Case 4: Consider $\beta > \alpha$, $\alpha > 1$ for $\beta > 3$. From (3), the equation becomes,

$$2^{2\alpha-1}(2^{3\beta-2\alpha}r^3 - s^2) = 7^7.$$

The equation above is contradicting since LHS is even while RHS is odd since $\alpha > 1$.

Case 5: Consider for the case $\alpha > \beta$ for $\beta = 1$ and substitute in (3), we will have

$$2^{2\beta}(2^{\beta}r^3 - 2^{2\alpha-2\beta}s^2) = 2^2.7^7.$$

By comparing both sides and since LHS=RHS, the above equation holds if $\beta = 1$ and it can be written as,

$$2r^3 - 2^{2\alpha-2}s^2 = 7^7.$$

Then, contradiction occurs since LHS is even while RHS is odd.

Lastly, we consider Case 6 from Table 1.
Case 6: Consider $\alpha > \beta$ for $\beta > 1$ and equation (3), becomes

$$2^{3\beta-2}(r^3 - 2^{2\alpha-3\beta}s^2) = 7^7.$$

The equation above is contradicting since LHS is even while RHS is odd since $\beta > 1$.

Secondly, we consider for the parity of x and y are both odd integers. We have the following result

Theorem 2 Let be a, b, x, y, n be positive integer, then there is no integral solution to Diophantine equation $x^2 + 2^a.7^b = y^n$ where $(a, b, n) = (2,7,3)$ and both x and y are odd.
Proof:
Suppose $x = 2^{\gamma}k + 1$ and $y = 2^{\delta}j + 1$, with $(2,k)$=1 and $(2,j)$ = 1 where $\gamma \geq 1$, $\delta \geq 1$ and $k, j \in \mathbb{N}$.
From (2), we have

$$(2^{\delta}j + 1)^3 - (2^{\gamma}k + 1)^2 = 2^2.7^7. \qquad (24)$$

In order to solve (24), we will consider the possibilities of γ and δ. That is, either $\gamma = \delta$, $\gamma > \delta$ or $\gamma < \delta$.

Now, we consider the first case where $\gamma = \delta = 1$. Then substitute these values in (24), we have

$$(2j + 1)^3 - (2k + 1)^2 = 2^2.7^7$$

By expanding and simplifying the equation above, we obtain

$$2^2j^3 + 3 \cdot 2j^2 + 3j - 2k^2 - 2k = 2 \cdot 7^7$$

Since (2, *j*) =1, then we obtain LHS is odd and RHS is even. Thus, contradiction occurs. That is, LHS ≠ RHS. By using the same method and arguments, contradiction also occurs for the case $\gamma > \delta$ and $\gamma < \delta$.

III. CONCLUSION

From this study, we found that the integral solution for positive integers *x* and *y* to the Diophantine equation $x^2 + 2a \cdot 7^b = y^n$ are $(x, y) = (2058, 196)$, $(7546, 392)$ where $(a, b, n) = (2,7,3)$.

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