# Boundary condition effect on the dynamics of micro-beams using Newton Raphon Method

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***Abstract*— In this paper, we have studied free vibration analysis of micro-Euler-Bernoulli beam carrying an attached mass. Based on Hamilton's principle dimensionless frequency is obtained. Micro-beam is considered as a cantilever carrying an attached mass. We have developed a similar study carried out by Ghanbari and Babaei by using a numerical method names as Newton-Raphson method. and compare our results with their results. Comparison demonstrates a pretty high level of accuracy. In other words, increment in the value of the attached mass leads to decrement in the value of dimensionless natural frequency. Increasing the inertia effects of the whole system including the micro-cantilever-beam and the attached mass roles as a structural damper and suppresses the vibration frequencies and amplitudes.**

***Keywords—Micro-electro-mechanical system, frequency, Newton Raphson***

## I. INTRODUCTION (*Heading 1*)

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## II. MODEL

Consider the following micro-beam with length $L$, width $w$ and uniform thickness $h$ which is shown in Figure 1. According to the modified couple stress theory (Ghanbari and Babaei (2015)) the stored strain energy $U_s$ of an elastic linear isotropic Euler-Bernoulli micro-beam is expressed in terms of the classic strain tensor and the symmetric curvature tensor which is related to the micro-scale rotation of material particles.

**Figure 1. Schematic of the micro-beam with attached mass**

$$U_s = \frac{1}{2}\int_V (\sigma_{ij}:\varepsilon_{ij} + m_{ij}:\chi_{ij})\, dV \qquad (1)$$

where $V$ is volume, $\sigma_{ij}$ is the Cauchy stress tensor, $\varepsilon_{ij}$ is the strain tensor, $m_{ij}$ stands for the deviatoric part of the couple stress tensor, and $\chi_{ij}$ is the symmetric curvature tensor. The tensors $\varepsilon_{ij}$ and $\chi_{ij}$ are defined by following relations:

$$\varepsilon_{ij} = \frac{1}{2}(u_{ij} + u_{ji}) \qquad (2)$$

$$\chi_{ij} = \frac{1}{2}(e_{ipq}\varepsilon_{qj,p} + e_{jpq}\varepsilon_{qi,p}) \qquad (3)$$

$u$ in Eq. (2) shows the displacement vector; $e_{ipq}$ in Eq. (3) is the alternating tensor; and comma refers to differentiation. Constitutive relations regarding the Cauchy stress tensor and the deviatoric part of the couple stress tensor are:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk} \quad (4)$$

$$m_{ij} = 2\mu l^2 \chi_{ij} \quad (5)$$

$\lambda$ and $\mu$ represent Lame's parameters. $\nu$ is the Poisson's ratio and $E$ is the modulus of elasticity. Besides following constraint equations are helpful in reducing the number of unknowns:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} \quad (6)$$

$l$ in Eq. (5) is the material length scale parameter of the MCST.

The displacement components in an Euler-Bernoulli beam can be represented by

(Ghanbari and Babaei (2015)):

$$u_1(x_1, x_3, t) = -x_3 \frac{\partial w}{\partial x_1} \quad (7)$$

$$u_2(x_1, x_3, t) = 0 \quad (8)$$

$$u_3(x_1, x_3, t) = w(x_1, t) \quad (9)$$

In the above equations $u_i$, $(i = 1,2,3)$ are the general displacement components in $x_1$, $x_2$, $x_3$ directions.

Using this displacement fields and Eqs. (2) - (5), elements of $\varepsilon_{ij}$, $\chi_{ij}$, $\sigma_{ij}$ and $m_{ij}$ are as follows:

$$\varepsilon_{11} = -x_3 \frac{\partial}{\partial x_1}\left(\frac{\partial w}{\partial x_1}\right) \quad (10)$$

$$\chi_{12} = \chi_{21} = -\frac{1}{2}\frac{\partial}{\partial x_1}\left(\frac{\partial w}{\partial x_1}\right) \quad (11)$$

$$\sigma_{11} = -Ex_3 \frac{\partial}{\partial x_1}\left(\frac{\partial w}{\partial x_1}\right) \quad (12)$$

$$m_{12} = m_{21} = -\mu l^2 \frac{\partial}{\partial x_1}\left(\frac{\partial w}{\partial x_1}\right) \quad (13)$$

Substitution of Eqs. (10)-(13) into Eq. (1) results:

$$U_s = \frac{1}{2}\int_V \{E\left(\frac{\partial}{\partial x_1}\left(\frac{\partial w}{\partial x_1}\right)\right)^2 ({x_3}^2 + \frac{l^2}{2})\}\, dV \quad (14)$$

Moreover, maximum kinetic energy of the Euler-Bernoulli beam carrying an attached mass can be expressed as:

$$T = \frac{1}{2}\int_0^L \rho A(\frac{\partial w}{\partial t})^2\, dx_1 + \frac{1}{2}M(\frac{\partial w(L,t)}{\partial t})^2 \quad (15)$$

Based on Hamilton's principle (Ghanbari and Babaei (2015)), the governing equation of the beam along with initial conditions and boundary conditions can be determined by using the following equation:

$$\delta\left[\int_{t_1}^{t_2} (T - U_s)\, dt\right] = 0 \quad (16)$$

Substituting the Eqs.(14),(15) into the Eq.(16), one can obtain the governing equation as following:

$$Q\frac{\partial^4 w}{\partial {x_1}^4} + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \quad (17)$$

It seems reasonable to expect that transverse displacement varies harmonically with respect to time variable. Also due to homogeneity of the partial differential equation obtained, it is acceptable to use the method of separation of variables like:

$$w(x_1, t) = W(x_1)T(t) \quad (18)$$

Substituting of the Eq.(18) into the Eq.(17) yields the following ordinary differential equation:

$$Q\frac{d^4 W}{d^4 x_1} - \rho A \omega^2 W = 0 \quad (19)$$

$$Q = EI_{x_2 x_2} + GAl^2 \quad (20)$$

Where $\omega$ denotes the natural frequency of vibration, $W$ is the displacement amplitude at $x_1 = 0$ and $I_{x_2x_2}$ is the second moment of mass inertia about the $x_2$ direction.

Using the following parameters helps in the analysis steps:

$$x = \frac{x_1}{L} \quad (21) \qquad , \qquad \beta^4 = \frac{(\rho A \omega^2 L^4)}{Q} \quad (22)$$

substituting above parameters into the Eq.(19), one can obtain

$$\frac{d^4 W}{d^4 x} - \beta^4 W = 0, \quad 0 \le x \le 1 \quad (23)$$

Besides, the boundary conditions of the clamped-free beam carrying an attached mass at the free end is:

$$W(0) = 0 \qquad (24)$$

$$\frac{dW}{dx}(0) = 0 \qquad (25)$$

$$\frac{d^2W}{dx^2}(1) = 0 \qquad (26)$$

$$\frac{d^3W}{dx^3}(1) + \left(\frac{M}{Q}\omega^2\right)W(1) = 0 \qquad (27)$$

In order to go on, general solution of Eq. (23) is compulsory. This general solution is proposed by Ghanbari and Babaei as the following linear combination of trigonometric and hyperbolic eqauations:

$$W(x) = C_1 \cosh\beta x + C_2 \sinh\beta x + C_3 \cos\beta x + C_4 \sin\beta x \qquad (28)$$

Applying the boundary conditions one can obtain four equations with corresponding values for each. After some mathematical operations, matrix of the coefficients is:

$$A(1,1) = 1, \quad A(1,2) = 0, \quad A(1,3) = 1, \quad A(1,4) = 0 \qquad (30)$$

$$A(2,1) = 0, \quad A(2,2) = 1, \quad A(2,3) = 0, \quad A(2,4) = 1 \qquad (31)$$

$$A(3,1) = \beta^2 \cosh\beta L, \quad A(3,2) = \beta^2 \sinh\beta L, \quad A(3,3) = -\cos\beta L \qquad (32)$$

$$A(3,4) = -\sin\beta L \qquad (33)$$

$$A(4,1) = \beta^3 \sinh\beta L + \left(\frac{M}{Q}\omega^2\right)\cosh\beta L,$$

$$A(4,2) = \beta^3 \cosh\beta L + \left(\frac{M}{Q}\omega^2\right)\sinh\beta L,$$

$$A(4,3) = \beta^3 \sin\beta L + \left(\frac{M}{Q}\omega^2\right)\cos\beta L,$$

$$A(4,4) = -\beta^3 \cos\beta L + \left(\frac{M}{Q}\omega^2\right)\sin\beta L \qquad (34)$$

And the coefficient vector matrix including $C_i, (i = 1,2,3,4)$.

Based on the fundamentals of linear algebra, determinant of the coefficient matrix is zero in the case of homogenous system of equations; this leads to a non-transcendental equation as the following one:

$$1 + \frac{1}{(\cosh\beta L)(\cos\beta L)} + R\beta(\tanh\beta L - \tan\beta L) = 0 \qquad (35)$$

In the above non-transcendental equation $R$ is the mass ratio which can be defined as:

$$R = \frac{M}{\rho AL} \qquad (36)$$

Eq. (35) is solved using the Newton-Raphson method. This is a numerical method which is mostly established on the initial guess of the root. Newton-Raphson method is among the iterative methods which tries to get the approximate estimate of the answer to a algebraic (transcendental or non-transcendental) equation by iterations and minimizing the error. Consequently, initial guess is highly vital in this method and a good guess leads to better approximations. Table 1 shows the current results compared to the results reported by Ghanbari and Babaei (2015). First point is the accuracy of the Newton-Raphson method. Other point is the approval of the result reported by Ghanbari and Babaei (2015) somehow by increasing the mass ratio (which is the ratio of the attached mass to the ration of the beam), vibration frequency decreases.

Newton-Raphson method is based on the initial guess. Base on this method, by having a good initial guess, and iteration methods, one can obtain the approximate root of an algebraic based on the consecutive process. Suppose that $f$ is a function of variable $x$ and $x_i$ is the initial guess, then first iteration leads to the first obtained root or the second approximate root $x_{i+1}$ as following:

$$x_{i+1} = x_i - \frac{f(x)|_{x=x_i}}{\frac{df(x)}{dx}\big|_{x=x_i}}$$

Table . comparison of natural dimensionless frequency with Newton-Raphson method

| $R$ | Ghanbari and Babaei (2015) | Newton-Raphson (Present) | Percent error (%) |
|---|---|---|---|
| 0.01 | 3.4477 | 3.5192 | 2.07 |
| 0.1 | 2.9678 | 2.3136 | 22 |
| 1 | 1.5573 | 1.5680 | 0.68 |
| 10 | 0.5414 | 0.5976 | 10 |

**Conclusion**

Model of the current study pertains to the system of micro-cantilever carrying an attached mass. Frequency decrement is of the concern which is mainly caused by the presence of the attached mass. The inertia derived from the attached mass rides the whole system to decrease in vibrations. A micro-cantilever beam carrying an attached mass is being investigated based on the Euler-Bernoulli beam theories and modified couple stress theory (MCST). Hamilton's principle is adopted to govern the equation of motion. Key points of the current study compromise using the numerical method to verify its applications in more challenging cases and applying the specific boundary conditions. This boundary condition implies the effects of the attached inertia on the frequency of the system. It is good to mention that the Newton-Raphson method shows pretty accurate results and as a result can be a good method in solving the other cases. Based on the mentioned results, as the mass ratio increases, vibration frequencies decrease and vice versa. So, the attached mass works like a structural damper.

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