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Local Integrability of Weak Solution to Obstacle Problems with Nonstandard Growth

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***Abstract*—The integrability properties of solution to the single obstacle problem with non standard growth is considered, and the result of local integrability is obtained by Hölder inequality, Young inequality and other methods.**

Keywords—obstacle problem; nonstandard growth; intergrability

I. INTRODUCTION

The obstacle problem is, roughly speaking, solving a partial differential equation with the additional constraint that the solution is required to stay above a given function, the obstacle. This leads to a variational inequality. From a minimization point of view, the problem is to find a minimizer with fixed boundary values in the set of functions lying above the obstacle function.

Let $\psi:\Omega\to[-\infty,\infty)$ be a function, called an obstacle; let $\theta\in W^{1,p(\cdot)}(\Omega)$ be a function which gives the boundary values. Define

$$\mathbf{K}_{\psi}^{\theta,p(\cdot)}(\Omega) = \{u\in W^{1,p(\cdot)}(\Omega): u-\theta\in W_0^{1,p(\cdot)}(\Omega),\ u\ge\psi,\ a.e.\ in\ \Omega\}$$

To avoid trivialities, we always assume that the set $\mathbf{K}_{\psi}^{\theta,p(\cdot)}(\Omega)$ is not empty. Note also the interpretation of the boundary values: u having the boundary values given by θ means that $u-\theta\in W_0^{1,p(\cdot)}(\Omega)$, i.e. $u-\theta$ has zero boundary values.

In this paper we deal with the single obstacle problems associated to quasi-linear elliptic equations

$$-div\mathbf{A}(x,\nabla u)=-div\left(F^{p(x)-2}F\right) \tag{1.1}$$

with non-standard structural conditions, where $F(x)\in L_{loc}^{p(\cdot)(1+\delta)}(\Omega)$ for a small $\delta>0$. These conditions involve a variable growth exponent $p(\cdot)$.

In this article, we always assume that $p(\cdot)$ is log-Hölder continuous with $1<p^-\le p^+<\infty$ and that Ω is a bounded open set in R^n .We need the following assumptions, with strictly positive constants α and β, to hold for the operator $\mathbf{A}:\Omega\times\mathbf{R}^n\to\mathbf{R}^n$

(H1) $x\mapsto\mathbf{A}(x,\xi)$ is measurable for all $\xi\in\mathbf{R}^n$,

(H2) $\xi\mapsto\mathbf{A}(x,\xi)$ is continuous for almost all $x\in\Omega$,

(H3) $\mathbf{A}(x,\xi)\cdot\xi\ge\alpha|\xi|^{p(x)}$ for almost all $x\in\Omega$ and for all $\mathbf{R}^n$,

(H4) $|\mathbf{A}(x,\xi)|\le\beta|\xi|^{p(x)-1}$ for almost all $x\in\Omega$ and for all $\mathbf{R}^n$.

We may assume that $\alpha\le\beta$ by choosing β larger if necessary. These are called the structure conditions of $\mathbf{A}$.

The above structural conditions imply that we can define solutions in the weak sense in the space $W^{1,p(\cdot)}(\Omega)$. More precisely, a function $u\in W^{1,p(\cdot)}(\Omega)$ is a weak solution to

$$-div\mathbf{A}(x,\nabla u)=-div\left(F^{p(x)-2}F\right)$$

if

$$\int_\Omega\langle\mathbf{A}(x,\nabla u),\nabla\varphi\rangle\,dx=\int_\Omega\left\langle|F(x)|^{p(x)-2}F(x),\nabla\varphi\right\rangle dx$$

for all test functions $\varphi\in C_0^\infty(\Omega)$.

Definition 1.1 We say that a function $u\in\mathbf{K}_{\psi}^{\theta,p(\cdot)}(\Omega)$ is a solution to the obstacle problem $\mathbf{K}_{\psi}^{\theta,p(\cdot)}(\Omega)$ if

$$\begin{aligned}&\int_\Omega\mathbf{A}\langle(x,\nabla u),\nabla(v-u)\rangle dx\\ &\ge\int_\Omega\left\langle|\mathrm{F}(x)|^{p(x)-2}\mathrm{F}(x),\nabla(v-u)\right\rangle dx\end{aligned} \tag{1.2}$$

for every $v\in\mathbf{K}_{\psi}^{\theta,p(\cdot)}(\Omega)$.

In recent years there has been a growing interest in nonlinear equations with nonstandard growth, which are related in a natural way to spaces $L^{p(\cdot)}$ with variable exponent. The first investigations of such problems were by the Italian school and had as their starting point the calculus of variations; see, e.g., [1, 2]. Recently, several authors have approached the same problem more in the spirit of nonlinear differential equations, e.g. in [3, 4, 5, 6, 7, 8].

The current paper aims at prove local higher integrability of the gradient of the solutions to the single obstacle problem with respect to continuous perturbations in the growth exponent $p(x)$.

Theorem 1.2 Let u be the solution to the $\mathbf{K}_{\psi}^{\theta,p(\cdot)}(\Omega)$ obstacle problem. If $2B\in\Omega$ is an open subset, where $\psi,\theta\in W^{1,p(\cdot)}(\Omega)$ and $|\nabla\psi|,|\nabla\theta|\in L^{p(\cdot)(1+\delta)}(\Omega)$ for a small $\delta>0$. Then exist a constant C depending only on n,p and α,β,M in Lemma 2.3, $|\nabla u|\in L^{p(\cdot)(1+\varepsilon)}(\Omega)$ for some $\varepsilon>0$, we have

$$\fint_{B}|\nabla u|^{p(x)(1+\varepsilon)}dx\le C\left[\left(\fint_{2B}|\nabla u|^{p(x)}dx\right)^{1+\varepsilon}+\fint_{2B}|\nabla\theta|^{p(x)(1+\varepsilon)}dx\right.$$
$$\left.+\fint_{2B}|\nabla\psi|^{p(x)(1+\varepsilon)}dx+1\right].$$

II. PRELIMINARY KNOWLEDGE AND LEMMAS

In this section, we introduce some notation, and lemmas.

We call a bounded measurable function $p:\mathbf{R}^n\to(1,\infty),n\ge 2$ a variable exponent. We denote

$$p_E^-=\inf_{x\in E}p(x)\text{ and }p_E^+=\sup_{x\in E}p(x),$$

where E is a measurable subset of $\mathbf{R}^n$. We assume that $1<p_\Omega^-\le p_\Omega^+<\infty$, where Ω is an open, bounded subset of $\mathbf{R}^n$. We abbreviate $p^-:=p_{\mathbf{R}^n}^-$ and $p^+:=p_{\mathbf{R}^n}^+$.

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ consists of all measurable functions f defined on Ω for which

$$\int_\Omega|\theta|^{p(x)}dx<\infty.$$

The Luxemburg norm on this space is defined as

$$\|\theta\|_{p(\cdot)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{\theta(x)}{\lambda}\right|^{p(x)}dx\le 1\right\}.$$

Equipped with this norm $L^{p(\cdot)}(\Omega)$ is a Banach space.

It is well known that regularity results for problems of $p(\cdot)$-growth require some assumption on the function $p(\cdot)$. We make the standard assumption on $p(\cdot)$, the so called logarithmic Hölder continuity condition. Indeed, the condition turns up quite naturally in the estimates of the De Giorgi and Moser methods, and there are very few regularity results that do not assume logarithmic Hölder continuity.

An interesting feature of variable exponent Sobolev spaces is that smooth functions need not to be dense.This was observed by Zhikov in connection with Lavrentiev phenomenon; see [9]. However, when the exponent satisfies a logarithmic Hölder continuity property, or briefly "p is log-Hölder continuous", then the maximal operator is bounded and consequently smooth functions are dense; see [10]. Recall that the log-Hölder condition means that, $\omega:\mathrm{R}^n\to\mathrm{R}^n$ is a nondecreasing continuous function, vanishing at zero, which represents the modulus of continuity of p:

$$|p(x)-p(y)|\le\omega(|x-y|). \tag{2.1}$$

We will assume that ω satisfies the following condition:

$$\limsup_{R\to 0}\omega_1(R)\log\left(\frac{1}{R}\right)<\infty, \tag{2.2}$$

thus in particular, without loss of generality, we may assume that

$$\omega_1(R)\le L|\log R|^{-1} \tag{2.3}$$

for all $R<1$ and $x,y\in\Omega$ with $|x-y|\le 1/2$. Under this condition smooth functions are dense in variable exponent Sobolev spaces and there is no confusion in defining the Sobolev space with zero boundary values, $W_0^{1,p(\cdot)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p(\cdot)}$,see [11].

We will use the following criterion to verify that various functions below belong to $W_0^{1,p(\cdot)}(\Omega)$. See [12].

Lemma 2.1 Suppose that a function v belongs to $W^{1,p(\cdot)}(\Omega)$. If there is a function $u\in W_0^{1,p(\cdot)}(\Omega)$ such that $|v|\le|u|$ a.e. in Ω, then $v\in W_0^{1,p(\cdot)}(\Omega)$.

Remark 2.2 Let us notice that, by replacing θ by $\theta_1=\max\{\theta,\psi\}$, we may assume that the boundary value function θ satisfies $\theta\ge\psi$ in Ω. Indeed $\theta_1=(\psi-\theta)^++\theta$ and since

$$0\le(\psi-\theta)^+\le(u-\theta)^+\in W_0^{1,p(\cdot)}(\Omega),$$

the function $(\psi-\theta)^+$, and hence $u-\theta_1$ belongs to $W_0^{1,p(\cdot)}(\Omega)$ by Lemma 2.1.

We consider the higher integrability for the single obstacle problem. More precisely, we show that under some natural assumptions, the solution u to the $\mathbf{K}_{\psi}^{\theta,p(\cdot)}(\Omega)$ obstacle problem, of which we a priori only know that $|\nabla u|^{p(\cdot)}\in L^1(\Omega)$, actually satisfies $|\nabla u|^{p(\cdot)(1+\varepsilon)}\in L^1(\Omega)$, for a small $\varepsilon>0$, assuming that the boundary values and the obstacle are sufficiently regular. This result can be used to study the corresponding stability yields a reverse Hölder inequality. Higher integrability then follows from a suitable version of Gehring's lemma. See [13].

For all balls $B=B(x_0,r)$ centered at x_0 and radius r. This condition is widely used in regularity theory, and it is also fairly weak.

Lemma 2.3 Let $u,\psi,\theta_1\in W^{1,p(\cdot)}(\Omega)$ and $|\nabla u|,|\nabla\psi|,|\nabla\theta|\in L^{p(\cdot)}(\Omega)$, then there has a number M such that

$$\left\{\int_\Omega|\nabla u|^{p(x)}dx,\ \int_\Omega|\nabla\psi|^{p(x)}dx,\ \int_\Omega|\nabla\theta|^{p(x)}dx\right\}\le M. \quad (2.4)$$

Proof Let's take $v=\max\{\psi,\theta\}=(\psi-\theta)^++\theta$ as an admissible function, for $v\in\mathbf{K}_\psi^{\theta,p(\cdot)}(\Omega)$; Indeed, $v\ge\psi$ and $v-\theta=(\psi-\theta)^+\le(u-\theta)^+\in W_0^{1,p(\cdot)}(\Omega)$. Then by (1.1), we get

$$\int_\Omega\langle\mathbf{A}(x,\nabla u),\nabla(v-u)\rangle dx \ge\int_\Omega\left\langle|F(x)|^{p(x)-2}F(x),\nabla(v-u)\right\rangle dx.$$

That is

$$\int_\Omega\langle\mathbf{A}(x,\nabla u),\nabla u\rangle dx\le\int_\Omega\langle\mathbf{A}(x,\nabla u),\nabla v\rangle dx -\int_\Omega\left\langle|F(x)|^{p(x)-2}F(x),\nabla(v-u)\right\rangle dx.$$

On the left-hand side of this inequality, we have

$$\int_\Omega\langle\mathbf{A}(x,\nabla u),\nabla u\rangle dx\ge\alpha\int_\Omega|\nabla u|^{p(x)}dx.$$

On the right-hand side of this inequality, by using Young inequality, we have

$$\begin{aligned}&\int_\Omega\langle\mathbf{A}(x,\nabla u),\nabla v\rangle dx+\int_\Omega|F(x)|^{p(x)-1}\nabla(u-v)dx\\ &\le\beta\int_\Omega|\nabla u|^{p(x)-1}|\nabla v|dx+\int_\Omega|F(x)|^{p(x)-1}(|\nabla u|+|\nabla v|)dx\\ &\le\varepsilon\int_\Omega|\nabla u|^{p(x)}dx+c(\varepsilon)\int_\Omega\left(|\nabla\varphi|^{p(x)}+|\nabla\theta|^{p(x)}\right)dx\\ &+\varepsilon\int_\Omega|F(x)|^{p(x)}dx+c(\varepsilon)\int_\Omega\left(|\nabla u|^{p(x)}+|\nabla\varphi|^{p(x)}+|\nabla\theta|^{p(x)}\right)dx.\end{aligned}$$

Hence,we have

$$\int_\Omega|\nabla u|^{p(x)}dx\le C\int_\Omega\left(|\nabla\varphi|^{p(x)}+|\nabla\theta|^{p(x)}\right)dx\le M,$$

so any M larger than the right hand side will do.

Lemma 2.4 Assume that $p(x)\equiv$ constant and $1<\gamma_1\le p(x)\le\gamma_2$, then

$$⨍_{2B}\left|\frac{u-c_u}{\operatorname{diam}(B)}\right|^{p(x)}dx\le C\left(⨍_{2B}|\nabla u|^{\frac{p(x)}{\bar\theta}}dx\right)^{\bar\theta}+C, \quad (2.5)$$

where C only depends on $n,\gamma_1,\gamma_2,L,M_1$.

Proof We fix $\bar\theta=\min\left(\sqrt{\frac{n+1}{n}},p^-\right)$; and we take $R<R_0/16$ where R_0 is small enough to have $\omega(8nR_0)\le\bar\theta-1$. By $p_{2B}^+\ge p^-$, we get $\frac{p_{2B}^+\bar\theta}{p^-}\ge\bar\theta>1$, and by

$$p_{2B}^+-p^-\le\omega_1(2\sqrt{n}R)\le\omega_1(2nR)\le\omega_1(8nR_0)\le\bar\theta-1$$

and

$$\frac{p_{2B}^+-p^-}{p^-}\le\frac{p_{2B}^+-p^-}{1}\le\bar\theta-1,$$

we get

$$1\le\frac{p_{2B}^+\bar\theta}{p^-}\le\bar\theta^2\le\frac{n+1}{n}.$$

By using the usual constant exponent Sobolev-Poincaré inequality and log-Hölder continuity, it yields that

$$\begin{aligned}&⨍_{2B}\left|\frac{u-c_u}{\operatorname{diam}(B)}\right|^{p(x)}dx\\ &\le1+⨍_{2B}\left|\frac{u-c_u}{\operatorname{diam}(B)}\right|^{p_{2B}^+}dx\\ &\le1+C\left(⨍_{2B}|\nabla u|^{\frac{p(x)}{\bar\theta}}dx\right)^{\frac{p_{2B}^+}{p(x)}\bar\theta}\\ &=1+C\left(⨍_{2B}|\nabla u|^{\frac{p(x)}{\bar\theta}}dx\right)^{\frac{p_{2B}^+-p(x)}{p(x)}\bar\theta}\left(⨍_{2B}|\nabla u|^{\frac{p(x)}{\bar\theta}}dx\right)^{\bar\theta}\\ &\le1+C\left(⨍_{2B}\left(1+|\nabla u|^{p(x)}\right)dx\right)^{\frac{p_{2B}^+-p(x)}{p(x)}\bar\theta}R^{\frac{-(p_{2B}^+-p(x))n}{p(x)}\bar\theta}\left(⨍_{2B}|\nabla u|^{\frac{p(x)}{\bar\theta}}dx\right)^{\bar\theta}\\ &\le C(M_1)\left(⨍_{2B}|\nabla u|^{\frac{p(x)}{\bar\theta}}dx\right)^{\bar\theta}+C,\end{aligned}$$

where $M_1=C\left(⨍_{2B}\left(1+|\nabla u|^{p(x)}\right)dx\right)^{\frac{p_{2B}^+-p(x)}{p(x)}\bar\theta}R^{\frac{-(p_{2B}^+-p(x))n}{p(x)}\bar\theta}$ and by (2.3), $R^{\frac{-(p_{2B}^+-p(x))n}{p(x)}\bar\theta}$ is bounded.

III. THE PROOF OF THEOREM 1.2

Proof Let B_0 be a ball with $\Omega\subset\frac{1}{2}B_0$. Due to Remark 2.2, we can assume that there exists a compact set $K\subset\Omega$ such that $\theta\ge\psi$ in $\Omega\setminus K$. Let $r_0:=dist\{\partial\Omega,K\}$. Let $B\equiv B(x,r),x\in\Omega$, and assume that $0<r<\frac{1}{4}r_0$ and $2B\subset B_0$. Now let's think about case $2B\subset\Omega$.

Let $\eta\in C_0^\infty(2B)$ be a cut-off function such that $\eta=1$ in $\bar B$, $0\le\eta\le1$ and $|\nabla\eta|\le C/\operatorname{diam}(B)$. We would like to test (1.2) with

$$v:=u-c_u-\eta^{p_{2B}^+}\left(u-c_u-(\psi-c_\psi)\right),$$

where c_u and c_ψ denote the mean value of the functions u and ψ respectively in $2B$, i.e.

$$c_u:=⨍_{2B}u\,dx:=\frac{1}{|2B|}\int_{2B}u\,dx\quad c_\psi:=⨍_{2B}\psi\,dx:=\frac{1}{|2B|}\int_{2B}\psi\,dx$$

To this aim, we need to show that v is an admissible test function, for a suitable obstacle problem. We notice that $v\in \mathbf{K}_{\psi-c_u}^{\theta-c_u,p(\cdot)}$, $v-(\theta-c_u)\in W_0^{1,p(\cdot)}(\Omega)$ because $\eta\in C_0^\infty(\Omega)$.

Since $c_u\ge c_\psi$ we obtain

$$\begin{aligned} v&=\left(1-\eta^{p_{2B}^+}\right)(u-c_u)+\eta^{p_{2B}^+}(\psi-c_\psi)\\ &\ge\left(1-\eta^{p_{2B}^+}\right)(\psi-c_u)+\eta^{p_{2B}^+}(\psi-c_u)\\ &=\psi-c_u \end{aligned}$$

a.e. in Ω. We calculate

$$\begin{aligned}\nabla v=&\left(1-\eta^{p_{2B}^+}\right)\nabla(u-c_u)+\eta^{p_{2B}^+}\nabla(\psi-c_\psi)\\ &+p_{2B}^+\eta^{p_{2B}^+-1}\nabla\eta\times\left[(\psi-c_\psi)-(u-c_u)\right].\end{aligned}$$

Since $u-c_u$ is a solution to the $\mathbf{K}_{\psi-c_u}^{f-c_u,p(\cdot)}$ obstacle problem and v is a test function, we have

$$\begin{aligned}&\int_\Omega\left\langle |F(x)|^{p(x)-2}F(x),\nabla(v-u)\right\rangle dx\\ &\le\int_\Omega\langle\mathbf{A}(x,\nabla u),\nabla(v-u)\rangle dx\\ &=\int_{2B}\langle\mathbf{A}(x,\nabla u),\nabla(v-u)\rangle dx,\end{aligned}$$

and thus

$$\begin{aligned}&\int_{2B}\langle\mathbf{A}(x,\nabla u),\nabla u\rangle dx\\ &\le\int_{2B}\langle\mathbf{A}(x,\nabla u),\nabla v\rangle dx+\int_{2B}\left\langle|F(x)|^{p(x)-2}F(x),\nabla(u-v)\right\rangle dx\\ &\le\int_{2B}\left(1-\eta^{p_{2B}^+}\right)\langle\mathbf{A}(x,\nabla u),\nabla u\rangle dx+\int_{2B}\eta^{p_{2B}^+}\langle\mathbf{A}(x,\nabla u),\nabla\psi\rangle dx\\ &\quad+\beta\int_{2B}p_{2B}^+|\nabla u|^{p(x)-1}\eta^{p_{2B}^+-1}|\nabla\eta|\times\left[(\psi-c_\psi)-(u-c_u)\right]dx\\ &\quad+\int_{2B}\eta^{p_{2B}^+}\left\langle|F(x)|^{p(x)-1},\nabla u\right\rangle dx+\int_{2B}\eta^{p_{2B}^+}\left\langle|F(x)|^{p(x)-1},\nabla\psi\right\rangle dx\\ &\quad+p_{2B}^+\int_{2B}|F(x)|^{p(x)-1}\eta^{p_{2B}^+-1}|\nabla\eta|\times\left[(\psi-c_\psi)-(u-c_u)\right]dx\\ &=I_1+I_2+I_3+I_4+I_5+I_6,\end{aligned}$$

where, in the last line, we used the structure conditions on $\mathbf{A}$. Simplifying and using again the structure conditions of $\mathbf{A}$, we have

$$\begin{aligned}I_1&=\int_{2B}\left(1-\eta^{p_{2B}^+}\right)\langle\mathbf{A}(x,\nabla u),\nabla u\rangle dx\\ &\ge\alpha\int_{2B}\left(1-\eta^{p_{2B}^+}\right)|\nabla u|^{p(x)}dx.\end{aligned}$$

On the other hand, using Young's inequality, for some suitable $\varepsilon\in(0,1)$, we get

$$\begin{aligned}I_2&=\int_{2B}\eta^{p_{2B}^+}\langle\mathbf{A}(x,\nabla u),\nabla\psi\rangle dx\\ &\le\int_{2B}\beta|\nabla u|^{p(x)-1}|\nabla\psi|dx\\ &\le\varepsilon\int_{2B}|\nabla u|^{p(x)}dx+c_\varepsilon\int_{2B}|\nabla\psi|^{p(x)}dx,\end{aligned}$$

and

$$\begin{aligned}I_3&=\beta\int_{2B}p_{2B}^+|\nabla u|^{p(x)-1}\eta^{p_{2B}^+-1}|\nabla\eta|\left[(\psi-c_\psi)-(u-c_u)\right]dx\\ &\le\varepsilon\int_{2B}\eta^{p_{2B}^+}|\nabla u|^{p(x)}dx\\ &\quad+c_\varepsilon\int_{2B}\left(\left|\frac{u-c_u}{\operatorname{diam}(B)}\right|^{p(x)}+\left|\frac{\psi-c_\psi}{\operatorname{diam}(B)}\right|^{p(x)}\right)dx.\end{aligned}\tag{3.1}$$

In the estimation of I_4, I_5 and I_6, we also adopted the method of Young's inequality,

$$\begin{aligned}I_4&=\int_{2B}\eta^{p_{2B}^+}\left\langle|F(x)|^{p(x)-1},\nabla u\right\rangle dx\\ &\le\varepsilon\int_{2B}\eta^{p_{2B}^+}|\nabla u|^{p(x)}dx+c_\varepsilon\int_{2B}|F(x)|^{p(x)}dx,\end{aligned}$$

$$\begin{aligned}I_5&=\int_{2B}\eta^{p_{2B}^+}\left\langle|F(x)|^{p(x)-1},\nabla\psi\right\rangle dx\\ &\le\varepsilon\int_{2B}\eta^{p_{2B}^+}|\nabla\psi|^{p(x)}dx+c_\varepsilon\int_{2B}|F(x)|^{p(x)}dx,\end{aligned}$$

$$\begin{aligned}I_6&=\int_{2B}p_{2B}^+\eta^{p_{2B}^+-1}|F(x)|^{p(x)-1}|\nabla\eta|\times\left[(\psi-c_\psi)-(u-c_u)\right]dx\\ &\le\varepsilon\int_{2B}\eta^{p_{2B}^+}|F(x)|^{p(x)}dx\\ &\quad+c_\varepsilon\int_{2B}\left(\left|\frac{u-c_u}{\operatorname{diam}(B)}\right|^{p(x)}+\left|\frac{\psi-c_\psi}{\operatorname{diam}(B)}\right|^{p(x)}\right)dx.\end{aligned}$$

Observe that we used the definition of p_{2B}^+ to get

$$\tilde p:=\frac{p(x)\left(p_{2B}^+-1\right)}{p(x)-1}\ge p_{2B}^+,\quad\forall\, x\in 2B.\tag{3.2}$$

and to estimate $\eta^{\tilde p}\le\eta^{p_{2B}^+}$ in the second inequality. Now, we choose ε, which depends on n,p^-,p^+,α,β, small enough to absorb the gradient of u to the left hand side in (3.1). We connect all the previous estimates and take the mean values, and get the Caccioppoli type inequality

$$\begin{aligned}⨍_B|\nabla u|^{p(x)}dx&\le C⨍_{2B}\left|\frac{u-c_u}{\operatorname{diam}(B)}\right|^{p(x)}dx+C⨍_{2B}\left|\frac{\psi-c_\psi}{\operatorname{diam}(B)}\right|^{p(x)}dx\\ &\quad+C⨍_{2B}(|\nabla\psi|)^{p(x)}dx+C⨍_{2B}|F(x)|^{p(x)}dx,\end{aligned}$$

where C only depends on n,p^-,p^+,α,β.

By using lemma 2.4, we get

$$⨍_{2B}\left|\frac{u-c_u}{\operatorname{diam}(B)}\right|^{p(x)}dx\le C\left(⨍_{2B}|\nabla u|^{\frac{p(x)}{\theta}}dx\right)^{\theta}+C$$

and

$$⨍_{2B}\left|\frac{\psi-c_\psi}{\operatorname{diam}(B)}\right|^{p(x)}dx\le C\left(⨍_{2B}|\nabla\psi|^{p(x)}+1\right)dx.$$

From this we can deduce the following reverse Hölder estimate

$$\fint_B |\nabla u|^{p(x)}\,dx \le C\left(\fint_{2B} |\nabla u|^{p(x)/\bar{\theta}}\,dx\right)^{\bar{\theta}} + C\fint_{2B} |\nabla \psi|^{p(x)}\,dx$$
$$+C\fint_{2B} |F(x)|^{p(x)}\,dx + C,$$

with $C \equiv C(n, p^-, p^+, \alpha, \beta, M)$ whenever $2B \subset B_0$ is sufficiency small. Now we can use a standard version of Gehring's lemma (see for example [14], Chap. V, or [15], Theorem 6.6), and find a number $\varepsilon > 0$ and a constant C such that

$$\fint_B |\nabla u|^{p(x)(1+\varepsilon)}\,dx \le C\left[\left(\fint_{2B} |\nabla u|^{p(x)}\,dx\right)^{1+\varepsilon} + \fint_{2B} |\nabla \psi|^{p(x)(1+\varepsilon)}\,dx\right.$$
$$\left.+\fint_{2B} |\nabla \theta|^{p(x)(1+\varepsilon)}\,dx + 1\right].$$

Hence，the proof of Theorem 1.2 is completed.

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