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Chaos Control and Hopf Bifurcation Analysis of a Carbon Price Dynamic System

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***Abstract*—Chaos control can eliminate chaotic behavior and suppress chaos to unstable equilibria or unstable periodic orbits. In this paper, a Carbon Price chaotic dynamic system has been studied to investigate the problem of controlling chaos. The time-delayed feedback control method is applied to the Carbon Price system. Based on the corresponding characteristic equation, the linear stability of the equilibrium points and the existence of Hopf bifurcation of the system are analyzed. Further, we derive the interval value of the stable(unstable) time delay. By establishing the appropriate time delay and feedback strength ranges, one of the unstable equilibrium points of the system can be controlled to be stable. Finally, numerical simulations are carried out to verify the effectiveness of the time-delayed feedback control method.**

Keywords—Carbon Price dynamic system; Chaos; Stability; Hopf bifurcation; Time-delayed feedback control

I. Introduction

Chaos has great potential applications in many technological and engineering disciplines [1-3]. However, the emergence of chaos often leads to the unpredictability of the development of the system, resulting in abnormal dynamic behavior. In many cases, this dynamic behavior is not what we want. Therefore, how to control and utilize chaos in many disciplines has attracted widespread attention from scholars [4-6].

In recent years, the methods and techniques of chaotic control have made breakthroughs. Shukla and Sharma [7] proposed the stability and synchronization problems for a class of fractional order chaotic systems and used the backstepping strategy to explain the systematic step by step approach to obtain the control results. Mahmoud and Abood [8] demonstrated the adaptive complex anti-lag synchronization (CALS) of two indistinguishable complex chaotic nonlinear systems. Aghababa and Heydari [9] studied chaotic synchronization between two different uncertain chaotic systems with input nonlinearities. A robust adaptive sliding mode control law is designed to ensure the existence of the sliding motion. Córdoba and Eduardo [10] analyzed the stabilization of an unstable periodic orbit through periodic prediction-based control. Deng et al. [11] proposed a new method of robust controller design and used this method to design new chaos robust controller, which can be used to control transient air-fuel ratio of the gasoline engine. Most of these methods are developments of two basic approaches: the Ott-Grebogi-Yorke method (OGY) [12] and time delay feedback control [13]. Chaos control is slowly forming a systematic theoretical system.

Compared with the above two basic methods, the time-delayed feedback control method is more simple and convenient in controlling chaos for a continuous dynamical system. The OGY method is based on the invariant manifold structure of unstable orbits. It only produces small time-dependent perturbations to the parameters of the chaotic systems and is difficult to change the form of the desired unstable periodic orbits. The time-delayed feedback control method applies a feedback signal which is proportional to the difference between the dynamic variable and its delayed value. By selecting the appropriate time delay, this difference is almost zero as the system approaches the required steady state or periodic orbit, which means stability. Moreover, the advantage of this method is that the delayed control does not require a reference system because it generates control from the information of the system itself.

Time-delayed feedback control has been used for a variety of applications such as biology, medicine, chemistry, engineering and physics. Taher et al. [14] proposed the time-delayed feedback control to solve the dangerous synchronization deviation, which is a useful control concept in nonlinear dynamic systems. Guo et al. [15] put forward a time-delayed feedback control method to improve collection performance of the multiple attractors wind-induced vibration energy harvester system. Guan and Qin [16] applied distributed delay as self-controlling feedback to realize the continuous control of a new butterfly-shaped chaotic system. Postlethwaite et al. [17] exploited the spatiotemporal symmetries to design non-invasive feedback controls to select and stabilize a targeted solution branch to prevent it from bifurcating unstably. Zakharova et al. [18] studied the influence of time-delayed feedback on coherence resonance chimeras. They showed that the chimera region may get enlarged or shrunk with proper tuning of the delayed feedback strength. Mahmoud et al. [19] investigated the control of chaotic Burke-Shaw system via the time-delayed feedback control.

In 2018, Fan et al. [20] discovered a new chaotic attractor for the following Carbon Price system:

$$\begin{cases} \dot{x} = a_1 x(\frac{z}{M} - 1) + a_2 y - a_3 z, \\ \dot{y} = b_1 y - b_2 x + b_3 z(\frac{z}{L} - 1), \\ \dot{z} = -c_1 x + c_2 y + c_3 z(1 - \frac{z}{N}), \end{cases} \quad (1)$$

where $x(t)$ is the time-dependent variable of carbon price, $y(t)$, of energy price, $z(t)$, of economic growth; $a_i, b_i, c_i, (i = 1,2,3), M, L$ and N are positive constants. When $a_1 = 0.048, a_2 = 0.012, a_3 = 0.001,$ $b_1 = 0.02, b_2 = 0.024, b_3 = 0.12, c_1 = 0.012, c_2 = 0.013,$ $c_3 = 0.008, M = 0.9, L = 1.7$ and $N = 0.3,$ system (1) has a chaotic attractor (Fig. 1).

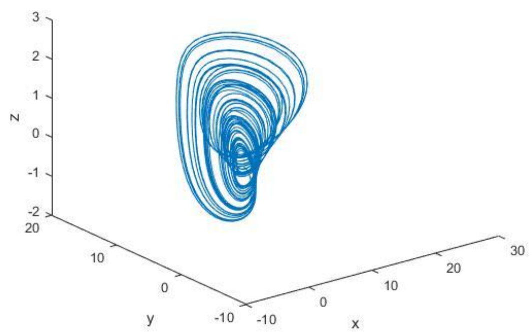

Fig. 1. *A chaotic attractor of the Carbon Price system.*

The purpose of this paper is to study the system (1) using the time-delayed feedback control method. First, we add time-delayed force to the original Carbon Price system. Then, an explicit formula for determining the direction of the Hopf bifurcation and the stability of the bifurcation periodic solution is derived by analyzing the corresponding characteristic equations. Finally, the Carbon Price chaotic system is controlled to a stable state under the appropriate feedback intensity. The main contribution can be divided into three aspects. Firstly, this paper controls the chaotic system of Carbon Price via time-delayed feedback control. Secondly, the local stability and the existence of Hopf bifurcation of the system are studied by theoretical analysis. Thirdly, it is proved that chaos vanishes when the time delay reaches a certain value.

The remainder of this paper is organized as follows: In Section 2, we investigate the problem of controlling the chaos of the Carbon Price system. We consider the stability of one of the equilibrium points and determine the ranges of delay τ at which the equilibrium point of the chaotic Carbon Price system can be controlled to be stable. Section 3 carries out the numerical simulations to verify the theoretic analysis. Finally, a conclusion is given in Section 4.

II. DELAYED FEEDBACK CONTROL METHOD FOR CONTROLLING CHAOS

In this section, we use the time-delayed feedback control strategy to perform chaotic control on the system (1). According to [13], we add a time-delayed force $k[y(t) - y(t-\tau)]$ (k is a constant) to the second equation of the system (1), then the system (1) is described as:

$$\begin{cases} \dot{x} = a_1 x(\frac{z}{M} - 1) + a_2 y - a_3 z, \\ \dot{y} = b_1 y - b_2 x + b_3 z(\frac{z}{L} - 1) + k[y(t) - y(t-\tau)], \\ \dot{z} = -c_1 x + c_2 y + c_3 z(1 - \frac{z}{N}), \end{cases} \quad (2)$$

where τ is a positive constant and $k \in R$.

From the idea of Fan [20], the system (2) has three equilibrium points: E_0, E_1 and E_2. $E_0(0,0,0)$ is obvious, while $E_1(x_1, y_1, z_1)$ and $E_2(x_2, y_2, z_2)$ are generally determined by system parameters due to the high nonlinearity. We just study the stability of the equilibrium point $E_0(0,0,0)$, the other two equilibrium points can be similarly discussed.

The linearization equation of the system (2) at the equilibrium point $E_0(0,0,0)$ is

$$\begin{cases} \dot{x} = -a_1 x + a_2 y - a_3 z, \\ \dot{y} = b_1 y - b_2 x - b_3 z + k[y(t) - y(t-\tau)], \\ \dot{z} = -c_1 x + c_2 y + c_3 z. \end{cases} \quad (3)$$

The characteristic equation of the system (3) is

$$\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 + (B_2\lambda^2 + B_1\lambda + B_0)e^{-\lambda\tau} = 0, \quad (4)$$

where $A_2 = a_1 - b_1 - c_3 - k,$ $A_1 = a_2 b_2 + b_3 c_2 + c_3 b_1 - a_1 c_3 - a_1 b_1 - a_3 c_1 + (c_3 - a_1)k,$ $A_0 = a_1 b_1 c_3 + a_1 b_3 c_2 + a_3 b_1 c_1 - a_2 b_3 c_1 - a_2 b_2 c_3 - a_3 b_2 c_3 + (a_1 c_3 + a_3 c_1)k,$ $B_2 = k, B_1 = (a_1 - c_3)k, B_0 = -(a_1 c_3 + a_3 c_1)k.$ In order to analyze the distribution of roots of the transcendental equation (4), we introduce the following lemma due to [21].

Lemma 1. *Consider the transcendental equation:*

$$P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m}) = \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} + [p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}]e^{-\lambda\tau_1} + \cdots + [p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_{n-1}^{(m)}]e^{-\lambda\tau_m} = 0, \quad (5)$$

where $\tau_j > 0 (j = 1,2,\cdots,m)$ *and* $p_k^{(j)}(j = 0,1,2,\cdots m; k = 1,2,\cdots,n)$ *are constants. As* $(\tau_1, \tau_2, \cdots, \tau_m)$ *vary, the sum of orders of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m})$ *in the open right half plane can change, and only a zero appears on or crosses the imaginary axis.*

Equation (4) has the form when $\tau = 0$:

$$\lambda^3 + (A_2 + B_2)\lambda^2 + (A_1 + B_1)\lambda + A_0 + B_0 = 0. \quad (6)$$

By the Routh-Hurwitz criterion, all the roots of (6) have negative real parts if and only if:

$$A_2 + B_2 > 0, A_0 + B_0 > 0, (A_2 + B_2)(A_1 + B_1) > A_0 + B_0, \quad \text{(H1)}$$

so the point $E_0(0,0,0)$ is locally asymptotically stable when the condition (H1) is satisfied.

For $\omega > 0$, $i\omega$ is a root of (4), then

$$-\omega^3 i - A_2\omega^2 + A_1\omega i + A_0 + (-B_2\omega^2 + B_1\omega i + B_0)e^{-\omega\tau i} = 0. \quad (7)$$

Separating the real and imaginary parts,

$$\begin{cases} (B_0 - B_2\omega^2)\cos\omega\tau + B_1\omega\sin\omega\tau = A_2\omega^2 - A_0, \\ B_1\omega\cos\omega\tau - (B_0 - B_2\omega^2)\sin\omega\tau = \omega^3 - A_1\omega. \end{cases} \quad (8)$$

Add the squares of both sides to get:

$$(B_0 - B_2\omega^2)^2 + (B_1\omega)^2 = (A_2\omega^2 - A_0)^2 + (\omega^3 - A_1\omega)^2,$$

which is equivalent to

$$\omega^6 + p\omega^4 + q\omega^2 + r = 0, \quad (9)$$

where

$$p = A_2^2 - B_2^2 - 2A_1, q = A_1^2 - 2A_0A_2 + 2B_0B_2 - B_1^2, r = A_0^2 - B_0^2.$$

Denote $z = \omega^2$, then (9) becomes

$$z^3 + pz^2 + qz + r = 0. \quad (10)$$

Let

$$h(z) = z^3 + pz^2 + qz + r. \quad (11)$$

We shall employ the ideas of [22-24] to discuss the zero distribution of (4). Since $\lim_{t\to+\infty} h(z) = +\infty$ and $h(0) = r = A_0^2 - B_0^2$, we assume that

$$\Delta = p^2 - 3q > 0, z^* = \frac{-p + \sqrt{\Delta}}{3} > 0, h(z^*) \le 0. \quad \text{(H2)}$$

Then (10) has at least one positive real root. Without loss of generality, suppose that (10) has three positive roots, which are denoted as $z_1^{(0)}, z_2^{(0)}, z_3^{(0)}$. The corresponding (9) has three positive roots $\sqrt{z_k^{(0)}} (k = 1,2,3)$. Then through (8), we derive

$$\tau_k^{(j)} = \frac{1}{\omega_k}\{\arccos[\frac{(A_2\omega_k^2 - A_0)(B_0 - B_2\omega_k^2) + B_1\omega_k(\omega_k^3 - A_1\omega_k)}{(B_0 - B_2\omega_k^2)^2 + (B_1\omega_k)^2}] + 2j\pi\}, \quad (12)$$

where $k = 1,2,3; j = 0,1,2,\cdots$, then $\pm i\omega_k$ are a pair of purely imaginary roots of (4) when $\tau = \tau_k^{(j)}$. Define

$$\tau_0 = \tau_{k_0}^{(0)} = \min_{k\in\{1,2,3\}}\{\tau_k^{(0)}\}. \quad (13)$$

The following results are obtained from the above analysis.

Lemma 2. *If the conditions (H1) and (H2) hold, then all the roots of (4) have a negative real part when* $\tau \in [0, \tau_0)$ *and (4) has a pair of purely imaginary roots* $\pm\omega_k i$ *when* $\tau = \tau_k^{(j)}(k = 1,2,3; j = 0,1,2,\cdots)$.

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be a root of (4) at $\tau = \tau_k^{(j)}$, and $\alpha(\tau_k^{(j)}) = 0, \omega(\tau_k^{(j)}) = \omega_k$. Differentiating both sides of (4) with respect to τ, it follows that

$$[\frac{d\lambda}{d\tau}]^{-1} = \frac{3\lambda^2 + 2A_2\lambda + A_1 + (2B_2\lambda + B_1)e^{-\lambda\tau}}{\lambda(B_2\lambda^2 + B_1\lambda + B_0)e^{-\lambda\tau}} - \frac{\tau}{\lambda}$$

$$= \frac{(3\lambda^2 + 2A_2\lambda + A_1)e^{\lambda\tau} + (2B_2\lambda + B_1)}{\lambda(B_2\lambda^2 + B_1\lambda + B_0)} - \frac{\tau}{\lambda}. \quad (14)$$

Then

$$[\frac{d(\mathrm{Re}\,\lambda(\tau))}{d\tau}]^{-1}_{\tau=\tau_k^{(j)}} = \mathrm{Re}\{\frac{(3\lambda^2 + 2A_2\lambda + A_1)e^{\lambda\tau} + (2B_2\lambda + B_1)}{\lambda(B_2\lambda^2 + B_1\lambda + B_0)}\}_{\tau=\tau_k^{(j)}}$$

$$= \mathrm{Re}\{\frac{(-3\lambda_k^2 + 2A_2\lambda_k i + A_1)(\cos\omega_k\tau_k^{(j)} + i\sin\omega_k\tau_k^{(j)}) + 2B_2\lambda_k i + B_1}{\lambda_k i(-B_2\lambda_k^2 + B_1\lambda_k i + B_0)}\}$$

$$= \mathrm{Re}\{\frac{M_1 + M_2 i}{N_1 + N_2 i}\} = \frac{M_1N_1 + M_2N_2}{N_1^2 + N_2^2}, \quad (15)$$

where

$$M_1 = (A_1 - 3\omega_k^2)\cos\omega_k\tau_k^{(j)} - 2A_2\omega_k\sin\omega_k\tau_k^{(j)} + B_1,$$

$$M_2 = 2A_2\omega_k\cos\omega_k\tau_k^{(j)} + (A_1 - 3\omega_k^2)\sin\omega_k\tau_k^{(j)} + 2B_2\omega_k,$$

$$N_1 = -B_1\omega_k^2, N_2 = \omega_k(A_0 - B_2\omega_k^2).$$

Assume that the following condition holds

$$M_1N_1 + M_2N_2 \ne 0. \quad \text{(H3)}$$

Based on the above analysis and the results of Hassard and Hale [25,26], we obtain the following theorem.

Theorem 1. *If the conditions (H1) and (H2) are true, then the equilibrium point* $E_0(0,0,0)$ *of the system (2) is asymptotically stable when* $\tau \in [0, \tau_0)$. *In the case where the conditions (H1) and (H2) are satisfied, if the* *condition (H3) holds, then the system (2) undergoes a Hopf bifurcation at the equilibrium point* $E_0(0,0,0)$ *when* $\tau = \tau_k^{(j)} (k = 1,2,3;\ j = 0,1,2,\cdots)$.

III. NUMERICAL SIMULATIONS

In this section, the time-delayed feedback control method is applied to suppress chaos to unstable equilibrium points or unstable periodic orbits. We perform numerical simulations to verify the analytical predictions obtained in the previous section. For the purpose of controlling the chaos, we take $k = -0.01$. Considering the following system

$$\begin{cases} \dot{x} = -0.048x + 0.012y - 0.001z + 0.053xz, \\ \dot{y} = 0.02y - 0.024x - 0.12z + 0.071z^2 - 0.01[y(t) - y(t-\tau)], \\ \dot{z} = -0.012x + 0.013y + 0.008z - 0.027z^2. \end{cases} \quad (16)$$

Obviously, the system (16) has an equilibrium point $E(0,0,0)$. We can get that the conditions (H1)-(H3) are satisfied. Let $j = 0$ and by means of Matlab 7.0 software, we gain $\omega_0 \approx 0.0329$, $\tau_0 \approx 90.0583$. Therefore, $E(0,0,0)$ is asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0 \approx 90.0583$. Fig. 2 shows that when $\tau = 83.77 < \tau_0$, the system chaos disappears and the equilibrium point $E(0,0,0)$ is asymptotically stable. When $\tau = 104.56 > \tau_0$, the system (16) undergoes a Hopf bifurcation at the equilibrium point $E(0,0,0)$, i.e. a periodic solution with a small amplitude appears near $E(0,0,0)$ when τ is close to τ_0, which can be shown in Fig. 3.

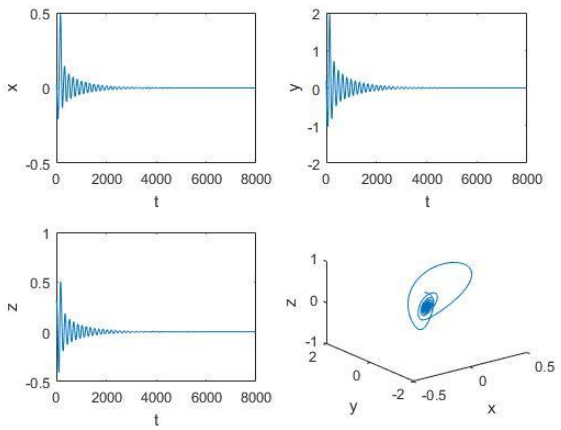

Fig. 2. *The equilibrium point* $E(0,0,0)$ *is asymptotically stable for* $\tau = 83.77 < \tau_0 \approx 90.0583$.

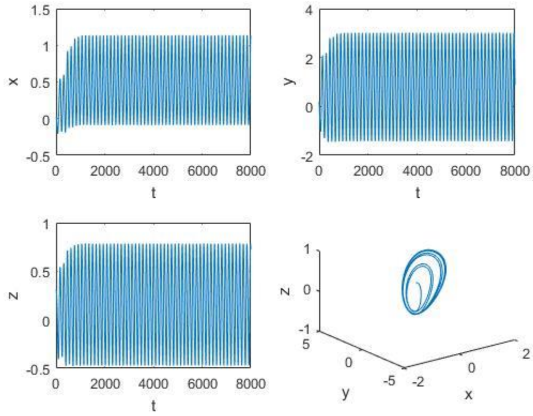

Fig. 3. *The equilibrium point* $E(0,0,0)$ *undergoes a Hopf bifurcation for* $\tau = 104.56 > \tau_0 \approx 90.0583$.

IV. CONCLUSIONS

In this paper, a time-delayed feedback control method is used to control the chaotic behavior of the Carbon Price chaotic dynamic system. By adding a time-delayed force to the second equation of the Carbon Price system, the sufficient conditions of the local asymptotic stability of the equilibrium point $E_0(0,0,0)$ and local Hopf bifurcation of the delayed Carbon Price system are obtained. It is shown that if the conditions (H1) and (H2) are satisfied, then the Carbon Price system is asymptotically stable when $\tau \in [0, \tau_0)$ and unstable when $\tau > \tau_0$; if the conditions (H1)-(H3) are true, a series of Hopf bifurcations occur near the equilibrium point $E_0(0,0,0)$. At this point, chaos disappears and chaos is suppressed. The chaotic system exhibits stable and bifurcation periodic solutions under the effect of the delayed force. Numerical simulations prove the correctness of the theoretical analysis.

This paper provides a theoretical basis for chaotic control of the Carbon Price system with time-delayed feedback, which can increase our understanding of chaotic control of Carbon Price system, and control energy price through delayed feedback, thus slowing down economic growth. I hope to get a better conclusion in further promoting the control of other three-dimensional chaotic systems. Due to the nature and controllability of chaotic systems, it can be applied to radio hyperchaotic secure communication systems and other security systems, which makes the research of chaotic time-delayed feedback control rich in theory and application.

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