Effect Of Nonlinear Intensity On Orbital Stability And Chaos Control In The Generalized Camassa-Holm Equation

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***Abstract*—Existence and stability of solitary waves in the generalized Camassa-Holm equation is considered. The nonlinear intensity has important influence on the shape and stability of solitary waves. When the power of nonlinear term is odd, the equation admits positive solitary waves which are also proved to be orbitally stable when the wave velocity exceeds a critical value. When the power of nonlinear term is even, the equation admits positive and negative solitary waves which are proved to orbitally stable for any wave velocity. Using the Menikov method, all solitary waves turn to chaos under the external periodic perturbation with arbitrary nonlinear intensity. By applying a feedback controller, chaos can be controlled into a stable state. Results show that the uncontrollable region becomes smaller and the awful frequency appears less with the increase of nonlinear intensity.**

Keywords—Camassa-Holm equation; Stability; Chaos; Solitary waves

I. INTRODUCTION

The generalized Camassa-Holm equation

$$u_t - u_{xxt} + au^n u_x = 2u_x u_{xx} + uu_{xxx}, \qquad (1.1)$$

is a famous shallow water equation [1]. Here n denotes nonlinear intensity and $a(>0)$ is the parameter of nonlinear term. Eq. (1.1) has three important models and it has been also widely studied by many researchers[2-4].

When $n=1$ and $a=3$, Eq. (1.1) becomes the famous Camassa-Holm equation [5-7], which has a nonsmooth solitary wave $ce^{-|x-ct|}$ (where c is the wave speed) which is called a peakon. This special solution is proved to be orbitally stable for any wave speed [8].

Eq. (1.1) is reduced to the modified Camassa-Holm equation as $n=2$ [9-13], which admits negative and positive smooth solitary waves. This negative smooth solution is proved to be orbitally stable for any wave speed [10]. Eq. (1.1) becomes the Camassa-Holm equation with quartic nonlinearity when $n=3$ [14-15], which admits a positive smooth solitary wave. This positive smooth solution is proved to be orbitally stable when the speed exceeds a critical value [15]. More property of generalized Camassa-Holm equation has been extensively studied [16-17].

It is obviously that the nonlinear intensity n has a great influence on the shape and stability of the solitary wave as the above facts shown. Hence our first objective is to carry out a further study on the existence and stability of solitary waves as the nonlinear $n>3$ strength . Moreover, the propagation of solitary waves is not in a pure environment, and it is easy to be affected by external perturbation. It is nature that the second objective of this paper is to study evolution progress and control problem of solitary wave under the external perturbation.

The rest of the paper is organized as follows. In Section 2, existence of solitary wave and homoclinic orbits of Eq. (1.1) are given. In Section 3, stability of solitary waves is considered. In Section 4, dynamics behavior for the perturbed and controlled system are studied.

II. EXISTENCE OF SOLITARY WAVE

The solitary wave of Eq. (1.1) has the form as

$$u(x,t)=\phi_c(x-ct) \tag{2.1}$$

and the profile ϕ_c propagating at speed $c>0.$

Then Eq. (1.1) becomes

$$-c\phi_{cx}+c\phi_{cxxx}+a\phi_c{}^n\phi_{cx}=2\phi_{cx}\phi_{cxx}+\phi_c\phi_{cxxx} \tag{2.2}$$

Using the decay of ϕ_c at infinity, we obtain

$$c\phi_c-c\phi_{cxx}-\frac{a}{n+1}\phi_c{}^{n+1}+\phi_c\phi_{cxx}+\frac{\phi_{cx}^2}{2}=0 \tag{2.3}$$

Multiplying by ϕ_{cx} and integrating Eq. (2.3), we have

$$\phi_{cx}^2=\frac{\phi_c^2(c-\frac{a\phi_c^n}{2(n+2)})}{c-\phi_c}. \tag{2.4}$$

Remark 1. We can deduce that solitary waves from Eq. (2.4) exist for $c>\sqrt[n-1]{\frac{2(n+2)}{a}}$.

Theorem 1. For any $n\in Z^+$, Eq. (1.1) admits homoclinic orbits associated with the solitary waves as $a>0.$

Proof. Letting $y=\phi'$, Eq. (2.3) becomes

$$\begin{cases}\phi'=y,\\ y'=\dfrac{-c\phi+\frac{a}{n+1}-\frac{y^2}{2}}{\phi-c}.\end{cases} \tag{2.5}$$

Taking the commutation $d\xi=(\phi-c)d\tau,$ Eq. (2.5) is rewritten as

$$\begin{cases}\dfrac{d\phi}{d\tau}=(\phi-c)y,\\ \dfrac{dy}{d\tau}=-c\phi+\dfrac{a}{n+1}\phi^{n+1}-\dfrac{y^2}{2}.\end{cases} \tag{2.6}$$

Hence Eq. (2.6) has the Hamiltonian function as follows

$$H(\phi,y)=(c-\phi)y^2+\frac{a}{2(n+2)}\phi^{n+2}-c\phi^2. \tag{2.7}$$

Case I. When n is even number, Eq. (2.6) has three equilibrium points: A(0,0), B $(-\sqrt[n]{\frac{(n+1)c}{a}},0)$, C $(\sqrt[n]{\frac{(n+1)c}{a}},0)$.

Case II. When n is odd, Eq.(2.6) has two equilibrium points: D(0,0), E $(\sqrt[n]{\frac{(n+1)c}{a}},0)$.

According to the dynamics theory, we know the system (2.6) admits homoclinic orbits (see Fig. 1(a), Fig. 2(c)). Furthermore, system (2.6) also has the solitary waves (see Fig. 1(b), Fig. 2(d)).

We find that Eq. (1.1) only admits positive solitary waves when n is odd, while both positive and negative solitary waves appear when n is even. It is noted that the height of the solitary wave is smaller and tends to $\sqrt[n]{\frac{(n+1)c}{a}}$ with the increasing n.

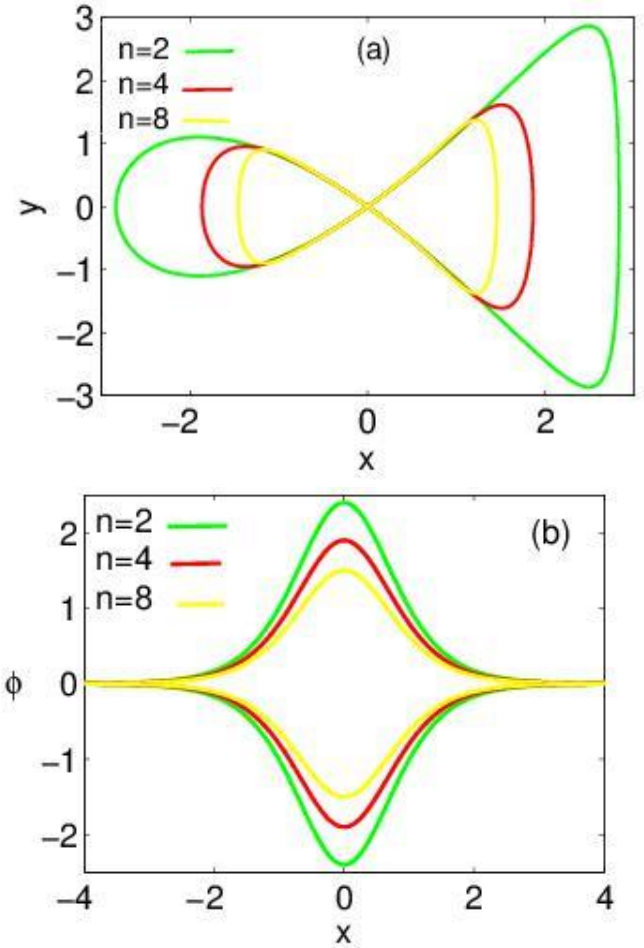

Fig. 1. *(a) phase portrait of Eq. (2.7); (b)The negative and positive solitary wave.*

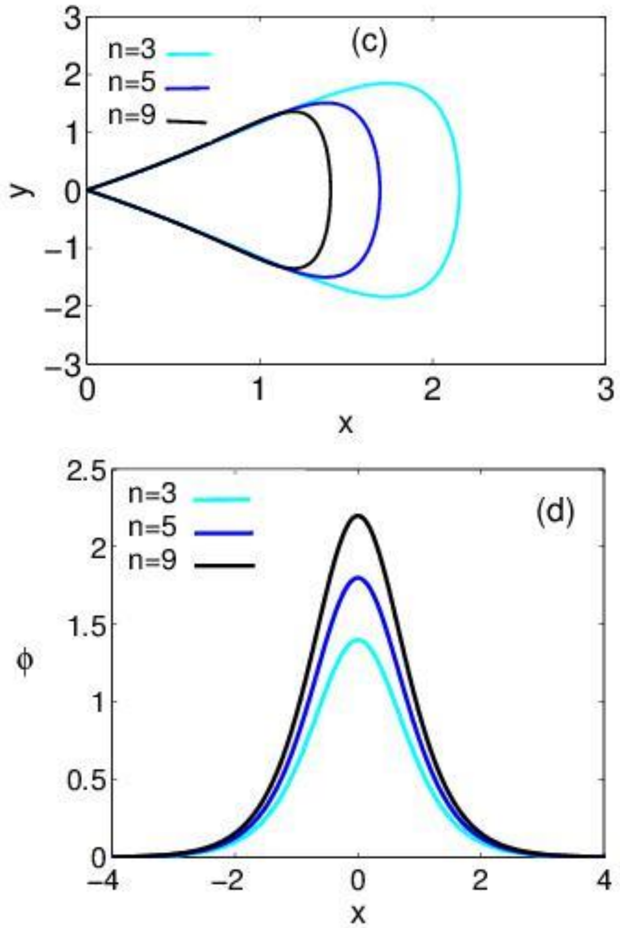

Fig. 2. *(c) phase portrait of Eq. (2.7); (d)The positive solitary wave.*

III. STABILITY OF SOLITARY WAVE

Eq. (1.1) has two invariants as follows

$$\begin{cases} E(u) = \frac{1}{2}\int_R (u^2 + u_x^2)dx, \\ F(u) = -\frac{1}{2}\int_R \frac{au^{n+2}}{2(n+2)} + uu_x^2 dx. \end{cases} \tag{3.1}$$

Eq. (2.3) can be also described as

$$F'(\varphi_c) + cE'(\varphi_c) = 0, \tag{3.2}$$

Where represents the Frechet derivatives of E and F and the linearized operator is given as

$$H_c = F''(\phi_c) + cE''(\phi_c) = -\partial_x(c - \phi_c)\partial_x - a\phi_c^n + \phi_{cxx} + c.$$

It is noted that the functions ϕ_c , ϕ_{cx} , $\phi_{cxx} \to 0$

exponentially fast when $|x| \to \infty$. The Liouville transformation admits the following expression

$$\psi(y) = (2c - 2\phi_c(z))^{\frac{1}{n+1}} v(x),$$

$$y = \int_0^x \frac{1}{\sqrt{c - \phi_c(z)}} dz.$$

Taking advantage of spectral equation $H_c v = \lambda v$ and Liouville transformation, we have

$$F_c \psi(y) = (-\partial_y^n + p_c(y) + c)\psi(y) = \lambda \psi(y),$$

where $p_c(y) = -a\phi_c^n(x) + \frac{3}{4}\phi_c'' - \frac{(\phi_c'(x))^2}{8(c - \phi_c(x))}$.

The stability depends on the convexity properties of the function $d(c) = F(\phi_c) + cE(\phi_c)$. We have the following lemma.

Lemma 1. According to the literature, the stability we know that the solitary wave ϕ_c is unstable if $d''(c) < 0$ and stable if $d''(c) > 0$.

By differentiating $d(c)$ once, we get

$$d'(c) = \left\langle F'(\phi_c) + cE'(\phi_c), \frac{\partial \phi_c}{\partial c} \right\rangle + E(\phi_c) = E(\phi_c). \tag{3.3}$$

Case 1. Positive solitary wave

In this case, the solitary wave $\phi_c \in [0, \sqrt[n]{\frac{(n+1)c}{a}}]$ and ϕ_c is an even function. So we have

$$d''(c) = \frac{d}{dc} \int_R \frac{1}{2}(\phi_c^2 + \phi_{cx}^2) dx$$

$$= \frac{d}{dc} \int_0^{+\infty} \phi_c^2 \frac{2c - \phi_c - \frac{a\phi_c^n}{2(n+2)}}{c - \phi_c} dx$$

$$= \frac{d}{dc} \int_0^{+\infty} \phi_c (-\phi_{cx}) \sqrt{\frac{c - \phi_c}{c - \frac{a\phi_c^n}{2(n+2)}}} \cdot \frac{2c - \phi_c - \frac{a\phi_c^n}{2(n+2)}}{c - \phi_c} dx$$

$$= \frac{d}{dc} \int_0^{+\infty} \phi_c (-\phi_{cx}) \frac{4(n+2)c - 2(n+2)\phi_c - a\phi_c^n}{\sqrt{2(n+2)}\sqrt{2(n+2)c - a\phi_c^n}\sqrt{c - \phi_c}} dx$$

$$= \frac{d}{dc} \int_{\sqrt[n]{\frac{2(n+2)c}{a}}}^0 y \frac{-4(n+2)c + 2(n+2)y + ay^n}{\sqrt{2(n+2)}\sqrt{2(n+2) - ay^n}\sqrt{c - y}} dy.$$

We can get a simpler form if letting $y = \sqrt[n]{\frac{2(n+2)c}{a}} s$ as follows

$$d''(c) = \frac{d}{dc} \int_0^1 \frac{(\frac{2(n+2)c}{a})^{\frac{2}{n}} s[4(n+2)c - 2(n+2)\sqrt[n]{\frac{2(n+2)c}{a}} s - 2(n+2)cs^n]}{\sqrt{2(n+2)}\sqrt{2(n+2)c - 2(n+2)cs^n}\sqrt{c - (\frac{2(n+2)c}{a})^{\frac{1}{n}} s}} ds$$

$$= \frac{d}{dc} \int_0^1 \frac{(\frac{2(n+2)c}{a})^{\frac{2}{n}} s(2c - \sqrt[n]{\frac{2(n+2)c}{a}} s - cs^n]}{\sqrt{c - cs^n}\sqrt{c - (\frac{2(n+2)c}{a})^{\frac{1}{n}} s}} ds. \tag{3.4}$$

Case 2. Negative solitary wave

In this case, the solitary wave $\phi_c \in [-\sqrt[n]{\frac{(n+1)c}{a}}, 0]$ and ϕ_c is an even function. So we have the following fact

$$d''(c) = \frac{d}{dc} \int_R \frac{1}{2}(\phi_c^2 + \phi_{cx}^2) dx$$

$$= \frac{d}{dc} \int_0^{+\infty} \phi_c^2 \frac{2c - \phi_c - \frac{a\phi_c^n}{2(n+2)}}{c - \phi_c} dx$$

$$=\frac{d}{dc}\int_0^{+\infty}\phi_c(-\phi_{cx})\frac{4(n+2)c-2(n+2)\phi_c-a\phi_c^n}{\sqrt{2(n+2)}\sqrt{2(n+2)c-a\phi_c^n}\sqrt{c-\phi_c}}dx$$

We can also get a simpler form if letting $y=-\sqrt[n]{\frac{2(n+2)c}{a}}s$ as follows

$$d''(c)=\frac{d}{dc}\int_0^1\frac{(\frac{2(n+2)c}{a})^{\frac{2}{n}}s(2c+\sqrt[n]{\frac{2(n+2)c}{a}}s-cs^n]}{\sqrt{c-cs^n}\sqrt{c+(\frac{2(n+2)c}{a})^{\frac{1}{n}}s}}ds.$$

(3.5)

It is hard to get the integrable value according to the Eq.(3.4) or Eq. (3.5). So we will use MATLAB to compute the value of $d''(c)$.

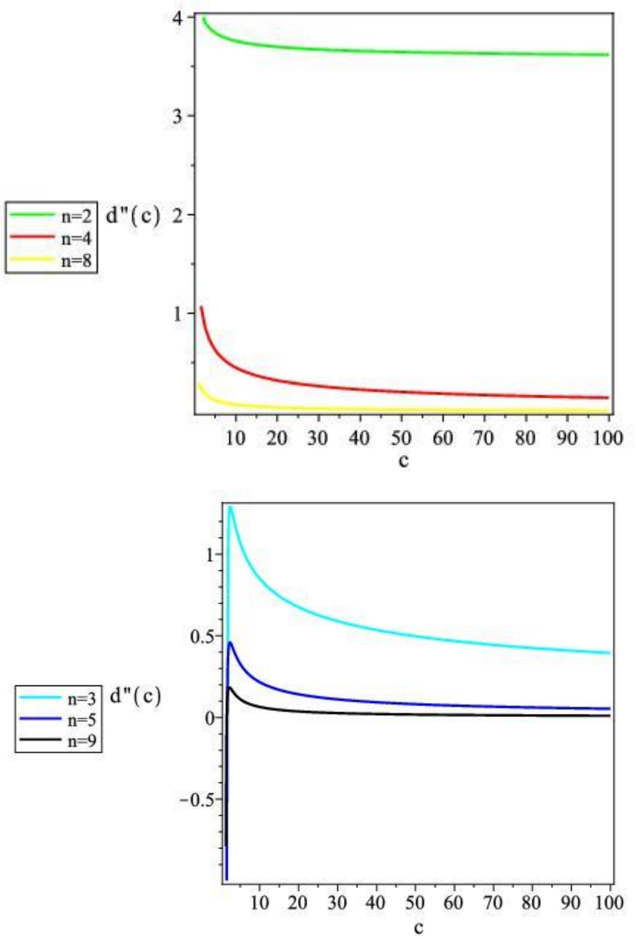

Fig. 3. *(a) The integration value of Eq.(3.4); (b) The integration value of Eq.(3.5).*

Remark. Fig. 3(a) shows that the positive solitary wave is unstable when the speed c is close to the critical value $\sqrt[n-1]{\frac{2(n+2)}{a}}$, which connects with the bifurcation condition contributes. And the positive solitary wave is stable when the speed is slightly greater than the critical value. Meanwhile, Fig.3(b) shows that the negative solitary wave is stable for any wave speed.

IV. Dynamics behavior for the perturbed and controlled system

The perturbed equation (2.7) is given as:

$$\frac{d\phi}{d\tau}=(c-\phi)y,$$
$$\frac{dy}{d\tau}=-c\phi+\frac{a}{n+1}\phi^{n+1}-\frac{y^2}{2}+r\cos\omega\tau, \tag{4.1}$$

where r and ω denote the amplitude and the frequency of perturbation term respectively. The unperturbed system permits a homoclinic orbit given by Eq. (4.1). It is noted that the closed homoclinic orbits perhaps break under the perturbation. So we want to find the criteria of the existence of homoclinic bifurcation and chaos by the melnikov method.

Supposing the unperturbed homoclinic orbits written as $(x_0,y_0)=(x^{\pm}(\tau),y^{\pm}(\tau))$, the Melnikov function for system (4.1) can be given by

$$M(\tau_0)=\int_{-\infty}^{+\infty}y_0(\tau)r\cos\omega(\tau+\tau_0)d\tau$$

$$=-2r\sin\omega\tau_0\int_0^{+\infty}y_0(\tau)\sin\omega\tau d\tau$$

$$=-2rI\sin\omega\tau_0,$$

Where $I=\int_0^{\infty}y_0(\tau)\sin\omega\tau d\tau$ is a function of frequency ω.

By using the Melnikov's theorem [18], the chaos occurs if $M(\tau_0)=0$ and $M'(\tau_0)\neq 0$for some τ_0. It is easy to find that the Melnikov function has the value $\tau_0=0$ such that $M(0)=0$ and $\frac{\partial M}{\partial \tau_0}\Big|_{\tau_0=0}=-2rI\neq 0$. Hence we can deduce that chaos would occur under the one external perturbations.

In order to suppressing chaos, the controlled system of Eq. (4.1) is devised as

$$\frac{d\phi}{d\tau}=(c-\phi)y,$$
$$\frac{dy}{d\tau}=-c\phi+\frac{a}{n+1}\phi^{n+1}-\frac{y^2}{2}+r\cos\omega\tau-ky. \quad (4.2)$$

And the relative Melnikov function is yielded as

$$M(\tau_0)=-k\int_{-\infty}^{+\infty}y_0^2(\tau)d\tau+\int_{-\infty}^{+\infty}y_0(\tau)r\cos\omega(\tau+\tau_0).$$

Considering $y_0(\tau)$ is odd, we obtain

$$M(\tau_0)=-2k\int_0^{+\infty}y_0^2(\tau)d\tau-2r\sin\omega\tau_0\int_0^{+\infty}y_0(\tau)\sin w\tau d\tau$$
$$=-2kC_{h0}-2rI_1\sin\omega\tau_0,$$

where $C_{h0}=\int_0^{\infty}y_0^2(\tau)d\tau$ and $I_1=\int_0^{\infty}y_0(\tau)\sin\omega\tau d\tau$.

Based the analysis in Ref. [15], the chaos may be controlled as $k\geq\frac{|rI_1(\omega)|}{C_{h0}}=R(\omega)$. In order to investigate the influence of different parameters on control ability, some numerical results are given in Fig. 4 and Fig. 5. The graph of $k=\frac{|rI_1(\omega)|}{C_{h0}}$ shows the following facts:

(1) The uncontrollable region becomes larger with the increase of perturbation amplitude r.

(2) The awful frequency appears periodicity where chaos is extremely difficult to be controlled.

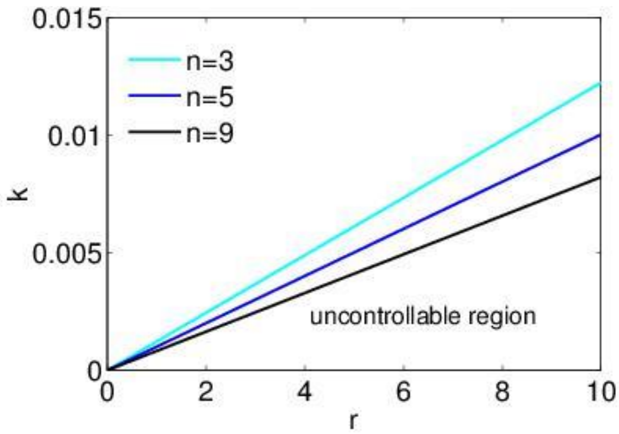

Fig. 4. *Chaos threshold for the perturbed system with* $a=3, c=4, \omega=0.5$.

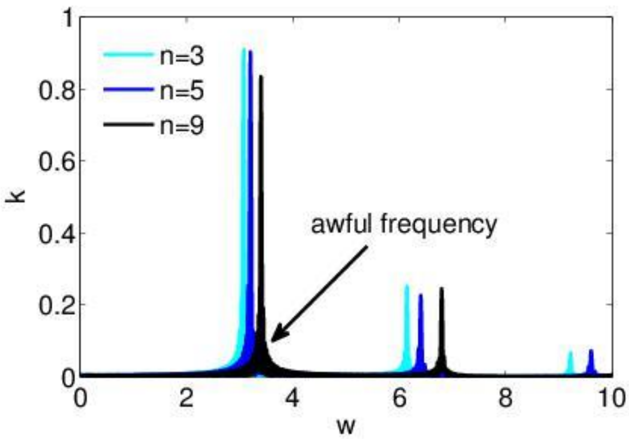

Fig. 5. *Chaos threshold for the perturbed system with* $a=3, c=4, r=1.2$.

(3) The uncontrollable region becomes smaller and the awful frequency appears less with the increase of nonlinear intensity n.

V. CONCLUSION

Based on the generalized Camassa-Holm equation, we study the existence and stability of solitary waves and find the nonlinear intensity has important influence on the shape and stability of solitary waves. Following conclusions could be obtained,

(1) When the power of nonlinear term is odd, the equation admits positive solitary waves which are also

proved to be orbitally stable when the wave velocity exceeds a critical value.

(2) When the power of nonlinear term is even, the equation admits positive and negative solitary waves which are proved to orbitally stable for any wave velocity.

(3) All solitary waves turn to chaos under the external periodic perturbation with arbitrary nonlinear intensity.

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References

[1] S. Zheng, Z. Ouyang, Stability of Negative Solitary Waves for a Generalized Camassa-Holm Equation with Quartic Nonlinearity, Advances in Mathematical Physics 2015;78926.

[2] C.H. Yu, T. W. Sheu, Numerical study of long-time Camassa–Holm solution behavior for soliton transport, Mathematics and Computers in Simulation 2016; vol. 128,pp. 1–12.

[3] H. Hu, W. Yin, H. Wu, Bilinear equations and new multi-soliton solution for the modified Camassa–Holm equation, Applied Mathematics Letters 2016; vol. 59 , pp 18–23.

[4] S. Yu, Wave-breaking phenomenon for a generalized spatially periodic Camassa-Holm system, Mathematical Methods in the Applied Sciences 2015; vol.38, pp. 1405-1417.

[5] Y. Zhu, F. Fu, Persistence properties of the solutions to a generalized two-component Camassa–Holm shallow water system, Nonlinear Analysis2015;vol.128,pp.77–85.

[6] B. Moon, Global solutions to a special case of the generalized weakly dissipative periodic two-component Camassa–Holm system, Nonlinear Analysis 2015; vol. 117, pp. 38–46.

[7] Y. Mi, C.i Mu, On the Cauchy problem for the generalized Camassa–Holm equation, Monatsh Math 2015;vol. 176, pp. 423–457.

[8] Y. Guo, Y. Tang, Blow-up for the weakly dissipative generalized Camassa-Holm equation, Journal of Inequalities and Applications 2014;2014:514.

[9] B. Xia , Z. Qiao, Multi-component generalization of the Camassa–Holm equation, Journal of Geometry and Physics 2016;vol. 107, pp. 35–44.

[10] J. Yin, L. Tian, X. Fan, Stability of negative solitary waves for an integrable modified Camassa–Holm equation, Journal of Mathematical Physics 2010; vol. 51:pp. 053515.

[11] X. Chang, X. Hu, J. Szmigielski, Multipeakons of a two-component modified Camassa–Holm equation and the relation with the finite Kac–van Moerbeke lattice,Advances in Mathematics 2016 ;vol. 299, pp. 1–35.

[12] Z. Guo, M. Zhu, L. Ni, Blow-up criteria of solutions to a modified two-component Camassa–Holm system, Nonlinear Analysis: Real World Applications 2011; vol. 12, pp. 3531–3540.

[13] R. M. Chen, Y. Liu, C. Qu, S. Zhang, Oscillation-induced blow-up to the modified Camassa–Holm equation with linear dispersion, Advances in Mathematics 2015;vol. 272,pp. 225–251.

[14] X. Tu, Z. Yin , Blow-up phenomena and local well-posedness for a generalized Camassa-Holm equation in the critical Besov space, Nonlinear

Analysis-Theory Methods& Applications 2015;vol. 128, pp. 1-19.

[15] J. Yin , Q. Xing, L. Tian, Orbital stability and dynamical behaviors of solitary waves for the Camassa–Holm equation with quartic nonlinearity, Chaos, Solitons & Fractals 2015;vol. 76, pp. 40–46.

[16] Y. Mi, C. Mu, On the Cauchy problem for the generalized Camassa-Holm equation, Monatshefte fur Mathematik 2015; vol. 176, pp. 423-457.

[17] Z. Guo, On an integrable Camassa-Holm type equation with cubic nonlinearity, Nonlinear Analysis-Real World Applications, 2017;vol. 34, pp. 225-232.

[18] J. Kuang, P. Meehan, A. Leung, On the chaotic rotation of a liquid-filled gyrostat via the Melnikov-Holmes-Marsden integral, International Journal of Nonlinear Mechanics , 2006; vol. 41, pp. 475-490.