Journal of Multidisciplinary Engineering Science and Technology (JMEST)
ISSN: 2458-9403
Vol. 6 Issue 3, March - 2019

Solving Fuzzy Linear Regression With ST Decomposition

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***Abstract*—Fuzzy systems have an essential role in the modeling of many real-world systems and problems, which can formulate uncertainty in actual environment. The purpose of this article is to present a new view of the ST decomposition, which is applicable for solving fuzzy linear regression and its realization through a direct application of LU decomposition without any iterative techniques. MATLAB implementation of the ST decomposition is given and some numerical examples clarify the ability of our method.**

Keywords—Fuzzy linear regression; ST decomposition; LU decomposition.

I. INTRODUCTION

Several methods for solving fuzzy linear regression systems have been suggested. In the paper of Abbasbandy et al. [1] LU decomposition method, for solving fuzzy system of linear equations is considered. They consider the method in a spatial case when the coefficient matrix is symmetric positive definite. In the work of Matinfar et al. [9], Householder decomposition method for solving fuzzy system of linear equations is suggested. If *A* is an *m x k* matrix with full column rank, QR decomposition is applied by Nasseri et al. [14].

A new decomposition of a nonsingular matrix, the Symmetric times Triangular (ST) decomposition, is proposed by Golub and Yuan [5]. By this decomposition, every nonsingular matrix can be represented as a product of a symmetric matrix *S* and a triangular matrix *T*. Furthermore, *S* can be made positive definite. Two numerical algorithms for computing the ST decomposition with positive definite *S* are presented.

In this section, some primary definitions and notes, which are required in this work, are given.

Definition 1. Following Zimmermann [20], a fuzzy number may be defined as *F = (b, g, h);* where *b* denotes the center (or mode), *g* and *h* are the left spread (L) and right spread (R), respectively, L and R denote the left and right shape functions. A popular fuzzy number is the triangular fuzzy number (see Figure 1).

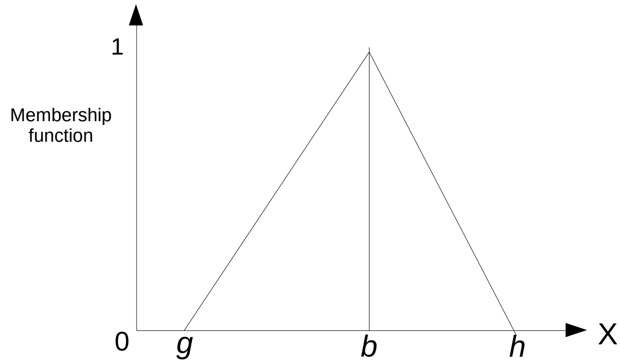

Fig. 1. Triangular fuzzy number.

The membership function of a triangular fuzzy number is defined by:

$$u(x) = \begin{cases} \dfrac{x-b}{g} + 1, & b - g \le x \le b, \\ \dfrac{b-x}{h} + 1, & b \le x \le b + h, \\ 0, & otherwise. \end{cases} \qquad (1)$$

Definition 2. A matrix $\tilde{A} = (\tilde{a}_{ij})_{m,n}$ is called a fuzzy matrix, if each its element is a fuzzy number. We may represent m×n fuzzy matrix, such that $\tilde{A} = (A, M, N)$, $(\tilde{a}_{ij}) = (b_{ij}; g_{ij}; h_{ij})$, with the new notation, where A = (b_{ij}), M = (g_{ij}) and N = (h_{ij}) are three m×n crisp matrices. Crisp means - something clearly defined, deterministic in character.

The purpose of this article is to present a new view of the ST decomposition and its realization through a direct application of LU decomposition without any iterative techniques.

II. FUZZY LINEAR MODELS AND METHOD

The different types of fuzzy linear regressions (FLR) have been classified by Maturo [11] in:

1. Partially fuzzy linear regression (PFLR), that can be further divided into:
- PFLR with fuzzy parameters and crisp data;
- PFLR with fuzzy data and crisp parameters.
2. Totally fuzzy linear regression (TFLR) where data and parameters are both fuzzy.

A numerical method for finding minimal solution of a m×n fully fuzzy linear system (TFLR) based on pseudo inverse calculation, is given in [12], when the central matrix of coefficients is row full rank or column full rank, and where is a non-negative fuzzy m×n matrix, the unknown vector x is a vector consisting of n non-negative fuzzy numbers and the constant b is a vector consisting of m non-negative fuzzy numbers.

For calculation of we $\tilde{A}\tilde{x}=\tilde{b}$ use Definition 2, therefore

$Ax = b$, (2)

$Ay + Mx = g$, (3)

$Az + Nx = h$. (4)

The singular value decomposition

$$A = U\Sigma V^T \quad (5)$$

has the unique pseudo inverse matrix

$$A^+ = V\Sigma^+ U^T. \quad (6)$$

Thus we easily have by equations (2) - (4)

$x = A^+b$, (7)

$y = A^+(g - MA^+b)$, (8)

$z = A^+(h - NA^+b)$. (9)

It should be noted here that, if $y \geq 0$, $z \geq 0$ and $x - y \geq 0$

$$\tilde{x} = (x, y, z)$$

is a nonnegative fuzzy solution of nonnegative full fuzzy linear system.

The comparison made by Mosleh et al. in [12] shows that the presented here method solve the fully fuzzy linear system where A is a nonnegative fuzzy matrix, while other authors considered only the case of symmetric and positive fuzzy matrices.

Proposition: If A, M and N are n×n nonsingular crisp matrices of rank = n, and the systems $Ax = b$, $My = g$, $Nz = h$ are consistent, then the solution of the totally fuzzy linear regression (A, M, N) = (b, g, h) according Definition 2 is given by

$x = A^{-1}b$; (10)

$y = M^{-1}g$; (11)

$z = N^{-1}h$. (12)

Proof. If there is at least one solution, the linear system is consistent. The linear system $Ax = b$ is consistent if and only if rank [A b] = rank A. The matrix [A b] is the augmented matrix. The augmented matrix and the coefficient matrix A of a linear system have the same rank. In this case, appending b to the columns of A does not increase the rank. If matrix A is not singular, then the inverse matrix A^{-1} exists. A solution of the linear system $Ax = b$ is a vector x whose entries are the coefficients in a representation of b as a linear combination of the columns of A. And the solution is $x = A^{-1}b$.

III. ST DECOMPOSITION

The parameters of fuzzy linear regression based on the least squares approach is computed by ST-decomposition method. This method is not an iterative technique; however, it is a powerful method for nonsingular coefficient matrices [19].

In his book Strang [17, Exercise 36, p. 108] suggests a new decomposition, which results in a triangular matrix multiplied by symmetrical. He puts the following questions:

If the matrix A is represented as $A = LDU$, why the matrix $L(U^T)^{-1}$ is triangular with all 1's on the main diagonal and why U^TDU is symmetric? The answers to these questions are positive when the matrix has the same number of rows and columns, but in the case of a rectangular matrix this is not true in the presence of the matrix D. The important conclusion of this suggestion is that the LU decomposition can be used to obtain ST decomposition.

For every nonsingular matrix A with nonsingular leading principal submatrices, there exist a triangular matrix T and a symmetric and positive definite matrix S such that $A = TS$. Since the decomposition is not unique, we can make correct choices such that the decomposition is stable. Some numerical algorithms were given in [6]. An algorithm is introduced to solve triangular fuzzy matrices by S_ST decomposition method [7]. The paper of Mosleh et al. [13] mainly discusses the new ST decomposition. By this decomposition, every nonsingular fuzzy matrix can be represented as a product of a fuzzy symmetric matrix S and a fuzzy triangular matrix T.

Cordeiro and Yuan [2] suggested new row-wise algorithms for the ST decomposition. The new algorithms require just a row of A and two triangular solvers at each step instead of three triangular solvers in Golub–Yuan algorithms [5, 6]. Cases with a symmetrical matrix of 4th order for trapezoidal fuzzy matrices are considered [18]. Recently, some modifications of the ST algorithms have been made by Santiago and Yuan [16]. They present preliminary investigations on the numerical behavior of the ST decomposition. They also propose modifications (modified algorithm) to improve the algorithm's numerical stability. Numerical tests of the Golub–Yuan algorithm and their modified algorithm are given for some famous test matrices. All tests include comparisons with the LU (or Cholesky) decomposition without pivoting. These numerical tests indicate that the Golub–Yuan algorithm and its modified version possess reasonable numerical stability.

Let's look at the properties of the LU decomposition.

It is applicable to every positive semi definite matrix A, $A = LU$, where L is lower triangular invertible matrix and U is upper triangular matrix. For matrices that are not square, LU decomposition still makes sense. Given an m×n matrix A, we could write $A = LU$ with L a square lower unit triangular matrix, and U a rectangular matrix. Then L will be an m×m matrix, and U will be an m×n matrix (of the same shape as A) and $LL^{-1} = I$. From here, the process is exactly the same as for a square matrix.

It is now clear that an LU factorization of a given matrix may or may not exist, and if it exists, it need not be unique. Much of the trouble arises from singularity, either of A or of its leading principal sub matrices.

For a given matrix A is known that:

(a) A has an LU factorization in which L is nonsingular if and only if A has the row inclusion property: For each i = 1, ..., n−1, *A*[{i+1; 1, ..., i}] is a linear combination of the rows of *A*[{1, ..., i}];

(b) *A* has an LU factorization in which U is nonsingular if and only if *A* has the column inclusion property: For each j = 1, ..., n−1, *A*[{1, ..., j; j+1}] is a linear combination of the columns of *A*[{1, ..., j}].

Using this theorem [7], we can give a full description in the nonsingular case, and we can impose a normalization that makes the factorization unique.

The ST decomposition could also be obtained using singular value decomposition, which is applicable to m×n matrix *A*. The decomposition is $A = UDV^T$, where *D* is a nonnegative diagonal matrix, and *U* and *V* are unitary matrices, and V^T denotes the conjugate transpose of *V* (or simply the transpose, if *V* contains real numbers only). From here you can find the pseudo-inverse matrix $A^+ = VD^+U^T$ to be used in the equations (7) - (9).

MATLAB implementation [10] of the ST decomposition to solve totally fuzzy linear regression problems:

```
Input A, M, N;
Input b, g, h;
        [L, U] = lu(A);
        T = L*(U')^(-1)
or (T = L/U'; if U' is not square matrix);
        S = U'*U;
Check A = T*S;
        AI = pinv(A);
        x = AI*b;
        y = AI*(g - M*AI*b);
        z = AI*(h - N*AI*b);
Output x, y, z.
```

IV. EXAMPLES

To illustrate the technique proposed in this paper, consider the following examples.

Example 1: Consider the 3 x 3 fully fuzzy linear system [8, Example 4.1]:

$$\begin{cases}(1,2,5)\otimes\tilde{x}_1\oplus(3,4,4)\otimes\tilde{x}_2\oplus(0,1,2)\otimes\tilde{x}_3=(19,68,115)\\(2,3,5)\otimes\tilde{x}_1\oplus(0,1,11)\otimes\tilde{x}_2\oplus(4,5,6)\otimes\tilde{x}_3=(30,77,261)\\(2,5,7)\otimes\tilde{x}_1\oplus(4,6,6)\otimes\tilde{x}_2\oplus(5,7,10)\otimes\tilde{x}_3=(61,167,253\end{cases}$$

$$A=\begin{pmatrix}1&3&0\\2&0&4\\2&4&5\end{pmatrix},\quad M=\begin{pmatrix}2&4&1\\3&1&5\\5&6&7\end{pmatrix},\quad N=\begin{pmatrix}5&4&2\\5&11&6\\7&6&10\end{pmatrix}$$

and $$b=\begin{pmatrix}19\\30\\61\end{pmatrix},\quad g=\begin{pmatrix}68\\77\\167\end{pmatrix},\quad h=\begin{pmatrix}115\\261\\253\end{pmatrix}$$

Let apply ST decomposition in the following order:

1. Performing LU decomposition of *A*.

We have matrices

$$L=\begin{pmatrix}0.50&0.75&1.00\\1.00&0.00&0.00\\1.00&1.00&0.00\end{pmatrix}\qquad U=\begin{pmatrix}2.00&0.00&4.00\\0.00&4.00&1.00\\0.00&0.00&-2.75\end{pmatrix}.$$

2. The matrix *T* and symmetric positive matrix *S* are

$$T=\begin{pmatrix}0.9773&0.2784&-0.3636\\0.5000&0.0000&0.0000\\0.5000&0.2500&0.0000\end{pmatrix}\qquad S=\begin{pmatrix}4.0000&0.0000&8.0000\\0.0000&16.0000&4.0000\\8.0000&4.0000&24.5625\end{pmatrix}.$$

So we found the ST decomposition. It is easy to check that *T.S* = *A*.

3. We find the pseudo inverse matrix A^+

$$A^+=\begin{pmatrix}0.7273&0.6818&-0.5454\\0.0909&-0.2273&0.1818\\-0.3636&-0.0909&0.2727\end{pmatrix}$$

4. With its help the unknown *x* is calculated by equation (7)

$$\tilde{x}=(1,\ 6,\ 7).$$

To find *y* and *z* we have to use the secondary symmetric matrix S_S, as shown in [8]. A simple way is to apply equations (10), (11) and (12). So we receive

$$\tilde{x}=A^{-1}b=(1,\ 6,7)$$
$$\tilde{y}=M^{-1}g=(5,\ 12,10)$$
$$\tilde{z}=N^{-1}h=(7,\ 14,12)$$

Example 2: Consider the 2 x 3 fully fuzzy linear system [12, Example 1]:

$$\begin{cases}(0.3,0.1,0.2)\otimes\tilde{x}_1\oplus(0.2,0.1,0.3)\otimes\tilde{x}_2\oplus(0.1,0.05,0.2)\otimes\tilde{x}_3=(2,1,3)\\(0.3,0.2,0.1)\otimes\tilde{x}_1\oplus(0.2,0.1,0.1)\otimes\tilde{x}_2\oplus(0.1,0.03,0.3)\otimes\tilde{x}_3=(3,2,1.5)\end{cases}$$

where

$$A=\begin{pmatrix}0.3&0.2&0.1\\0.3&0.2&0.1\end{pmatrix},\quad M=\begin{pmatrix}0.1&0.1&0.05\\0.2&0.1&0.03\end{pmatrix},\quad N=\begin{pmatrix}0.2&0.3&0.2\\0.1&0.1&0.3\end{pmatrix}$$

and $$b=\begin{pmatrix}2\\3\end{pmatrix},\quad g=\begin{pmatrix}1\\2\end{pmatrix},\quad h=\begin{pmatrix}3\\1.5\end{pmatrix}.$$

We apply ST decomposition to matrix *A* and consequently receive

$$L=\begin{pmatrix}1&0\\1&1\end{pmatrix}\quad U=\begin{pmatrix}0.3&0.2&0.1\\0.0&0.0&0.0\end{pmatrix}.$$

The operation $T = L(U')^{-1}$ cannot be performed, as only square matrix arguments are permitted in rising to power. So we shall use *T* = *L/U'* and find

$$T=\begin{pmatrix}2.14286&1.42857&0.71429\\2.14286&1.42857&0.71429\end{pmatrix},\quad S=\begin{pmatrix}0.09&0.06&0.03\\0.06&0.04&0.02\\0.03&0.02&0.01\end{pmatrix}.$$

The pseudo inverse matrix A^+ is

$$A^+=\begin{pmatrix}1.07143&1.07143\\0.71429&0.71429\\0.35714&0.35714\end{pmatrix}.$$

Using Equations (7), (8) and (9) from above we receive

$$\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} 5.3571, \ 0.57398, \ 0.61224 \\ 3.5714, \ 0.38265, \ 0.40816 \\ 1.7857, \ 0.19133, \ 0.20408 \end{pmatrix}.$$

It is the minimal solution of fully fuzzy linear system. The calculations are done with free software package GNU Octave, version 4.4.1.

V. CONCLUSION

In this paper we present a new view of the ST decomposition and its realization through a direct application of LU decomposition without any iterative techniques. By ST decomposition, every nonsingular matrix can be represented as a product of a symmetric matrix *S* and a triangular matrix *T*. Analytical model is used for solving a system of m×n totally fuzzy linear regression (TFLR), where *A* is a nonnegative fuzzy matrix. In particular case when the three matrices *A*, *M* and *N* are n×n nonsingular crisp matrices of rank = n, then a simple solution was given. Two numerical examples clarify the ability of our method. The proposed method is easily applicable with MATLAB or GNU Octave software.

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