Bifurcation Analyse And Control For A Delayed Predator-Prey System With Disease In The Predator

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***Abstract*—In this paper, a predator-prey system with a transmissible disease spreading in the predator population and a time delay due to the feedback period of the prey is considered. By analyzing the corresponding characteristic equations, the explicit formulas that the local stability of each of feasible equilibria and the existence of the Hopf bifurcations at the disease-free equilibrium and the interior equilibrium are established. Furthermore, based on the Lyapunov functional and LaSalle's invariance principle, sufficient conditions are derived for the global stability of interior equilibrium. In addition, by using parameter disturbance and state feedback control to act on the system, the Hopf bifurcation of the system is controlled. Numerical simulations are conducted to validate the previous analytical results.**

Keywords: Predator-prey system; Delay; Hopf bifurcation; Global stability; Hybrid control

I. INTRODUCTION

Dynamics of predator-prey models are one of the important subjects in ecology and mathematical ecology due to its wide existence and importance, these complex biological phenomenons are understood by means of biological models. Common models are as follows: predator–prey harvesting model, predator–prey disease model, predator–prey refuge model and other models. Since the predator-prey system with disease was proposed by Anderson and May in [1], great concern has been put on researching actual mathematical models for transmission dynamics of infectious disease in [2,3,4,5,6]. So far most of the above studies concerned their models that the prey or the predator is infectious only in [7,8,9,10,11,12]. Reference [10], the authors paid attention to the following phenomenons that infectious predators would die of disease and the healthy predators had predation capacity, in addition, the predator would not be able to recover under the predator infected with the disease. Reference [12], formulated the following eco-epidemiological model:

$$\begin{cases} \dfrac{dx(t)}{dt} = x(t)\big(r - a_{11}x(t)\big) - \dfrac{a_{12}x(t)S(t)}{1+mx(t)}, \\[2ex] \dfrac{dS(t)}{dt} = \dfrac{a_{21}x(t)S(t)}{1+mx(t)} - r_1 S(t) - \beta S(t)I(t), \\[2ex] \dfrac{dI(t)}{dt} = \beta S(t)I(t) - r_2 I(t). \end{cases} \quad (1.1)$$

Where $x(t)$ represents the density of the prey at time t, $S(t)$ and $I(t)$ are the density of the susceptible and infected predator at time t respectively. We assume the parameters r, r_1, r_2, a_{11}, a_{12}, a_{21}, m, β are positive constants, in which r denotes the intrinsic growth rate, a_{11} can be defined as internal competition rate, a_{12} is regarded as the capturing rate of the susceptible predator, a_{21}/a_{12} is the conversion rate of nutrients of the predator by consuming prey, only the susceptible predators have ability to capture prey , $m>0$ is the half saturation rate of the predator and the infected predators are unable to catch prey due to a high infection, β is called the disease transmission coefficient, the disease can not be transmitted vertically only by contacting, the infected predator do not recover or become immune, r_1 and r_2 look upon as the death rate of the susceptible predator and the infected predator, we assume $r_1 \le r_2$.

While the effect of time delay in predator-prey systems received a lot of attention, this factor has induced more complicated dynamic characteristics than that without time delay, because the delay could cause stable system to turn to unstable and cause the species to fluctuate. In real situations, there are different time delays of species that affect the predator-prey systems. For instance, the species feedback time delay, the hunting delay, the gestation period of prey or predator. The researches of dynamical behavior of the predator-prey systems with time delay are complicated in [13-20]. In nature, sometimes it is necessary to keep the population at a reasonable level, otherwise this population may lead to the reduction or even extinction of other populations. Hybrid control methods have been widely applied used by researchers in [24-31]. Reference [24], the authors introduced a hybrid control strategy that incorporates sliding mode control and state feedback control techniques via fuzzy logic for synchronization of a class of nonlinear chaotic systems. Reference [26], Liu and Chung investigated a hybrid control strategy for bifurcation in a continuous nonlinear dynamics system without time delay. Furthermore, we can adjust parameters to regulate the system.

Motivated by Guo, Li, Cheng and Li in [12] and based on a hybrid control by combining the state feedback control and perturbation parameter, a controlled system is considered:

$$\begin{cases} \dot{x}(t) = x(t)(r - a_{11}x(t-\tau)) - \dfrac{a_{12}x(t)S(t)}{1+mx(t)}, \\ \dot{S}(t) = \dfrac{a_{21}x(t)S(t)}{1+mx(t)} - r_1 S(t) - \beta S(t)I(t), \\ \dot{I}(t) = \beta S(t)I(t) - r_2 I(t). \end{cases} \quad (1.2)$$

Where the parameters r, r_1, r_2, a_{11}, a_{12}, a_{21}, m, β are defined as in system (1.1). τ represents the feedback time delay of the prey.

The initial conditions for system (1.2) take the form of

$$x(\theta) = \phi_1(\theta), \quad S(\theta) = \phi_2(\theta), \quad I(\theta) = \phi_3(\theta), \quad (1.3)$$

$$\phi_1(0) > 0, \ \phi_2(0) > 0, \ \phi_3(0) > 0,$$

where $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C([-\tau, 0], R^3_{+0})$, the space of continuous functions mapping in the interval $[-\tau, 0]$ into R^3_{+0}, here $R^3_{+0} = \{(x_1, x_2, x_3) : x_i \geq 0, i = 1,2,3\}$.

It is well known by the fundamental theory of functional differential equations in [21], the system (1.2) has a unique solution ($x(t)$, $S(t)$, $I(t)$) satisfying initial conditions (1.3). It is obvious that all solutions of system (1.2) corresponding to initial conditions (1.3) are defined on [0,+ ∞) and remain positive for all $t \geq 0$.

The organization of this paper is as follows. In Section 2, the stability problem of system is investigated, and the existence and direction of the Hopf bifurcation at the internal equilibrium are analyzed. In Section 3, by using the Lyapunov functionals and LaSalle's invariance principle, sufficient conditions are derived for the global stability of interior equilibrium E^*. In section 4, by using the parameter perturbation and the state feedback control method, we succeeded in controlling the Hopf bifurcation of the original system. In Section 5, numerical simulations are carried out to illustrate the theoretical results. Finally, a brief conclusion is concerning the whole analysis.

II. Local stability and Hopf bifurcation

In this section, the focus of our research is the conditions of the local stability and the Hopf bifurcation of each feasible equilibrium of system (1.2) by analyzing the corresponding characteristic equations, respectively.

System (1.2) always has a trivial equilibrium $E_0(0,0,0)$, we can obtain the linear part of system (1.2),

$$\begin{cases} \dot{x} = rx(t), \\ \dot{S} = -r_1 S(t), \\ \dot{I} = -r_2 I(t). \end{cases} \quad (2.1)$$

The linearized matrix of the system (2.1) at $E_0(0,0,0)$ is:

$$J_{(E_0)} = \begin{bmatrix} r & 0 & 0 \\ 0 & -r_1 & 0 \\ 0 & 0 & -r_2 \end{bmatrix}. \quad (2.2)$$

Hence, the corresponding characteristic equation of system (2.1) can be rewritten equivalently as:

$$(\lambda - r)(\lambda + r_1)(\lambda + r_2) = 0. \quad (2.3)$$

It's easy to find the (2.3) always has following eigenvalues $\lambda_1 = r > 0$, $\lambda_2 = -r_1 < 0$, $\lambda_3 = -r_2 < 0$.

That is to say that the trivial equilibrium $E_0(0,0,0)$ is always unstable.

Next, system (1.2) has the disease-free equilibrium $E_1(x_1, S_1, 0)$, if the following holds:

$$(H_1): a_{21}r > r_1(a_{11} + mr). \quad (2.4)$$

The linear part of system (1.2) at $E_1(x_1, S_1, 0)$ is:

$$\begin{cases} \dot{x} = (r - a_{11}x_1 - \dfrac{a_{12}S_1}{(1+mx_1)^2})x(t) - \dfrac{a_{12}x_1}{1+mx_1}S(t) \\ \quad - a_{11}x_1x(t-\tau), \\ \dot{S} = \dfrac{a_{21}S_1}{(1+mx_1)^2}x(t) + (\dfrac{a_{21}x_1}{1+mx_1} - r_1)S(t) \\ \quad - \beta S(t)I(t), \\ \dot{I} = (\beta S - r_2)I(t). \end{cases}$$

(2.5)

The linearized matrix of the system (2.1) at $E_1(x_1, S_1, 0)$ can be described as:

$$\begin{bmatrix} r - a_{11}x_1 - \dfrac{a_{12}S_1}{(1+mx_1)^2} - a_{11}x_1e^{-\lambda\tau} & \dfrac{-a_{12}x_1}{1+mx_1} & 0 \\ \dfrac{a_{21}S_1}{(1+mx_1)^2} & \dfrac{a_{21}x_1}{1+mx_1} - r_1 & -\beta S_1 \\ 0 & 0 & \beta S_1 - r_2 \end{bmatrix}$$

(2.6)

That, the corresponding characteristic equation of system (2.1) is:

$$(\lambda + r_2 - \beta S_1)(\lambda^2 + p_0\lambda + p_1 + (q_0\lambda + q_1)e^{-\lambda\tau}) = 0,$$

(2.7)

where

$$p_0 = -r + a_{11}x_1 + \frac{a_{21}S_1}{(1+mx_1)^2},$$

$$p_1 = \frac{a_{12}S_1r_1}{(1+mx_1)^2},$$

$$q_0 = a_{11}x_1, \quad q_1 = 0.$$

Clearly, (2.7) always has a root $\lambda_1 = \beta S_1 - r_2$. All other roots of (2.7) are determined by the following equation

$$\lambda^2 + p_0\lambda + p_1 + (q_0\lambda + q_1)e^{-\lambda\tau} = 0. \quad (2.8)$$

Next, let's discuss $\tau = 0$ and $\tau > 0$ separately.

Case I: $\tau = 0$.

The (2.8) reduces to

$$\lambda^2 + (p_0 + q_0)\lambda + p_1 = 0.$$

(2.9)

It is easy to know that

$$p_1 = \frac{a_{12}S_1r_1}{(1+mx_1)^2} > 0,$$

$$p_0 + q_0 = -r + 2a_{11}x_1 + \frac{a_{12}S_1}{(1+mx_1)^2}.$$

Hence, if $\beta S_1 < r_2$ and the following holds:

$$(H_2): -r + 2a_{11}x_1 + \frac{a_{12}S_1}{(1+mx_1)^2} > 0,$$

the equilibrium E_1 is locally asymptotically stable when $\tau = 0$.

Case II: $\tau > 0$.

When $\tau > 0$, if $iw(w > 0)$ is a solution of (2.8), separating real and imaginary parts, we can conclude

$$w^2 - p_1 = q_0 w \sin w\tau,$$

$$-p_0 w = q_0 w \cos w\tau. \quad (2.10)$$

From which it follows that

$$w^4 + ({p_0}^2 - 2p_1 - q_0^2)w^2 + p_1^2 = 0. \quad (2.11)$$

Let $h_{11} = p_0^2 - 2p_1 - q_0^2$, $h_{12} = p_1^2$

For convenience, let $v = w^2$, (2.11) becomes

$$v^2 + h_{11}v + h_{12} = 0. \quad (2.12)$$

Denote

$$f(v) = v^2 + h_{11}v + h_{12}. \quad (2.13)$$

We have $f'(v) = 2v + h_{11}$. Since $f(0) = h_{12}$, $\lim_{v\to+\infty} f(v) = +\infty$.

Lemma 2.1 For the polynomial (2.12), we conclude the following results.

a) If $(H_{21}): h_{12} \geq 0, \Delta = h_{11}^2 - 4h_{12} \leq 0$ holds, then (2.12) has no positive root.

b) If $(H_{22}): h_{12} > 0, \Delta = h_{11}^2 - 4h_{12} > 0$, $v^* = (-h_{11} + \sqrt{\Delta})/2$, $f(v^*) \leq 0$ or $(H_{23}): h_{12} < 0$ holds, then (2.12) has positive roots.

Therefore, we suppose that (2.12) has positive roots. Without loss of generality, we assume that it has two positive roots which are denoted by v_1, v_2, then (2.11) has two positive roots $w_n = \sqrt{v_n}, n = 1,2$.

Through the above analysis, one can obtain

$$\tau_n^{(j)} = \frac{1}{w_n}\arccos(-\frac{p_0}{q_0} + \frac{2j\pi}{w_n}), n = 1,2,3...;$$

$$j = 0,1,2...$$

Therefore, $\pm iw$ is a pair of purely imaginary roots of (2.8) with $\tau = \tau_n^{(0)}$, and let $\tau = \min_{n\in[1,2]}\{\tau_n^{(0)}\}$, $w_0 = w_{n_0}$.

Discussion of the roots is similar to [22], we need verify the transversality condition. For (2.8), $\lambda(\tau) = a(\tau) + iw(\tau)$ is the root at $a(\tau_0) = 0$ if $w(\tau_0) = w_0$ satisfied. The following lemma can be obtained.

Lemma2.2 If the condition $(H_{24}): f^{'}(w_0^2) \neq 0$ holds, then $\left(\frac{d(\mathrm{Re}\lambda)}{d\tau}\right)_{\lambda=iw_0} \neq 0$, $\left(\frac{d(\mathrm{Re}\lambda)}{d\tau}\right)_{\lambda=iw_0}$ and $f^{'}(w_0^2)$ have same sign.

Proof: Differentiating (2.8) with respect to τ, the λ is a function of τ, we can obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = -\frac{2\lambda + p_0}{\lambda(\lambda^2 + p_0\lambda + p_1)} + \frac{1}{\lambda^2} - \frac{\tau}{\lambda}.$$

Hence, a direct calculation shows that

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1} = \mathrm{Re}\left(-\frac{2\lambda + p_0}{\lambda(\lambda^2 + p_0\lambda + p_1)}\right)_{\lambda=iw_0} + \mathrm{Re}\left(\frac{1}{\lambda^2}\right)_{\lambda=iw_0}$$

$$= \frac{p_0^2 + 2w_0^2 - 2p_1}{p_0^2 w_0^2 + (p_1 - w_0^2)} - \frac{q_0^2}{w_0^2 q_0^2}.$$

Form (2.8) and (2.13), we have

$$(p_0 w_0^2)^2 + (p_1 - w_0^2)^2 = w_0^2 q_0^2,$$

we find that $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=iw_0}$ and $\left(\frac{d(\mathrm{Re}\lambda)}{d\tau}\right)_{\lambda=iw_0}$ have the same sign. Hence,

$$sign\left(\frac{d(\mathrm{Re}\lambda)}{d\tau}\right)_{\lambda=iw_0} = sign\left(\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right)_{\lambda=iw_0} = sign\left(\frac{2w_0^2 + p_0^2 - 2p_1 - q_0^2}{w_0^2 q_0^2}\right).$$

Therefore, $\left(\frac{d(\mathrm{Re}\lambda)}{d\tau}\right)_{\lambda=iw_0} \neq 0$ if $(H_{24}): f^{'}(w_0^2) \neq 0$ holds.

Theorem 2.1 For system (1.2), if (H_1) and $\beta S_1 < r_2$ satisfies, we address the following results.

a) If (H_2) satisfies, system (1.2) is locally asymptotically stable at disease-free equilibrium $E_1(x_1, S_1, 0)$ for $\tau = 0$.

b) If (H_{21}) satisfies, system (1.2) is locally asymptotically stable at disease-free equilibrium $E_1(x_1, S_1, 0)$ for $\tau \geq 0$; if (H_{22}) or (H_{23}) is satisfied, system (1.2) is locally asymptotically stable at disease-free equilibrium $E_1(x_1, S_1, 0)$ for $\tau \in [0, \tau_0)$; if (H_{22}) or (H_{23}) and (H_{24}) satisfy, system (1.2) is unstable when $\tau > \tau_0$. Furthermore, system (1.2) undergoes a hopf bifurcation at disease-free equilibrium $E_1(x_1, S_1, 0)$ when $\tau = \tau_0$.

Next, in view of the practical significance of biological field, it is necessary to consider the local stability at the interior equilibrium $E^*(x^*, S^*, I^*)$ and the existence of Hopf bifurcation for system (1.2), we imply that prey and susceptible predator and infected predator all exist. If the following holds: $(H_3): r > a_{12}S^*, r_1 < \frac{a_{21}x^*}{1 + mx^*}$, it can be achieved that system (1.2) has unique interior equilibrium $E^*(x^*, S^*, I^*)$, where $S^* = \frac{r_2}{\beta}$, $I^* = \frac{1}{\beta}(\frac{a_{21}x^*}{1 + mx^*} - r_1)$, $x^* = \frac{mr - a_{11} + \sqrt{(mr - a_{11})^2 + 4ma_{11}(r - a_{12}S^*)}}{2ma_{11}}$.

Using the transformation $\bar{x}(t) = x(t) - x^*$, $\bar{y}_1(t) = y_1(t) - y_1^*$, $\bar{y}_2(t) = y_2(t) - y_2^*$ and still denote $\bar{x}(t)$, $\bar{y}_1(t)$, $\bar{y}_2(t)$ by $x(t)$, $y_1(t)$, $y_2(t)$ respectively. Then system (1.2) converts to

$$\begin{cases}\dot{x}(t)=b_{11}x(t)+b_{12}S(t)+c_{11}x(t-\tau) \\ \qquad +\sum_{i+j+k\geq 2} f_1^{(ijk)}x^i(t)S^j(t)x^k(t-\tau), \\ \dot{S}(t)=b_{21}x(t)+b_{22}S(t)+b_{23}I(t) \\ \qquad +\sum_{i+j+k\geq 2} f_2^{(ijk)}x^i(t)S^j(t)I^k(t), \\ \dot{I}(t)=b_{32}S(t)+b_{33}I(t) \\ \qquad +\sum_{i+j\geq 2} f_3^{(ij)}S^i(t)I^j(t),\end{cases}$$

(2.14)

where

$$b_{11}=r-a_{11}x^*-\frac{a_{12}S^*}{(1+m(x^*))^2},$$

$$b_{12}=-\frac{a_{12}x^*}{1+m(x^*)},$$

$$c_{11}=-a_{11}x^*,$$

$$b_{21}=\frac{a_{21}S^*}{(1+mx^*)^2},$$

$$b_{22}=\frac{a_{21}x^*}{1+mx^*}-r_1-\beta I^*,$$

$$b_{23}=-\beta x^*,\quad b_{32}=\beta I^*,\quad b_{33}=\beta S^*-r_2,$$

$$f_1^{(ijk)}=\frac{1}{i!j!k!}\frac{\partial^{i+j+k}f_1}{\partial x^i(t)\partial S^j(t)\partial x^k(t-\tau)}\bigg|(x^*,S^*,I^*),$$

$$f_2^{(ijk)}=\frac{1}{i!j!k!}\frac{\partial^{i+j+k}f_2}{\partial x^i(t)\partial S^j(t)\partial I^k(t)}\bigg|(x^*,S^*,I^*),$$

$$f_3^{(ij)}=\frac{1}{i!j!}\frac{\partial^{i+j}f_3}{\partial S^i(t)I^*(t)}\bigg|(x^*,S^*,I^*),$$

$$f_1=x(t)(r-a_{11}x(t-\tau))-\frac{a_{12}x(t)S(t)}{1+mx(t)},$$

$$f_2=\frac{a_{21}x(t)S(t)}{1+mx(t)}-r_1S(t)-\beta S(t)I(t),$$

$$f_3=\beta S(t)I(t)-r_2I(t).$$

Then we can obtain the linear part of system (2.14) as follows:

$$\begin{cases}\dot{x}(t)=b_{11}x(t)+b_{12}S(t)+c_{11}x(t-\tau), \\ \dot{S}(t)=b_{21}x(t)+b_{22}S(t)+b_{23}I(t), \\ \dot{I}(t)=b_{32}S(t)+b_{33}I(t).\end{cases}$$

(2.15)

Hence, the corresponding characteristic equation of system (2.15) can be written as:

$$\lambda^3+m_0\lambda^2+m_1\lambda+m_2+(n_0\lambda^2+n_1\lambda+n_2)e^{-\lambda\tau}=0,$$

(2.16)

where

$$m_0=-(b_{11}+b_{22}+b_{33}),$$

$$m_1=b_{11}b_{22}+b_{11}b_{33}+b_{22}b_{33}-b_{12}b_{21}-b_{23}b_{32},$$

$$m_2=b_{12}b_{21}b_{33}+b_{11}b_{23}b_{32}-b_{11}b_{22}b_{33},\quad n_0=-c_{11},$$

$$n_1=c_{11}b_{22}+c_{11}b_{33},\quad n_2=c_{11}b_{23}b_{32}-c_{11}b_{22}b_{33}.$$

Further, we consider the local stability of the interior equilibrium $E^*(x^*,S^*,I^*)$ and the existence of Hopf bifurcation. The following two cases are considered.

Case I: $\tau=0$.

The corresponding characteristic (2.16) becomes

$$\lambda^3+m_{01}\lambda^2+m_{11}\lambda+m_{21}=0, \tag{2.17}$$

where

$$m_{01}=m_0+n_0,\quad m_{11}=m_1+n_1,\quad m_{21}=m_2+n_2.$$

It is easy to verify that if (H_3) holds, by using Routh-Hurwitz Theorem, the necessary and sufficient condition that all roots of (2.17) have negative real part is given in the following form: (H_4) $m_{01}>0, m_{01}m_{11}>m_{21}$. When the condition (H_4) holds, the interior equilibrium $E^*(x^*,S^*,I^*)$ is locally asymptotically stable when $\tau=0$.

Case II: $\tau>0$.

The corresponding characteristic equation is (2.16)

$$\lambda^3+m_0\lambda^2+m_1\lambda+m_2+(n_0\lambda^2+n_1\lambda+n_2)e^{-\lambda\tau}=0,$$

Let $iw_1(w_1>0)$ be a root of (2.16), separating real and imaginary parts,one can obtain

$$n_1w_1\sin w_1\tau+(n_2-n_0w_1^2)\cos w_1\tau=m_0w_1^2-m_2,$$

$$n_1w_1\cos w_1\tau-(n_2-n_0w_1^2)\sin w_1\tau=w_1^3-m_1w_1.$$

(2.18)

Squaring and adding the two equations of (2.18), which follows that

$$w_1^6+h_{21}w_1^4+h_{22}w_1^2+h_{23}=0, \tag{2.19}$$

where

$$h_{21}=m_0^2-2m_1-n_0^2,$$

$$h_{22}=m_1^2-2m_0m_2+2n_0n_2-n_1^2,\quad h_{23}=m_2^2-n_2^2.$$

For convenience, let $v_1=w_1^2$, then (2.19) becomes

$$v_1^3+h_{21}v_1^2+h_{22}v_1+h_{23}=0. \quad (2.20)$$

Denote

$$f_1(v_1)=v_1^3+h_{21}v_1^2+h_{22}v_1+h_{23}. \quad (2.21)$$

We have $f_1'(v_1)=3v_1^2+2h_{21}v_1+h_{22}$. Since $f_1(0)=h_{23}$, $\lim_{v_1\to+\infty} f_1(v_1)=+\infty$.

After discussion about the roots of (2.20) is similar to that of Song and Wei in [23], we have following lemma.

Lemma 2.3 For the polynomial (2.20), we have the following results.

a) If $(H_{21}'):h_{23}\ge 0,\Delta=h_{21}^2-3h_{22}\le 0$ satisfies, then (2.20) has no positive root.

b) If $(H_{22}'):h_{23}\ge 0,\Delta=h_{21}^2-3h_{22}>0,$ $v_2^*=\frac{-h_1+\sqrt{\Delta}}{3}>0,\ f_1(v_2^*)\le 0$ or $(H_{23}'):h_{23}<0$ holds, then (2.20) has positive root.

Therefore, we suppose that (2.20) has positive roots. Without loss of generality, we assume that it has three positive roots which are denoted by v_{11}, v_{12} and v_{13}. Then (2.19) has three positive roots $w_{1n}=\sqrt{v_{1n}}, n=1,2,3$.

Through the above analysis, we derive that

$$\tau_{1n}^{(j)}=\frac{1}{w_{1n}}\arccos\left\{\frac{M_1w_{1n}^4+M_2w_{1n}^2-M_3}{M_4w_{1n}^4+M_5w_{1n}^2}\right\}+\frac{2j\pi}{w_{1n}},$$
$$n=1,2,3\ldots j=0,1,2\ldots$$

where

$$M_1=n_1-n_0m_0,$$

$$M_2=m_0n_2+n_0m_2-m_1n_1,$$

$$M_3=m_2n_2,$$

$$M_4=n_0^2,\quad M_5=n_1^2+n_2^2-2n_0n_2.$$

Therefore, $\pm iw_{1n}$ is a pair of purely imaginary roots of (2.16) with $\tau=\tau_{1n}^{(j)}$, and let

$$\tau_{10}=\min_{n\in\{1,2,3\}}\{\tau_{1n}^{(0)}\},\ w_{10}=w_{1n_0}.$$

According to The Hopf bifurcation Theorem in [22], we need verify the transversality condition. For (2.16), $\lambda(\tau)=a(\tau)+iw(\tau)$ is the root at $\tau_1=\tau_{10}$ if $a(\tau_{10})=0$ and $w(\tau_{10})=w_{10}$ hold. We can get following lemma.

Lemma2.4 If the condition (H_{24}') $f_1'(w_{10}^2)\ne 0$ holds, then $\left(\frac{d(\mathrm{Re}\lambda)}{d\tau}\right)_{\lambda=iw_{10}}\ne 0$, $\left(\frac{d(\mathrm{Re}\lambda)}{d\tau}\right)_{\lambda=iw_{10}}$ and $f_1'(w_{10}^2)\ne 0$ have the same sign.

Proof: Differentiating (2.16) with respect to τ, and noticing that λ is a function of τ, it follows that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}=-\frac{3\lambda^2+2m_0\lambda+m_1}{\lambda(\lambda^3+m_0\lambda^2+m_1\lambda+m_2)}+\frac{2n_0\lambda+n_1}{\lambda(n_0\lambda^2+n_1\lambda+n_2)}-\frac{\tau}{\lambda}.$$

Hence, a direct calculation shows that

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=iw_{10}}=\mathrm{Re}\left(-\frac{3\lambda^2+2m_0\lambda+m_1}{\lambda(\lambda^3+m_0\lambda^2+m_1\lambda+m_2)}\right)_{\lambda=iw_{10}}+\mathrm{Re}\left(\frac{2n_0\lambda+n_1}{\lambda(n_0\lambda^2+n_1\lambda+n_2)}\right)_{\lambda=iw_{10}}$$
$$=\frac{3w_{10}^4+2(m_0^2-2m_1)w_{10}^2+m_1^2-2m_0m_2}{(w_{10}^3-m_1w_{10})^2+(m_0w_{10}^2-m_2)^2}-\frac{2n_0^2w_{10}^2+n_1^2-2n_0n_2}{n_1^2w_{10}^2+(n_2-n_0w_{10}^2)^2}.$$

Form (2.18) and (2.19), we derive that

$$(w_{10}^3-m_1w_{10})^2+(m_0w_{10}^2-m_2)^2=n_1^2w_{10}^2+(n_2-n_0w_{10}^2)^2,$$

We find that $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=iw_{10}}$ and $\left(\frac{d(\mathrm{Re}\lambda)}{d\tau}\right)_{\lambda=iw_{10}}$ have the same sign. Hence,

$$sign\left\{\left(\frac{d(\operatorname{Re}\lambda)}{d\tau}\right)\right\}_{\lambda=iw_{10}} = sign\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=iw_{10}} = \frac{3(w_{10}^2)^2+2h_1w_{10}^2+h_2}{n_1^2w_{10}^2+(n_2-n_0w_{10}^2)^2} = \frac{f_1^{'}(w_{10}^2)}{n_1^2w_{10}^2+(n_2-n_0w_{10}^2)^2}.$$

(2.17)

Therefore, $\left(\frac{d(\operatorname{Re}\lambda)}{d\tau_1}\right)_{\lambda=iw_{10}} \neq 0$ if $(H_{24}^{'})$ $f_1^{'}(w_{10}^2)\neq 0$ holds.

Theorem 2.2 For system (1.2). If (H_3) holds, we have the following results.

a) If (H_4) holds, system (1.2) is locally asymptotically stable at interior equilibrium $E^*(x^*,S^*,I^*)$ for $\tau=0$.

b) If $(H_{21}^{'})$ holds, system (1.2) is locally asymptotically stable at interior equilibrium $E^*(x^*,S^*,I^*)$ for all $\tau>0$.

c) If $(H_{22}^{'})$ or $(H_{23}^{'})$ holds, system (1.2) is locally asymptotically stable at interior equilibrium $E^*(x^*,S^*,I^*)$ for $\tau\in[0,\tau_{10})$, if $(H_{22}^{'})$ or $(H_{23}^{'})$ and $(H_{24}^{'})$ hold, the system (1.2) is unstable when $\tau>\tau_{10}$. Furthermore, system (1.2) undergoes a hopf bifurcation at $E^*(x^*,S^*,I^*)$ when $\tau=\tau_{10}$

III. GLOBAL STABILITY ANALYSIS

In this section, we only analyze the situation of interior equilibrium $E^*(x^*,S^*,I^*)$ when $\tau=0$ was established, other circumstances is similar to this case.

Theorem 3.1

If $\frac{a_{12}S^*m}{(1+mx)(1+mx^*)}<a_{11}$, $\frac{a_{21}x}{1+mx}<r_1\xi_2+\beta I^*$ and $\beta S<r_2$ hold, then the model (1.2) is globally asymptotically stable around the interior equilibrium $E^*(x^*,S^*,I^*)$.

Proof: To investigate the global stability of the system (1.2), we consider the following positive definite function around the interior equilibrium $E^*(x^*,S^*,I^*)$,

$$V(x,S,I)=\xi_1\int_{x^*}^{x}\left(\frac{x-x^*}{x}\right)dx+\xi_2\int_0^{S-S^*}\eta d\eta + \xi_3\int_0^{I-I^*}\gamma d\gamma,$$

(3.1)

we denote ξ_1, ξ_2 and ξ_3 are positive constants. From the $V(x,S,I)$, it is clear that the defined function $V(x,S,I)$ is vanish at the interior equilibrium $E^*(x^*,S^*,I^*)$ and is positive with initial points (1.3) in the positive quadrant. Therefore, $E^*(x^*,S^*,I^*)$ is global minimum value of $V(x,S,I)$. Now, taking the time derivative of $V(x,S,I)$ along the solution of (1.2), $\dot{V}(x,S,I)$ is given by

$$\frac{dV(x,S,I)}{dt}=\xi_1(x-x^*)\left(\frac{\dot{x}}{x}\right)+\xi_2(S-S^*)\dot{S}+\xi_3(I-I^*)\dot{I}$$

$$=\xi_1(x-x^*)\left[x(r-a_{11}x)-\frac{a_{12}xS}{1+mx}\right]+\xi_2(S-S^*)\left[\frac{a_{21}xS}{1+mx}-r_1S-\beta SI\right]+\xi_3(I-I^*)(\beta SI-r_2I)$$

$$=\xi_1\left[-a_{11}(x-x^*)+\frac{a_{12}S^*}{1+mx^*}-\frac{a_{12}S}{1+mx}\right](x-x^*)+\xi_2\left[-r_1(S-S^*)+\frac{a_{21}xS}{1+mx}-\frac{a_{21}x^*S^*}{1+mx^*}+\beta S^*I^*-\beta SI\right](S-S^*)+\xi_3\left[-r_2(I-I^*)+\beta SI-\beta S^*I^*\right](I-I^*)$$

$$=\xi_1\left[\frac{a_{12}S^*m}{(1+mx)(1+mx^*)}-a_{11}\right](x-x^*)^2+\xi_2\left[\frac{a_{21}x}{(1+mx)}-r_1-\beta I^*\right](S-S^*)^2+\xi_3(\beta S-r_2)(I-I^*)^2$$
$$+\left(\frac{\xi_2a_{21}S^*}{(1+mx)(1+mx^*)}-\frac{\xi_1a_{12}}{1+mx}\right)(x-x^*)(S-S^*)+(\xi_3\beta I^*-\xi_2\beta S)(S-S^*)(I-I^*).$$

Let us choose , $\xi_1 = a_{12}$; $\xi_2 = \frac{a_{21}S^*}{1+mx^*}$ and $\xi_3 = \frac{a_{12}S}{I^*}$. Then, we have provided $\frac{a_{12}S^*m}{(1+mx)(1+mx^*)} < a_{11}$, $\frac{a_{21}x}{1+mx} < r_1\xi_2 + \beta I^*$, $\beta S < r_2$.

$$\frac{dV(x,S,I)}{dt} = \xi_1\left[\frac{a_{12}S^*m}{(1+mx)(1+mx^*)} - a_{11}\right](x-x^*)^2$$

$$+\xi_2\left[\frac{a_{21}x}{1+mx} - r_1 - \beta I^*\right](S-S^*)^2$$

$$+\xi_3(\beta S - r_2)(I-I^*)^2 \le 0.$$

IV. HOPF BIFURCATION CONTROL

In this section, we will use hybrid control method of choosing a appropriate controller to come true the movement of the bifurcation point. Add the controller to system (1.2), then the controlled system is as follows:

$$\begin{cases} \dot{x}(t) = p(x(t)(r - a_{11}x(t-\tau)) - \frac{a_{12}x(t)S(t)}{1+mx(t)}) \\ \quad + qx(t), \\ \dot{S}(t) = p(\frac{a_{21}x(t)S(t)}{1+mx(t)} - r_1S(t) - \beta S(t)I(t)) \\ \quad + qS(t), \\ \dot{I}(t) = p[\beta S(t)I(t) - r_2I(t)], \end{cases}$$

(4.1)

where p and q are control parameters, $p > 0$, $q \in R$, other parameters are same as system (1.2). In this part, we consider only the relative properties of bifurcation point.

Similarly, if the following holds: $(H_5): a_{12}pr_2 < pr\beta + q\beta, a_{21}p > (pr_1 - q)m$, it is easy to see that system (4.1) has a bifurcation point $E^\circ(x^\circ, S^\circ, I^\circ)$, where $x^\circ = \frac{-C_1 + \sqrt{\Delta}}{2a_{11}pm}$, $S^\circ = \frac{r_2}{\beta}$,

$$I^\circ = \frac{1}{\beta}(\frac{a_{21}x^\circ}{1+mx^\circ} - r_1 + \frac{q}{p}).$$

And

$$C_1 = a_{11}p - (pr+q)m,$$

$$\Delta = C_1^2 - 4a_{11}pm(a_{12}pS^\circ - (pr+q)).$$

The linearized matrix of the system (4.1) at $E^\circ(x^\circ, S^\circ, I^\circ)$ is

$$\begin{bmatrix} L_1 + L_2e^{-\lambda\tau} & L_3 & 0 \\ L_4 & L_5 & L_6 \\ 0 & L_7 & L_8 \end{bmatrix},$$

(4.2)

where

$$L_1 = pr - a_{11}px^\circ - \frac{a_{12}S^\circ p}{(1+mx^\circ)^2} + q,$$

$$L_2 = -a_{11}px^\circ,$$

$$L_3 = -\frac{a_{12}x^\circ p}{1+mx^\circ},$$

$$L_4 = \frac{a_{21}pS^\circ}{(1+mx^\circ)^2},$$

$$L_5 = \frac{pa_{21}x^\circ}{1+mx^\circ} - pr_1 - p\beta I^0 + q,$$

$$L_6 = -p\beta S^0,$$

$$L_7 = p\beta I^0,$$

$$L_8 = p\beta S^0 - pr_2.$$

The characteristic equation of system (4.1) at $E^\circ(x^\circ, S^\circ, I^\circ)$ is

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 + (B_4\lambda^2 + B_5\lambda + B_6)e^{-\lambda\tau_1} = 0$$

(4.3)

where

$$B_1 = -(L_1 + L_5 + L_8),$$

$$B_2 = L_1L_5 + L_1L_8 + L_5L_8 - L_3L_4 - L_6L_7,$$

$$B_3 = L_1L_6L_7 + L_3L_4L_8 - L_1L_5L_8, \quad B_4 = -L_2,$$

$$B_5 = L_2(L_5 + L_8),$$

$$B_6 = L_2L_6L_7 - L_2L_5L_8.$$

Similar to the front, let $iw^\circ(w^\circ > 0)$ be a root of (4.3), separating real and imaginary parts, one can get,

$$\begin{aligned} & B_5 w^\circ \sin w^\circ \tau + (B_6 - B_4 w^2)\cos w^\circ \tau = \\ & B_1 (w^\circ)^2 - B_3 \\ & B_5 w^\circ \cos w^\circ \tau - (B_6 - B_4 (w^\circ)^2)\sin w^\circ \tau = \\ & (w^\circ)^3 - B_2 w^\circ \end{aligned} \quad (4.4)$$

Squaring and adding the two equations of (4.4), which follows that

$$(w^\circ)^6 + D_1 (w^\circ)^4 + D_2 (w^\circ)^2 + D_3 = 0, \quad (4.5)$$

where

$D_1 = B_1^2 - 2B_2 - B_4^2$,

$D_2 = B_2^2 - 2B_1 B_3 + 2B_4 B_6 - B_5^2$, $D_3 = B_3^2 - B_6^2$.

For convenience, let $v^\circ = (w^\circ)^2$, (4.5) becomes

$$(v^\circ)^3 + D_1 (v^\circ)^2 + D_2 v^\circ + D_3 = 0. \quad (4.6)$$

Denote

$$f^\circ(v^\circ) = (v^\circ)^3 + D_1 (v^\circ)^2 + D_2 v^\circ + D_3. \quad (4.7)$$

Since $f^\circ(0) = D_3$, $\lim_{v^\circ \to +\infty} (f^\circ(v^\circ)) = +\infty$. From (4.7), we have $f^{\circ'}(v^\circ) = 3(v^\circ)^2 + 2D_1 v^\circ + D_2$.

Lemma 4.1 For the polynomial (4.6), we have the following results.

a) If $(H_{31}^{'}): D_3 \geq 0, \Delta = D_1^2 - 3D_2 \leq 0$ satisfies, then (4.6) has no positive root.

b) If $(H_{32}^{'}): D_3 \geq 0, \Delta = D_1^2 - 3D_2 > 0, v_1^\circ = (-D_1 + \sqrt{D_1^2 - 3D_2})/3, f^\circ(v_1^\circ) \leq 0$ or $(H_{33}^{'}): D_3 < 0$ satisfies, then (4.6) has positive root.

Therefore, we suppose that (4.6) has positive roots. Without loss of generality, we assume that it has three positive roots which are denoted by v_1°, v_2° and v_3°, then (4.5) has three positive roots $w_n^\circ = \sqrt{v_n^\circ}, n = 1,2,3$.

Through the above analysis, one can get

$$\tau_n^{\circ(j)} = \frac{1}{w_n^\circ} \arccos\left\{ \frac{N_1 (w_n^\circ)^4 + N_2 (w_n^\circ)^2 + N_3}{N_4 (w_n^\circ)^4 + N_5 (w_n^\circ)^2 + N_6} \right\} + \frac{2j\pi}{w_n^\circ},$$

$n = 1,2,3 \ldots j = 0,1,2 \ldots$

where

$N_1 = B_5 - B_1 B_4$,

$N_2 = B_1 B_6 + B_3 B_4 - B_2 B_5$,

$N_3 = -B_3 B_6$,

$N_4 = B_4^2$,

$N_5 = B_5^2 - 2B_4 B_6$,

$N_6 = B_6^2$.

Therefore, $\pm i w_n^\circ$ is a pair of purely imaginary roots of (4.3) with $\tau = \tau_n^{\circ(j)}$, and let $\tau^\circ = \min_{n \in \{1,2,3\}} \{\tau_n^{\circ(0)}\}, w_0^\circ = w_{n_0}^\circ$. Similar to the previous situation, if $(H_{34}^{'}): f^{\circ'}(w_0^\circ)^2 \neq 0$ holds, we have the following results.

Theorem 4.1 For system (4.1), if the condition (H_5) holds, We have the following results.

a) If $(H_{31}^{'})$ holds, the bifurcation point $E^\circ(x^\circ, S^\circ, I^\circ)$ is locally asymptotically stable when $\tau^\circ \geq 0$.

b) If $(H_{32}^{'})$ or $(H_{33}^{'})$,the bifurcation point $E^\circ(x^\circ, S^\circ, I^\circ)$ is locally asymptotically stable when $\tau^\circ \in [0, \tau_0^\circ)$, if $(H_{32}^{'})$ or $(H_{33}^{'})$ and $(H_{34}^{'})$ hold, the bifurcation point $E^\circ(x^\circ, S^\circ, I^\circ)$ is unstable when $\tau_0 > \tau_0^\circ$, furthermore, system (4.1) undergoes a Hopf bifurcation at E° when $\tau_0 = \tau_0^\circ$.

V. NUMERICAL SIMULATION

In this section, some numerical simulations by using Matlab to illustrate the analytical results are displayed, we obtain the corresponding waveform and the phase plots of system (1.2) and system(4.1).

Let $r = 0.55$, $a_{11} = 0.125$, $a_{12} = 1.8$, $a_{21} = 1.35$, $m = 0.01$, $r_1 = 0.17$, $r_2 = 0.25$, $\beta = 0.9$. Then, we have the following particular example of system (1.2):

$$\begin{cases} \dot{x}(t) = x(t)(0.55 - 0.125x(t-\tau)) - \dfrac{1.8x(t)S(t)}{1 + 0.01x(t)}, \\ \dot{S}(t) = \dfrac{1.35x(t)S(t)}{1 + 0.01x(t)} - 0.17S(t) - 0.9S(t)I(t), \\ \dot{I}(t) = 0.9S(t)I(t) - 0.25I(t). \end{cases}$$

(5.1)

Through the above set of parameter values, it is no difficult to get the interior equilibrium point

$E^*(0.4166, 0.2778, 0.4344)$ if the condition (H_3) holds.

For interior equilibrium E^*, we can get $\omega_{10} = 0.5953$, $\tau_{10} = 2.6387$ for $\tau > 0$. From Theorem 2.2, we can obtain the interior equilibrium E^* is locally asymptotically stable when $\tau \in [0, \tau_{10})$, when the time delay τ passes through the critical value τ_{10}, the interior equilibrium E^* will lose its stability and a Hopf bifurcation occurs. The corresponding waveform and the phase plots are depicted in Fig.1 and 2.

Next, let $p = 0.2$, $q = 0.03$, the other parameters are similar as in system (5.1), we have the following particular example of system (4.1):

$$\begin{cases} \dot{x}(t) = 0.2\left(x(t)\left(0.55 - 0.125x(t-\tau)\right) - \dfrac{1.8x(t)S(t)}{1+0.01x(t)} \right) \\ \quad + 0.03x(t), \\ \dot{S}(t) = 0.2\left(\dfrac{1.35x(t)S(t)}{1+0.01x(t)} - 0.17x(t) - 0.9S(t)I(t) \right) \\ \quad + 0.03y_1(t), \\ \dot{I}(t) = 0.2[0.9S(t)I(t) - 0.25I(t)]. \end{cases}$$

(5.2)

Through the above set of parameter values, it is no difficult to get the bifurcation point $E^{\circ}(1.6655, 0.2777, 2.4351)$ if the condition (H_3) and (H_5) hold, then $w_0^{\circ} = 0.2746$, $\tau_0^{\circ} = 5.6857$ for $\tau > 0$. From Theorem 4.1, we can obtain the bifurcation point E° is locally asymptotically stable when $\tau \in [0, \tau_0^{\circ})$ and unstable when $\tau > \tau_0^{\circ}$, the corresponding waveform and the phase trajectory are drawn as shown in Fig.3 and 4.

By using the hybrid control method control the Hopf bifurcation of the system, the time delay is increased from $\tau_{10} = 2.6387$ to $\tau_0^{\circ} = 5.6857$, the bifurcation point change from $(0.4166, 0.2778, 0.4334)$ to $(1.6655, 0.2777, 2.4351)$. By observing Fig.1, Fig.2 and Fig.3, Fig.4, we can obviously observe the bifurcation delay of the system (1.2).

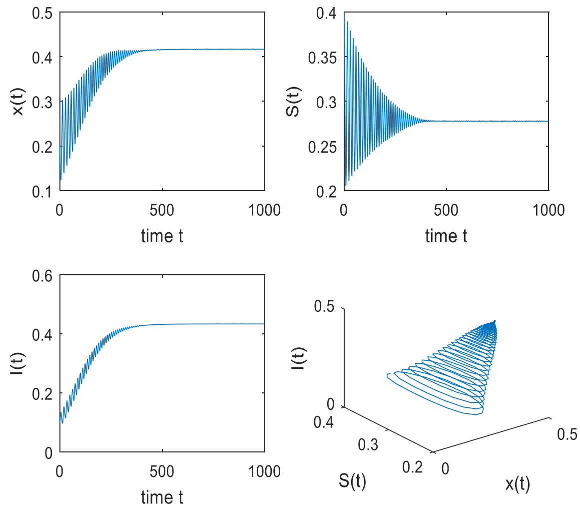

Fig.1 *The interior equilibrium* $E^*(0.4166, 0.2778, 0.4334)$ *is locally asymptotically stable for* $\tau = 2.301 < \tau_{10} = 2.6387$.

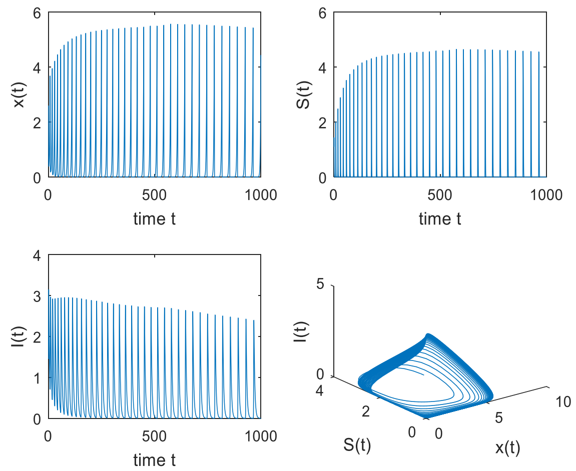

Fig.2 *The Hopf bifurcation occurs from interior equilibrium* $E^*(0.4166, 0.2778, 0.4334)$ *for* $\tau = 3.6671 > \tau_{10} = 2.6387$.

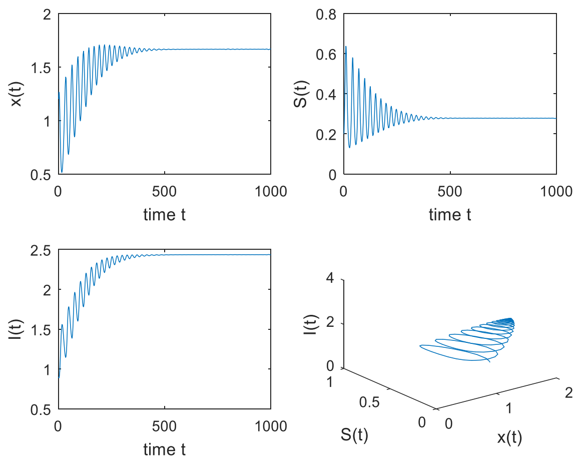

Fig.3 The new bifurcation point $E^{\circ}(1.6655, 0.2777, 2.4351)$ of the controlled system (4.1) is locally asymptotically stable for $\tau = 3.6671 < \tau_0^{\circ} = 5.6857$.

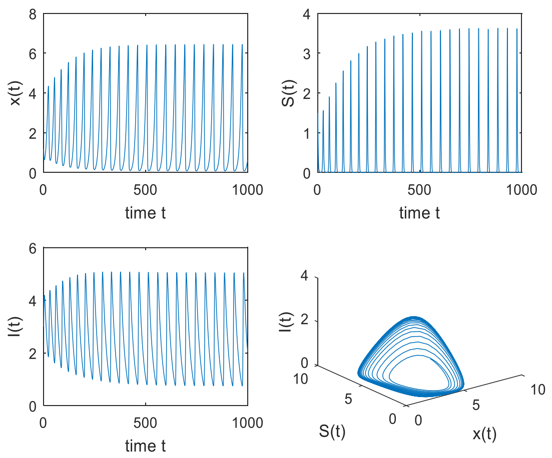

Fig.4 The new bifurcation point $E^{\circ}(1.6655, 0.2777, 2.4351)$ of the controlled system (4.1) is unstable for $\tau = 8.891 > \tau_0^{\circ} = 5.6857$.

VI. Conclusions

In this paper, we have incorporated the disease and time delay into predator-prey model with disease that can be transmitted by contacting spreading among the predator population and a time delay representing the feedback period for prey. By analyzing the corresponding characteristic equations, the local stability of each of feasible equilibria has been established, respectively. It has been shown that,under some conditions, the time delay due to feedback of the prey may destabilize both the disease-free equilibrium E_1 and the interior equilibrium E^* of the system and cause the population to fluctuate. By means of Lyapunov functional and LaSalle's principle, sufficient conditions were obtained for the global asymptotic stability of the interior equilibrium E^* of system (1.2). By choosing time delay as the bifurcation parameters and using the hybrid control method to act on the system, we successfully controlled the Hopf bifurcation of original system. Finally, numerical simulations are given to verity the theoretical analysis.

From the obtained results of this paper, we can predict the dynamical behavior of system by selecting suitable parameters for the predator-prey system with delay. Our research come form two aspects: if the delay is small enough, the system will keep stable in the long run, which suggests the state of system balance; if the delay is large enough, the system will lose stability, which indicates that the densities of the predator and prey population will fluctuate periodically in a range. We are able to control the Hopf bifurcation by determining an appropriate control parameter. For $\tau > 0$, let $p = 0.2$, $q = 0.03$, the critical value of delay increase from $\tau_{10} = 2.6387$ to $\tau_0^{\circ} = 5.6857$, the bifurcation point is changed from $(0.4166, 0.2778, 0.4334)$ to $(1.6655, 0.2777, 2.4351)$.

The model of ecosystem and validation with experimental work and the proper choice of functional response must receive great attention in near future, the analysis and model simulations of this article can be viewed as building blocks toward a comprehensive study and deeper understanding of the fundamental mechanisms in delayed predator-prey dynamics with disease. Furthermore, we can consider others response functions and the stage-structure for prey and predator. How will the dynamic behaviors of the system change? These effects and practical significance of these situations deserve our attention, this will be the focus of our next work.

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