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Analysis of the SIR epidemic model with time delay and nonlinear incidence rate

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***Abstract*—This paper is concerned with an SIR epidemic model with time delay and two different general nonlinear incidence rates. Existence and uniqueness of the global positive solution, extinction and persistence in mean of the epidemic are established. Furthermore, We obtain the threshold between persistence in the mean and extinction of the stochastic system by applying the Itˆo formula and comparison theorem. Compare with the corresponding deterministic system, The threshold of white noise interference is smaller than the basic reproduction number of deterministic system. Finally, numerical simulations shows that time delay has an important impact on the extinction and persistence of the epidemic.**

Keywords—epidemic model; Time delay; Nonlinear incidence; Noise interference; Itˆo formula

I. INTRODUCTION

Infectious disease is an important way to carry out theoretical quantitative research on infectious diseases because of its infectiousness, which seriously endangers people's health and life, causing immeasurable loss to human survival and national economy. Therefore, the study of infectious diseases has always been one of the objects of concern in the community, the control of infectious diseases has become an important topic in epidemiology. Many scholars have established the corresponding mathematical models by using the methodology of infectious diseases. One of the important mathematical models is the partition model, which is according to the disease status of the individual classification. Based on different transmission methods, Kermack and McKendrick are introduced into SIS and SIR model [1–3]. Since then, a number of other relevant models have been proposed to reveal the process of disease diffusion and to provide some relevant control strategies[4,5].

The incidence of diseases plays an important role in the infectious disease model[6–8,12,13].More and more scholars have studied nonlinear incidence which it can more accurately reveal the transmission mechanism of infectious diseases, the nonlinear incidence rates $\frac{\beta_1 S(t) I_1(t)}{\alpha_1 + I_1(t)}$ and $\frac{\beta_2 S(t) I_2(t)}{\alpha_2 + S(t)}$ have been adopted in[7,12],respectively. On the other hand, time delay plays an important role in the spread of infectious diseases, which can be used to describe the changes of individuals in infectious diseases in different periods. Therefore, many models of infectious diseases by using differential equation of nonlinear incidence rate and time delay of the two, so it can make the model better reflect the propagation of diseases, more realistic, so as to prevent and control diseases and provide targeted measures [9–11].

However, in the natural environment, the infectious disease model is always more or less affected by environmental noise [6,14–18]. Therefore, we consider the interference of environmental noise to the infectious disease model in this paper, and establish a stochastic model to predict the system dynamic behavior.

In this paper, we propose a new stochastic differential equation model with time delay and two different nonlinear rates, and prove the existence and uniqueness of the global solution, extinction and persistence in mean.

II. MODEL ESTABLISHMENT AND THEORETICAL ANALYSIS

In this paper, We propose a stochastic SIR model with two different nonlinear incidence rates and time delay. We assume that the contact rates in the model are disturbed and the interference can be described as white noise: $\beta_i \to \beta_i + \delta_i \dot{\mathrm{B}}(t) (i = 1,2)$. Where $\mathrm{B}(t) = (\mathrm{B}_1(t), \mathrm{B}_2(t))$ is the standard Brownian defined in a complete probability space and $\delta_i (i = 1,2)$ is noise intensity. Thus, we establish SIR model in stochastic differential equations:

$$\begin{cases} dS(t) = (A - dS(t) - \frac{\beta_1\langle k\rangle S(t)I_1(t)}{\mu_1 + I_1(t)} - \frac{\beta_2\langle k\rangle S(t)I_2(t)}{\mu_2 + S(t)} \\ \quad +\gamma_1 I_1(t-\tau_1)e^{-d\tau_1} + \gamma_2 I_2(t-\tau_2)e^{-d\tau_2})dt \\ \quad - \frac{\delta_1\langle k\rangle S(t)I_1(t)}{\mu_1 + I_1(t)} dB_1(t) - \frac{\delta_2\langle k\rangle S(t)I_2(t)}{\mu_2 + S(t)} dB_2(t) \\ dI_1(t) = \left(\frac{\beta_1\langle k\rangle S(t)I_1(t)}{\mu_1 + I_1(t)} - (d+\alpha_1+\gamma_1)I_1(t)\right)dt \\ \quad + \frac{\delta_1\langle k\rangle S(t)I_1(t)}{\mu_1 + I_1(t)} dB_1(t) \\ dI_2(t) = \left(\frac{\beta_2\langle k\rangle S(t)I_2(t)}{\mu_2 + S(t)} - (d+\alpha_2+\gamma_2)I_2(t)\right)dt \\ \quad + \frac{\delta_2\langle k\rangle S(t)I_2(t)}{\mu_2 + S(t)} dB_2(t) \\ dR(t) = (\gamma_1 I_1(t) + \gamma_2 I_2(t) - dR(t) - \gamma_1 I_1(t-\tau_1)e^{-d\tau_1} \\ \quad - I_2(t-\tau_2)e^{-d\tau_2})dt \end{cases} \quad (1)$$

Where A is the total input population size, $S(t)$ stands for the number of the individual susceptible to the disease, $I_1(t)$ and $I_2(t)$ represents the total population of infectives infected with virus A_1 and A_2 at time t ,respectively. $R(t)$ denotes the people who is immune to diseases. d is the natural death rate of population, $\langle k\rangle$ is the average degree. α_1 and α_2 are the death rate due to diseases, γ_1 and γ_2 are the immunization rate of I_1 and I_2 , β_1 and β_2 are the contact rates, respectively. $\tau_i(i=1,2)$ is the length of immunity period. $\frac{\beta_1 S(t)I_1(t)}{\mu_1+I_1(t)}$ and $\frac{\beta_2 S(t)I_2(t)}{\mu_2+S(t)}$ represents the different types saturated incidence rates, where μ_1 and μ_2 are the half-saturation constants.

Throughout this paper, we let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions which are increasing and right continuous while $\mathcal{F}_0$ contains all $\mathcal{P}-\mathrm{null}$ sets. In addition, $B_i(t)(i=1,2)$ is the standard Brownian motions defined on the complete probability space and $R_+^n = \{x \in R^n | x_i > 0, 1 \leq i \leq n\}$.

A. Existence and uniqueness of the global positive solution

Theorem 1. For any initial solution $S(0), I_1(0), I_2(0)$, there exists a unique positive solution $(S(t), I_1(t), I_2(t))$ of system (1) on $t \geq 0$, and the solution will remain in R_+^3 with probability one for all $t \geq 0$.

Proof First we set $inf\emptyset = \infty$ ($\emptyset$ is an empty set). Since the coefficients of system (1) satisfy the local Lipschitz conditions, then for any initial value $(S(0), I_1(0), I_2(0)) \in R_+^3$, there exists a unique solution on $t \in [0, \tau_e)$, where τ_e is the explosion time. In order to show the solution is global,we only need to prove $\tau_e = \infty$ almost surely(a.s.).To this end, let $k_0 > 1$ be enough large so that $S(t), I_1(t)$ and $I_2(t)$ all lie within the interval $[1/k_0, k_0]$. For every integer $k > k_0$,we define the stopping time

$$\tau_k = \inf\{t \in [0, \tau_e) | S(t) \in (1/k, k) \; or \; I_i(t) \notin (1/k, k), i = 1,2\}$$

Clearly, τ_k is increasing as $k \to \infty$. let $\tau_\infty = \lim_{k\to\infty} \tau_k$,whence $\tau_\infty \leq \tau_e$ a.s. If $\tau_\infty = \infty$ a.s. is true, then $\tau_e = \infty$a.s. and $(S(t), I_1(t), I_2(t)) \in R_+^3$ a.s. for all $t \geq 0$. If not, there exists a pair of constants $T \geq 0$ and $\varepsilon \in (0,1)$ such that

$$P(\tau_\infty \leq T) > \varepsilon$$

Define

$$V(S, I_1, I_2) = \left(S - a - a\ln\frac{S}{a}\right) + (I_1 - 1 - \ln I_1) + (I_2 - 1 - \ln I_2) + \gamma_1 e^{-d\tau_1}\int_{t-\tau_1}^{t} I_1(s)ds + \gamma_2 e^{-d\tau_2}\int_{t-\tau_2}^{t} I_2(s)ds$$

where a is a positive constant.

By applying the Itˆo formula we can get

$$\begin{aligned} dV(S, I_1, I_2) &= \left(1 - \frac{a}{S}\right)dS + \frac{a}{2S^2}(dS)^2 + \left(1 - \frac{1}{I_1}\right)dI_1 \\ &\quad + \frac{1}{2I_1^2}(dI_1)^2 + \left(1 - \frac{1}{I_2}\right)dI_2 + \frac{1}{2I_2^2}(dI_2)^2 \\ &\quad + \gamma_1 I_1 e^{-d\tau_1} - \gamma_1 I_1(t-\tau_1)e^{-d\tau_1} \\ &\quad + \gamma_2 I_2 e^{-d\tau_2} - \gamma_2 I_2(t-\tau_2)e^{-d\tau_2} \\ &= LV(S, I_1, I_2)dt + \frac{\delta_1\langle k\rangle S(I_1 - 1)}{\mu_1 + I_1}dB_1(t) \\ &\quad + \frac{\delta_2\langle k\rangle S(I_2 - 1)}{\mu_2 + S}dB_2(t) \\ &\quad - \frac{\delta_1\langle k\rangle (S - a)I_1}{\mu_1 + I_1}dB_1(t) \\ &\quad - \frac{\delta_2\langle k\rangle (S - 1)I_2}{\mu_2 + S}dB_2(t) \end{aligned}$$

Where

$$\begin{aligned} LV(S, I_1, I_2) &= \left(1 - \frac{a}{S}\right)\left(A - dS(t) - -\frac{\beta_1\langle k\rangle S(t)I_1(t)}{\mu_1 + I_1(t)} - \frac{\beta_2\langle k\rangle S(t)I_2(t)}{\mu_2 + S(t)} + \gamma_1 I_1(t-\tau_1)e^{-d\tau_1} + \gamma_2 I_2(t-\tau_2)e^{-d\tau_2}\right) \\ &\quad + \frac{a}{2}\left(\frac{\delta_1^2 I_1^2\langle k\rangle^2}{(\mu_1 + I_1)^2} + \frac{\delta_2^2 I_2^2\langle k\rangle^2}{(\mu_2 + S)^2}\right) \\ &\quad + \left(1 - \frac{1}{I_1}\right)\left(\frac{\beta_1\langle k\rangle S(t)I_1(t)}{\mu_1 + I_1(t)} - (d + \alpha_1 + \gamma_1)I_1(t)\right) \\ &\quad + \left(1 - \frac{1}{I_2}\right)\left(\frac{\beta_2\langle k\rangle S(t)I_2(t)}{\mu_2 + S(t)} - (d + \alpha_2 + \gamma_2)I_2(t)\right) \\ &\quad + \frac{1}{2}\left(\frac{\delta_1^2 S^2\langle k\rangle^2}{(\mu_1 + I_1)^2} + \frac{S^2\langle k\rangle^2}{(\mu_2 + S)^2}\right) \\ &\quad + \gamma_1 I_1 e^{-d\tau_1} - \gamma_1 I_1(t-\tau_1)e^{-d\tau_1} \\ &\quad + \gamma_2 I_2 e^{-d\tau_2} - \gamma_2 I_2(t-\tau_2)e^{-d\tau_2} \end{aligned}$$

Then we obtain

$$dV(S,I_1,I_2) \le Kdt + \frac{\delta_1\langle k\rangle S(I_1-1)}{\mu_1+I_1}dB_1(t) + \frac{\delta_2\langle k\rangle S(I_2-1)}{\mu_2+S}dB_2(t) - \frac{\delta_1\langle k\rangle (S-a)I_1}{\mu_1+I_1}dB_1(t) - \frac{\delta_2\langle k\rangle (S-1)I_2}{\mu_2+S}dB_2(t)$$

Integrating both sides from 0 to $\tau_k \wedge T$ and taking expectations yields

$$EV(S,I_1,I_2) \le V(S(0),I_1(0),I_2(0)) + KT$$

K is a constant. We set $\Omega_k = \{\ \tau_k \le T\}$ for any positive $k \ge k_1$,then $P(\Omega_k) \ge \varepsilon$, and there is at least one of $(S(\tau_k,v),I_1(\tau_k,v),I_2(\tau_k,v))$ equals k or $1/k$ for every $v \in \Omega_k$. Thus, $V(S(\tau_k,v),I_1(\tau_k,v),I_2(\tau_k,v))$ is no less than either $\min\{k-1-lnk, 1/k-1+lnk\}$.

Therefore, we get

$$(S(\tau_k,v),I_1(\tau_k,v),I_2(\tau_k,v)) \ge [k-1-lnk] \wedge [1/k-1+lnk]$$

Then, we obtain

$$V(S(0),I_1(0),I_2(0)) + KT \ge E[I_{\Omega_k}V(S(\tau_k,v),I_1(\tau_k,v),I_2(\tau_k,v))] \ge \varepsilon[k-1-lnk] \wedge [1/k-1+lnk]$$

Where I_{Ω_k} is the indicator function of Ω_k, then let $k \to \infty$, we can get

$$\infty > V(S(0),I_1(0),I_2(0)) = \infty$$

Which is a contradiction. Hence, we get $\tau_\infty = \infty$.The proof is completed.

B. Extinction

Lemma 1 Let $(S(t),I_1(t),I_2(t),R(t))$ be a solution of system (1) and $(S(t),I_1(t),I_2(t),R(t)) \in R_+^4$,then $\lim_{t\to+\infty}\frac{\int_0^t \frac{\delta_1 S(\tau)}{\mu_1+I_1(\tau)}dB_1(\tau)}{t} = 0$, $\lim_{t\to+\infty}\frac{\int_0^t \frac{\delta_2 S(\tau)}{\mu_2+S(\tau)}dB_2(\tau)}{t} = 0$,

$\lim_{t\to+\infty}\frac{\int_0^t \delta_1 S(\tau)dB_1(\tau)}{t} = 0$, a.s.

Proof The proof is similar to that in Chang etal.[19] and hence is omitted.

For the sake of simplicity, we define $\langle x(t)\rangle = \frac{1}{t}\int_0^t x(s)ds$. In addition

$$d\left(h + \gamma_1 e^{-d\tau_1}\int_{t-\tau_1}^{t} I_1(s)ds + \gamma_2 e^{-d\tau_2}\int_{t-\tau_2}^{t} I_2(s)ds\right)$$
$$= A - dS - (d+\alpha_1)I_1 + \gamma_1(1-\tau_1)e^{-d\tau_1}I_1 + (d+\alpha_2)I_2 + \gamma_2(1-\tau_2)e^{-d\tau_2}I_2$$

Therefore, we can get

$$\langle S(t)\rangle = \frac{A-(d+\alpha_2+\gamma_2(1-\tau_2)e^{-d\tau_2})}{d}\langle I_2(t)\rangle - \frac{(d+\alpha_1+\gamma_1(1-\tau_1)e^{-d\tau_1})}{d}\langle I_1(t)\rangle - \varphi(t) \quad (2)$$

Where $h = S(t)+I_1(t)+I_2(t)$;

$$\varphi(t) = \frac{S(t)+I_1(t)+I_2(t)+R(t)}{t} + \frac{\gamma_1 e^{-d\tau_1}\int_{t-\tau_1}^{t} I_1(s)ds}{t} + \frac{\gamma_2 e^{-d\tau_2}\int_{t-\tau_2}^{t} I_2(s)ds}{t} - \frac{S(0)+I_1(0)+I_2(0)+R(0)}{t} - \frac{\gamma_1 e^{-d\tau_1}\int_{-\tau_1}^{0} I_1(s)ds}{t} - \frac{\gamma_2 e^{-d\tau_2}\int_{-\tau_2}^{0} I_2(s)ds}{t}$$

Let

$$R_1^* = \frac{\beta_1 A\langle k\rangle}{d\mu_1(d+\alpha_1+\gamma_1)} - \frac{\delta_1^2 A^2\langle k\rangle^2}{2d^2\mu_1^2(d+\alpha_1+\gamma_1)},$$

$$R_2^* = \frac{\beta_2 A\langle k\rangle}{(d\mu_2+A)(d+\alpha_1+\gamma_1)} - \frac{\delta_2^2 A^2\langle k\rangle^2}{2(d\mu_2+A)^2(d+\alpha_1+\gamma_1)}$$

Where $R_i^*(i=1,2)$ is the thresholds of the stochastic system (1).

Theorem 2 Let $(S(t),I_1(t),I_2(t),R(t))$ be the solution of system(1) with initial value$(S(0),I_1(0),I_2(0),R(0)) \in R_+^4$. Then

$$(a)\ \lim_{t\to+\infty} I_i(t) < 0\ if\ \delta_i > \frac{\sqrt{2}\beta_i}{2(d+\mu_i+\gamma_i)}, i=1,2.$$

$$(b)\ \lim_{t\to+\infty} I_i(t) = 0\ if\ R_i^* < 1\ and\ \delta_i > \sqrt{\frac{2\mu_i\beta_i}{A\langle k\rangle}}, i=1,2.$$

Furthermore, $\lim_{t\to+\infty}\langle S(t)\rangle = \frac{A}{d}$.a.s.

Proof (a) Using Itˆo formula to system (1), we get

$$dlnI_1(t) = \left(\frac{\beta_1\langle k\rangle S(t)}{\mu_1+I_1(t)} - \frac{\delta_1^2 S^2(t)\langle k\rangle^2}{2(\mu_1+I_1(t))^2}\right)dt - (d+\alpha_1+\gamma_1)dt + \frac{\delta_1\langle k\rangle S(t)}{\mu_1+I_1(t)}dB_1(t) \quad (3)$$
$$\le (\frac{\sqrt{2}\beta_1}{2\delta_1} - (d+\alpha_1+\gamma_1))dt + \frac{\delta_1\langle k\rangle S(t)}{\mu_1+I_1(t)}dB_1(t)$$

Integrating from 0 to t and dividing by t on both sides, then

$$\frac{lnI_1(t)}{t} \le \frac{\sqrt{2}\beta_1}{2\delta_1} - (d+\alpha_1+\gamma_1) + \frac{M_1(t)}{t} + \frac{lnI_1(0)}{t}$$

Where $M_1(t) = \int_0^t \frac{\delta_1 S(t)}{\mu_1+I_1(t)}dB_1(t)$. By Lemma 1, we know that $\lim_{t\to+\infty}\frac{M_1(t)}{t} = 0$ a.s.

Since $\delta_1 > \frac{\sqrt{2}\beta_1}{2(d+\mu_1+\gamma_1)}$,taking the limit superior of both sides yields

$$\lim_{t\to+\infty} sup\frac{lnI_1(t)}{t} \leq -\left[(d+\alpha_1+\gamma_1)-\frac{\sqrt{2}\beta_1}{2\delta_1}\right] < 0, a.s.$$

Similarly, we obtain

$$dlnI_2(t) = \left(\frac{\beta_2\langle k\rangle S(t)}{\mu_2+S(t)} - \frac{\delta_2^2 S^2(t)\langle k\rangle^2}{2(\mu_2+S(t))^2}\right)dt$$

$$-(d+\alpha_2+\gamma_2)dt + \frac{\delta_2\langle k\rangle S(t)}{\mu_2+S(t)}dB_2(t) \qquad (4)$$

Integrating from 0 to t and dividing by t on both sides, then we can get

$$\frac{lnI_2(t)}{t} \leq \frac{\sqrt{2}\beta_2}{2\delta_2} - (d+\alpha_2+\gamma_2) + \frac{M_2(t)}{t} + \frac{lnI_2(0)}{t}$$

Where $M_2(t) = \int_0^t \frac{\delta_2 S(t)}{\mu_2+S(t)} dB_2(t)$, and $\lim_{t\to+\infty}\frac{M_2(t)}{t} = 0$ a.s.

Since $\delta_2 > \frac{\sqrt{2}\beta_2}{2(d+\mu_2+\gamma_2)}$,then taking the limit superior of both sides leads to

$$\lim_{t\to+\infty} sup\frac{lnI_2(t)}{t} \leq -\left[(d+\alpha_2+\gamma_2)-\frac{\sqrt{2}\beta_2}{2\delta_2}\right] < 0, a.s.$$

Theorem 2 shows that when $\delta_i > \frac{\sqrt{2}\beta_i}{2(d+\mu_i+\gamma_i)}, i = 1,2$ the two infectious diseases of system (1) die out almost surely, that is to say, the large white noise stochastic disturbance can cause the two epidemics to go to extinct.

This completes the proof.

Proof (b) For both sides of (3),integrating from 0 to t and dividing by t yields

$$\frac{lnI_1(t)}{t}$$

$$\leq \frac{\beta_1\langle k\rangle A}{\mu_1 d} - (d+\alpha_1+\gamma_1) - \frac{\delta_1^2 A^2\langle k\rangle^2}{2d^2\mu_1^2} + \frac{M_1(t)}{t} + \frac{lnI_1(0)}{t}$$

$$= (d+\alpha_1+\gamma_1)\left(\frac{\beta_1 A\langle k\rangle}{d\mu_1(d+\alpha_1+\gamma_1)} - \frac{\delta_1^2 A^2\langle k\rangle^2}{2d^2\mu_1^2(d+\alpha_1+\gamma_1)} - 1\right) + \frac{M_1(t)}{t} + \frac{lnI_1(0)}{t}$$

Since $R_1^* < 1$ *and* $\delta_1 > \sqrt{\frac{2\mu_1\beta_1}{A\langle k\rangle}}$,then taking the superior limit leads to

$$\lim_{t\to+\infty} sup\frac{lnI_1(t)}{t} \leq (d+\alpha_1+\gamma_1)(R_1^*-1) < 0$$

Which implies

$$\lim_{t\to+\infty} I_1(t) = 0 \qquad (5)$$

Similarly, we also can get

$$\lim_{t\to+\infty} I_2(t) = 0 \qquad (6)$$

Observing (2), (5) and (6),we obtain

$$\lim_{t\to+\infty}\langle S(t)\rangle$$

$$= \frac{A}{d} - \frac{d+\alpha_2+\gamma_2(1-\tau_2)e^{-d\tau_2}}{d}\lim_{t\to+\infty}\langle I_2(t)\rangle - \frac{(d+\alpha_1+\gamma_1(1-\tau_1)e^{-d\tau_1})}{d}\lim_{t\to+\infty}\langle I_1(t)\rangle - \lim_{t\to+\infty}\varphi(t)$$

Then $\lim_{t\to+\infty}\langle S(t)\rangle = \frac{A}{d}$.a.s.

C. Persistence in mean

Theorem 3 Let $(S(t), I_1(t), I_2(t))$ be a solution of system (1) and$(S(t), I_1(t), I_2(t)) \in R_+^3$,

(a) *if* $R_1^* > 1, R_2^* < 1$ *and* $\delta_2 \leq \sqrt{\frac{2\beta_2(\mu_2 d+A)}{A\langle k\rangle}}$,then I_1 is persistence in mean and I_2 go to extinct. Furthermore,

$$\lim_{t\to+\infty} inf\langle I_1(t)\rangle \geq \frac{d\mu_1(d+\alpha_1+\gamma_1)(R_1^*-1)}{\beta_1(d+\alpha_1+\gamma_1(1-e^{-d\tau_1})) + d(d+\alpha_1+\gamma_1)}, a.s.$$

(b) *if* $R_2^* > 1, R_1^* < 1$ *and* $\delta_1 \leq \sqrt{\frac{2\mu_1 d\beta_1}{A\langle k\rangle}}$, then I_2 is persistence in mean and I_1 goes extinct. Furthermore,

$$\lim_{t\to+\infty} inf\langle I_2(t)\rangle \geq \frac{(d\mu_2+A)(d+\alpha_2+\gamma_2)(R_2^*-1)}{\beta_2(d+\alpha_2+\gamma_2(1-e^{-d\tau_2}))}, a.s.$$

(c) *if* $R_1^* > 1, R_2^* > 1$, then I_1 and I_2 are persistent in mean. Furthermore,

$$\lim_{t\to+\infty} inf\langle I_1(t)+I_2(t)\rangle \geq \frac{1}{h_{max}}[\mu_1(d+\alpha_1+\gamma_1)(R_1^*-1) + (d+\alpha_2+\gamma_2)(R_2^*-1)] > 0, \text{a.s.}$$

Where

$$h_{max} = \max\{\langle k\rangle\left(\beta_1 + \frac{\beta_2 d}{\mu_2 d+A}\right)\frac{(d+\alpha_1+\gamma_1(1-e^{-d\tau_1}))(d+\alpha_1+\gamma_1)}{d}, \langle k\rangle\left(\beta_1+\frac{\beta_2 d}{\mu_2 d+A}\right)\frac{d+\alpha_1+\gamma_1(1-e^{-d\tau_1})}{d}\}$$

Proof (a) By Theorem 2,we can get $0 < I_2(t) < \varepsilon (0 < \varepsilon < 1)$,then

$$\langle S(t)\rangle \geq \frac{A-(d+\alpha_2+\gamma_2(1-\tau_2)e^{-d\tau_2})}{d}\varepsilon - \frac{(d+\alpha_1+\gamma_1(1-\tau_1)e^{-d\tau_1})}{d}\langle I_1(t)\rangle$$

An application of Itˆo formula yields

$$d(\mu_1 lnI_1(t) + I_1(t)) = [\beta_1\langle k\rangle S(t) - \mu_1(d+\alpha_1+\gamma_1) - (d+\alpha_1+\gamma_1)I_1(t) - \frac{\mu_1\delta_1^2 S^2(t)\langle k\rangle^2}{2(\mu_1+I_1(t))^2}]dt + \delta_1\langle k\rangle S(t)dB_1(t)$$

$$\geq [\beta_1\langle k\rangle S(t) - \mu_1(d+\alpha_1+\gamma_1) - (d+\alpha_1+\gamma_1)I_1(t) - \frac{\delta_1^2 A^2\langle k\rangle^2}{2\mu_1 d^2}]dt + \delta_1\langle k\rangle S(t)dB_1(t)$$

Integrating from 0 to$t$ and diving by t on both sides leads to

$$\frac{\mu_1(lnI_1(t) - lnI_1(0))}{t} + \frac{I_1(t) - I_1(0)}{t}$$

$$\geq \beta_1\langle k\rangle S(t)-\mu_1(d+\alpha_1+\gamma_1)-(d+\alpha_1+\gamma_1)\langle I_1(t)\rangle -\frac{\delta_1^2A^2\langle k\rangle^2}{2\mu_1 d^2}+\frac{N_1(t)\langle k\rangle}{t}$$

$$\geq \mu_1(d+\alpha_1+\gamma_1)[\frac{(A-(d+\alpha_2+\gamma_2(1-\tau_2)e^{-d\tau_2}))\varepsilon}{d\mu_1(d+\alpha_1+\gamma_1)} \beta_1\langle k\rangle-\frac{\delta_1^2A^2\langle k\rangle^2}{2\mu_1 d^2(d+\alpha_1+\gamma_1)}-1]$$
$$-[\frac{\beta_1\langle k\rangle}{d\left(d+\alpha_1+\gamma_1(1-e^{-d\tau_1})\right)}+(d+\alpha_1+\gamma_1)] \langle I_1(t)\rangle-\frac{\beta_1\langle k\rangle}{d}\varphi(t)+\frac{N_1(t)\langle k\rangle}{t}$$

Let $\varepsilon\to 0$, the above inequality can be written as follows

$$\langle I_1(t)\rangle \geq \frac{1}{h_1}[\mu_1(d+\alpha_1+\gamma_1)(R_1^*-1)-\frac{\beta_1\langle k\rangle}{d}\varphi(t)+\frac{N_1(t)\langle k\rangle}{t} - \frac{\mu_1(lnI_1(t)-lnI_1(0))}{t}-\frac{I_1(t)-I_1(0)}{t}$$

$$\geq \begin{cases} \frac{1}{h_1}[\mu_1(d+\alpha_1+\gamma_1)(R_1^*-1)-\frac{\beta_1\langle k\rangle}{d}\varphi(t)+\frac{N_1(t)\langle k\rangle}{t} - \frac{\mu_1 lnI_1(t)}{t}-\frac{I_1(t)-I_1(0)}{t}, & 0<I_1(t)<1 \\ \frac{1}{h_1}[\mu_1(d+\alpha_1+\gamma_1)(R_1^*-1)-\frac{\beta_1\langle k\rangle}{d}\varphi(t)+\frac{N_1(t)\langle k\rangle}{t} - \frac{\mu_1(lnI_1(t)-lnI_1(0))}{t}-\frac{I_1(t)-I_1(0)}{t}, & I_1(t)\geq 1 \end{cases}$$

Where $h_1=\frac{\beta_1\langle k\rangle\left(d+\alpha_1+\gamma_1(1-e^{-d\tau_1})\right)}{d}+(d+\alpha_1+\gamma_1)$,

$N_1(t)=\int_0^t \delta_1 S(\tau)dB_1(\tau)$.

$$\lim_{t\to+\infty} inf\langle I_1(t)\rangle \geq \frac{\mu_1(d+\alpha_1+\gamma_1)}{h}(R_1^*-1), a.s.$$

(b) Similarly, we have

$$\langle S(t)\rangle \geq \frac{A-(d+\alpha_2+\gamma_2(1-\tau_2)e^{-d\tau_2})}{d}\langle I_2(t)\rangle -\frac{(d+\alpha_1+\gamma_1(1-\tau_1)e^{-d\tau_1})}{d}\varepsilon$$

By Itˆo formula, we get

$$dlnI_2(t)\geq\left(\frac{\beta_2\langle k\rangle S(t)}{\mu_2+\frac{A}{d}}-\frac{\delta_2^2S^2(t)\langle k\rangle^2}{2\left(\mu_2+\frac{A}{d}\right)^2}\right)dt -(d+\alpha_2+\gamma_2)dt+\frac{\delta_2\langle k\rangle S(t)}{\mu_2+S(t)}dB_2(t)$$

Make integral from 0 to t and divide the two sides by t yields

$$\frac{lnI_2(t)-lnI_2(0)}{t}$$

$$\geq \frac{\beta_2\langle k\rangle A}{d\mu_2+A}-\frac{\delta_2^2A^2\langle k\rangle^2}{2\left(\mu_2+\frac{A}{d}\right)^2}-(d+\alpha_2+\gamma_2)+\frac{N_2(t)\langle k\rangle}{t}$$
$$-\frac{\beta_2\langle k\rangle\varphi(t)}{d\mu_2+A}-\frac{\beta_2\langle k\rangle A(d+\alpha_1+\gamma_1(1-\tau_1)e^{-d\tau_1})}{d\mu_2+A}\varepsilon -\frac{\beta_2\langle k\rangle (d+\alpha_2+\gamma_2(1-\tau_2)e^{-d\tau_2})}{d\mu_2+A}\langle I_2(t)\rangle$$

Let $\varepsilon\to 0$, then

$$\langle I_2(t)\rangle \geq \frac{1}{h_2}[(d+\alpha_2+\gamma_2)(R_2^*-1)+\frac{N_2(t)\langle k\rangle}{t} -\frac{\beta_2\langle k\rangle\varphi(t)}{d\mu_2+A}-\frac{lnI_2(t)-lnI_2(0)}{t}$$

$$\geq \begin{cases} \frac{1}{h_2}[(d+\alpha_2+\gamma_2)(R_2^*-1)+\frac{N_2(t)\langle k\rangle}{t}-\frac{\beta_2\langle k\rangle\varphi(t)}{d\mu_2+A} - \frac{lnI_2(0)}{t}, & 0<I_2(t)<1 \\ \frac{1}{h_2}[(d+\alpha_2+\gamma_2)(R_2^*-1)+\frac{N_2(t)\langle k\rangle}{t}-\frac{\beta_2\langle k\rangle\varphi(t)}{d\mu_2+A} - \frac{lnI_2(t)-lnI_2(0)}{t}, & I_2(t)\geq 1 \end{cases}$$

Where

$h_2=\frac{\beta_2\langle k\rangle\left(d+\alpha_2+\gamma_2(1-\tau_2)e^{-d\tau_2}\right)}{d\mu_2+A}$,

$N_2(t)=\int_0^t \delta_2 S(\tau)dB_2(\tau)$.

Therefore,

$$\lim_{t\to+\infty} inf\langle I_2(t)\rangle \geq \frac{(d+\alpha_2+\gamma_2)}{h_2}(R_2^*-1), a.s.$$

(c) Define $V(t)=\mu_1 lnI_1(t)+I_1(t)+lnI_2(t)$

$$dV(t)\geq\left[\beta_1\langle k\rangle S(t)-(\mu_1+I_1(t))(d+\alpha_1+\gamma_1)-\frac{\delta_1^2A^2\langle k\rangle^2}{2\mu_1 d^2}\right]dt$$
$$+\delta_1\langle k\rangle S(t)dB_1(t)+[\frac{\beta_2\langle k\rangle S(t)}{\mu_2+\frac{A}{d}}-\frac{\delta_2^2S^2(t)\langle k\rangle^2}{2\left(\mu_2+\frac{A}{d}\right)^2}$$
$$-(d+\alpha_2+\gamma_2)]dt+\frac{\delta_2\langle k\rangle S(t)}{\mu_2+S(t)}dB_2(t)$$

Make integral from 0 to t and divide the two sides by t leads to

$$\frac{V(t)}{t}-\frac{V(0)}{t}$$

$$\geq \beta_1\langle k\rangle S(t)-(\mu_1+I_1(t))(d+\alpha_1+\gamma_1)-\frac{\delta_1^2A^2\langle k\rangle^2}{2\mu_1 d^2} +\frac{N_1(t)\langle k\rangle}{t}+\frac{N_2(t)\langle k\rangle}{t}-\frac{\beta_1\langle k\rangle\varphi(t)}{d}-\frac{\beta_2\langle k\rangle\varphi(t)}{d\mu_2+A}$$

$$+\frac{\beta_2\langle k\rangle\langle s(t)\rangle}{\mu_2+\frac{A}{d}}-(d+\alpha_2+\gamma_2)$$

$$-\frac{\delta_2^2A^2\langle k\rangle^2}{2(d\mu_2+A)^2}+\frac{\delta_2\langle k\rangle S(t)}{\mu_2+S(t)}dB_2(t)$$

$$\geq\frac{\beta_1A\langle k\rangle}{d}+\frac{\beta_2A\langle k\rangle}{d\mu_2+A}-\frac{\delta_1^2A^2\langle k\rangle^2}{2\mu_1d^2}-\frac{\delta_2^2A^2\langle k\rangle^2}{2(d\mu_2+A)^2}$$

$$-(d+\alpha_1+\gamma_1)\mu_1-(d+\alpha_2+\gamma_2)+\frac{N_1(t)\langle k\rangle}{t}$$

$$+\frac{N_2(t)\langle k\rangle}{t}-h_{max}[\langle I_1(t)\rangle+\langle I_2(t)\rangle]$$

$$-\frac{\beta_1\langle k\rangle\varphi(t)}{d}-\frac{\beta_2\langle k\rangle\varphi(t)}{d\mu_2+A}$$

The inequality can be rewritten as

$$\langle I_1(t)\rangle+\langle I_2(t)\rangle$$

$$\geq\frac{1}{h_{max}}[\frac{\beta_1A\langle k\rangle}{d}+\frac{\beta_2A\langle k\rangle}{d\mu_2+A}-\frac{\delta_1^2A^2\langle k\rangle^2}{2\mu_1d^2}-\frac{\delta_2^2A^2\langle k\rangle^2}{2(d\mu_2+A)^2}$$

$$+\frac{N_2(t)\langle k\rangle}{t}-\left(\frac{\beta_1\langle k\rangle}{d}+\frac{\beta_2\langle k\rangle}{d\mu_2+A}\right)\varphi(t)-\frac{V(t)}{t}+\frac{V(0)}{t}$$

$$-(d+\alpha_1+\gamma_1)\mu_1-(d+\alpha_2+\gamma_2)+\frac{N_1(t)\langle k\rangle}{t}$$

Taking the inferior limit of both sides yields

$$\lim_{t\to+\infty} inf\langle I_1(t)+I_2(t)\rangle$$

$$\geq\frac{1}{h_{max}}[\mu_1(d+\alpha_1+\gamma_1)(R_1^*-1)$$

$$+(d+\alpha_2+\gamma_2)(R_2^*-1)]>0, a.s.$$

This completes the proof of Theorem 3.

III. NUMERICAL SIMULATION

In this section, we discuss the density changes of $I_1(t)$, $I_2(t)$ and $R(t)$ are in different immune time in the homogeneous network through numerical simulation. In order to carry out the simulation model better, we construct a homogeneous network with a population number of 9×10^6.

Choose the parameters in system (1) as follows: $A=1,\mu_1=0.9,\mu_2=0.9,d=0.25,\beta_1=0.58,\beta_2=0.78,\alpha_1=0.1,\alpha_2=0.1,\gamma_1=0.1,\gamma_2=0.15,\delta_1=\delta_2=0.$

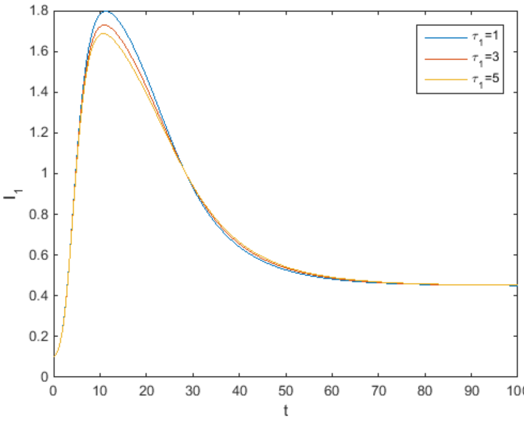

Fig. 1. *The density of $I_1(t)$ under different τ_1*

Figure 1 shows the changes of $I_1(t)$ under different τ_1. From the figure we can see that the smaller of τ_1, the shorter the time for $I_1(t)$ to reach its peak, and the smaller of τ_1, the greater the peak of $I_1(t)$. After $I_1(t)$ reaches its peak, it decreases gradually and finally tends to a stable state.

Figure 2 discusses the density of $I_2(t)$ under different τ_2. By observing, we can find that the larger the τ_2, the slower of $I_2(t)$, the smaller the peak. then the gradual stability after the peak of $I_2(t)$. From this we can conclude that $I_2(t)$ is negatively correlated with τ_2 before reaching the peak.

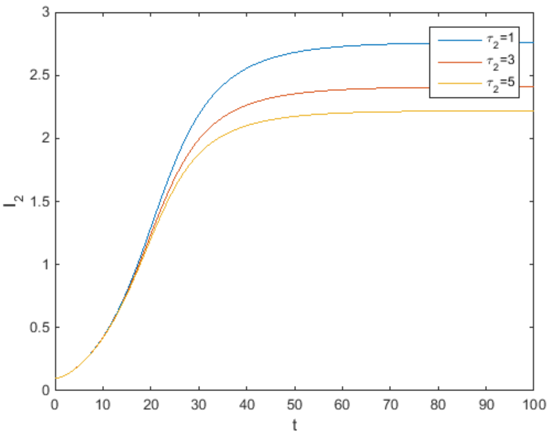

Fig. 2. *The density of $I_2(t)$under different τ_2*

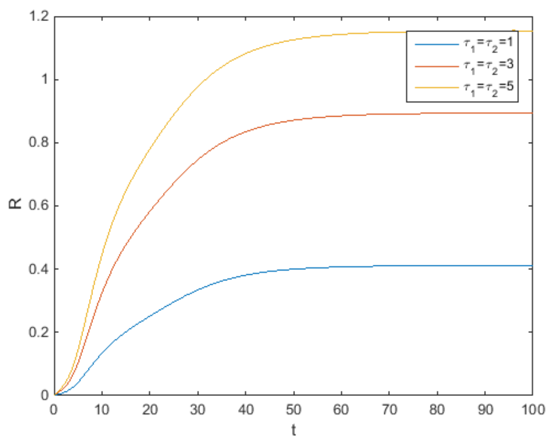

Fig. 3. *The density of $R(t)$under different τ_1 and τ_2*

Figure 3 describes the changes of $R(t)$ with different τ_1 *and* τ_2. From the graph, we can see: with the increase of τ_1 *and* τ_2, $R(t)$ also increases, and then reaches the peak value. At last, it tends to be stable. In addition, we find that when $R(t)$ reaches the stable state, the larger of τ_1 *and* τ_2, the greater of $R(t)$ in the stable state.

IV. CONCLUSION

In this paper, firstly, we propose a new stochastic differential equation model with time delay and two different nonlinear rates, then we complete the theoretical proof and numerical simulation. Finally, we draw the following conclusions:.

1) under the great noise interference, the two diseases of System (1) *eventually tend to become extinct.*

2) Before reaching the peak, the individual density of the two diseases was negatively correlated with the length of the immune period. After reaching the peak, the density of $I_1(t)$ *gradually descends and tends to be stable, and the density of* $I_2(t)$ *gradually tends to be stable after reaching the peak. So in order to better control the disease, we should appropriately increase the length of the immune period.*

3) The density of the immune individual is proportional to the immune time, which shows that the appropriate increase of the immune time can effectively control the disease.

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