Analysis of the stability and hopf bifurcation of a two-delayed SEIR model with general nonlinear incidence rate and saturated recovery rate

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***Abstract*—In this paper, we mainly investigated the two-delayed SEIR epidemiological model with general nonlinear incidence rate and saturated recovery rate. Using the basic reproduction number and the Liapunov function, we discussed the local and global asymptotic stability of the disease-free equilibrium. Further, by different values of the redefined basic reproduction numbers,we studied the local asymptotic stability and Hopf bifurcation at the endemic equilibrium in six different cases. Besides, using the center manifold and normal form theorem, we derived an explicit algorithm that determines the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions. Ultimately, some numerical simulations are performed to confirm the correctness of the theoretical analysis above.**

Keywords—epidemiological model; basic reproduction number; local asymptotically stability; global asymptotically stability; Hopf bifurcation

I. INTRODUCTION

In epidemiology, mathematical models have been proved significant in better understanding of transmissible nature and disease control for more than a century. In 1760, Daniel Bernoulli formulated and solved the first differential equation evaluating the effectiveness of smallpox virus. Follow Kermack and McKendrick [1] built up a system of ordinary differential equations to study disease transmission. And then, Anderson and May [2] summarized the basis and evolution of communicable diseases. What's more, in recent years, a tremendous number of the SIS, SIR, SEI, SEIR, SEIS, SIRS, SEIRS models have been established, analyzed and applied to a variety of infectious diseases[3-14].

When exploring the character of the disease propagation , Many mathematical researchers found causative agents would incubate in their host for a period of time before it is infected, this phenomenon is time delay. Whereupon, for practical consideration, it is suggested that time delay can be incorporated in related models, such as predator-prey models [15,16], critical care model [17], bio-economic phytoplankton zoo-plankton model [18] and epidemic models [6,8-10,12,19,20] extensively. Based on those work above, in this paper, we are absorbed in epidemic models with necessary delay.

In epidemic models, an incidence rate attracted considerable attention in verifying that the model can give better kinetic properties. A bilinear incidence rate [21] and a standard incidence rate [22] were usually used in some epidemic models. Specially, The Holling type II and Holling type III functions were introduced into some predator-prey models by Lianwang Deng [16] and Xuedi Wang [15]. Furthermore, Jiang and Ma [12] proposed a SEIR system with general nonlinear incidence rate, which is a class of nonlinear incidence rate. Jiang and Ma gave the detailed dynamic behavior, which showed that the applicability of the general nonlinear incidence rate is more wider. So it's meaningful to corporate with the general nonlinear incidence rate in the study of epidemic models.

Jingan Cui [23] found an interesting dynamical behavior, backward bifurcation, by proposing a SIS epidemic model with saturation recovery $cI(t)[b+I(t)]^{-1}$ from infected individuals, which made a better understanding of the effect of limited resources for treatment of the infected on the emergency disease control. Three years later, C and Z [11] proposed a SEIR epidemic model with saturated recovery rate, by compound matrices and geometric approaches, they further studied the existence of bi-stability by a backward bifurcation and global dynamical properties.

In the present paper, we will introduce three key factors (two delays, a general incidence rate and a saturate recovery rate) into a SEIR epidemic model, investigating the local and global stability and Hopf bifurcations. The detail arrangement is as follows :

In section 2, the author investigated the local and the global stability of the disease-free equilibrium by the defined basic reproduction number. Besides, in section 3, the locally asymptotic stability of the endemic equilibrium and Hopf bifurcation were certified. What's more, the direction of Hopf bifurcations and the stability of the bifurcating periodic solutions were studied in section 4. In addition, in section 5, some numerical simulations were carried out to support the theoretical results, and some conclusions were obtained in the last section.

II. Local and Global Asymptotic Stability of the Disease-free Equilibrium

Based on the former work made by Zhou [11] and Jiang [12], in this paper, the following two delayed SEIR epidemic model with general nonlinear incidence rate and saturated recovery rate will be discussed.

$$\begin{cases} \dot{S} = \mu - \beta f(I(t))S(t) + \gamma e^{-d_4\tau_1} I(t-\tau_1) - d_1 S(t) \\ \dot{E} = \beta f(I(t))S(t) - \beta e^{-d_2\tau_2} f(I(t-\tau_2))S(t-\tau_2) - d_1 E(t) \\ \dot{I} = \beta e^{-d_2\tau_2} f(I(t-\tau_2))S(t-\tau_2) - (d_3+d+\gamma) I(t) - cI(t)[b+I(t)]^{-1} \\ \dot{R} = \gamma I(t) + cI(t)[b+I(t)]^{-1} - \gamma e^{-d_4\tau_1} I(t-\tau_1) - d_4 R(t) \end{cases} \quad (2.1)$$

where μ represents the recruitment rate of the susceptible population by birth or migration, β stands for the infect rate from the susceptible class to the infected class, γ is the recovery rate, $d_i (i=1,2,3,4)$ denote the natural death rates of the suspected, exposed, infected and recovered population, respectively, where $d_1 \le \min\{d_2, d_3, d_4\}$, d is the extra disease-related death rate, $\beta f(I(t))S(t)$ denotes the general nonlinear incidence rate, $\gamma e^{-d_4\tau_1} I(t-\tau_1)$ indicates the individuals who survived from natural death in a recovery pool before becoming susceptible again. Further, τ_1, τ_2 stands for the immunity period and the latent period respectively. $cI(t)[b+I(t)]^{-1}$ denotes the saturated recovery rate from the infected compartment with hospital treatments, here b gives the infected size at which is 50% saturation ($h(b)=c/2$), and c denotes the maximum recovery per unite of time. And μ, β, γ, d_1, d_2, d_3, d_4, c, b, d are positive, τ_1, τ_2 are non-negative and the function $f(I)$ satisfies the following conditions :

(H_1) $f(0)=0$, $\lim_{I\to+\infty} f(I) = c < \infty$, $f'(I) > 0$, $f''(I) \le 0$ for $I \ge 0$.

In this section, we will prove the local and the global stability of the disease-free equilibrium by the basic reproduction number. For the dynamical properties of system (2.2) and system (2.1) are *parallel, we considering the following system :*

$$\begin{cases} \dot{S} = \mu - \beta f(I(t))S(t) + \gamma e^{-d_4\tau_1} I(t-\tau_1) - d_1 S(t) \\ \dot{I} = \beta e^{-d_2\tau_2} f(I(t-\tau_2))S(t-\tau_2) - (d_3+d+\gamma) I(t) - cI(t)[b+I(t)]^{-1} \end{cases} \quad (2.2)$$

The disease-free equilibrium of system (2.2) is $E_0(S_0, 0)$, where $S_0 = \mu d_1^{-1}$. Definite the basic reproduction number [24] as

$$R_{01} = \beta\mu d_1^{-1} e^{-d_2\tau_2} f'(0)[d_3+d+\gamma+cb^{-1}]^{-1},$$

we can conclude theorem 2.1 as follows :

Theorem 2.1. Consider system (2.2) . if the basic reproduction number $R_{01} < 1$ and $\tau_2 = 0$, the disease-free equilibrium E_0 is locally asymptotically stable; on the contrary, the disease-free equilibrium is unstable, if $R_{01} > 1$ when $\tau_2 \ge 0$.

Proof. Linearizing (2.2) at the disease-free equilibrium :

$$\begin{cases} \dot{S} = -d_1 S(t) + (-\beta\mu d_1^{-1} f'(0) + \gamma e^{-(d_4+\lambda)\tau_1}) I(t) \\ \dot{I} = (\beta\mu d_1^{-1} e^{-(d_2+\lambda)\tau_2} f'(0) - (d_3+d+\gamma+cb^{-1})) I(t) \end{cases} \quad (2.3)$$

Then, the eigenvalues of (2.3) can be worked out respectively as follows :

$$\lambda_1 = -d_1, \lambda_2 = \beta\mu d_1^{-1} e^{-(d_2+\lambda_2)\tau_2} f'(0) - (d_3+d+\gamma+cb^{-1}).$$

Since the eigenvalue $\lambda_1 = -d_1 < 0$, we investigate the other eigenvalue.

Let $f_1(\lambda) = \lambda - \beta\mu d_1^{-1} e^{-(d_2+\lambda)\tau_2} f'(0) + (d_3+d+\gamma+cb^{-1})$. When $\tau_2 = 0$, we can get that $\lambda_2 = (d_3+d+\gamma+cb^{-1})(R_{01} - 1) < 0$ if $R_{01} < 1$. Hence, the disease-free equilibrium E_0 is locally asymptotically stable. when $\tau_2 \ge 0$, it has $f_1(0) = (d_3+d+\gamma+cb^{-1})(1 - R_{01}) < 0$ if $R_{01} > 1$, in addition, $\lim_{\lambda\to+\infty} f_1(\lambda) = +\infty$, so the function $f_1(\lambda) = 0$ has at least a positive root, and the disease-free equilibrium is unstable if $R_{01} > 1$. This completes the proof.

Furthermore, the globally asymptotic stability of E_0 can be chalked up by setting up a Lyapunov function in the following.

Theorem 2.2. If (H_{21}) $h_2 < 0, h_1 > 0$, (H_{22}) $\mu + \gamma e^{-d_4\tau_1} I(t-\tau_1) < \beta S I f'(0)$ (H_{23}) $\beta e^{-d_2\tau_2} S(t-\tau_2) I(t-\tau_2) f'(I(t-\tau_2)) < (d_3+d+\gamma)I - cI(b+I)^{-1}$ where $h_1 = \gamma\, e^{-(d_4+\lambda)\tau_1} - \beta f'(0)\mu d_1^{-1}$, $h_2 = \beta f'(0)\mu d_1^{-1} e^{-(d_2+\lambda)\tau_2} - (d_3+d+\gamma+cb^{-1})$ .are satisfied, the disease-free equilibrium E_0 of subsystem (2.2) is globally *asymptotically stable when $R_{01} < 1$.*

Proof. Considering system (2.2) in Ω_1, where $\Omega_1 = \{(S, I) \in C \mid 0 \le S + I \le S_0\}$. Define the *Lyapunov function as follows :*

$$V(S,I)=(h_2S-h_1I)^2[h_2d_1(h_2-d_1)]^{-1}-S^2(h_2-d)^{-1}.$$

In fact, if $\lambda=0$, it can be verified that $V(S_0,0)=h_2S_0^2[d_1(h_2-d_1)]^{-1}-S_0^2\ (h_2-d)^{-1}>0$ when $R_{01}<1$. On the contrary, if $\lambda\neq 0$, we can obtain that $V(S_0,0)=-\ S_0^2(d_1-h_2)^{-1}(h_2d_1^{-1}-1)>0$ when $h_2\neq d_1$ and $\lim\limits_{|\varphi|\to+\infty}V(S,I)=+\infty$. Thus, $V(S,I)>0$ is satisfied .

Differentiating $V(S,I)$ *as the following formulation :*

$$\dot V(S,I)=[h_2d_1(h_2-d_1)]^{-1}[2h_2(h_2-h_1)S[\mu+\gamma e^{-d_4\tau_1}I(t-\tau_1)-d_1S-\beta f(I)S+2h_1h_2I[\beta SIf'(0)-\mu-\gamma e^{-d_4\tau_1}I(t-\tau_1)]+2h_1h_2S[(d_1+d_3+d+\gamma)I+cI(b+I)^{-1}-\beta e^{-d_2\tau_2}S(t-\tau_2)f(I(t-\tau_2))]+2h_1^2I[\beta e^{-d_2\tau_2}S(t-\tau_2)I(t-\tau_2)f'(0)-(d_1+d_3+d+\gamma)I-cI(b+I)^{-1}]].$$

For that (H_{21}), (H_{22}) *and* (H_{23}) *are contented, we can get* $\dot V(S,I)<0$ *. Consequently,* E_0 *is globally asymptotically stable. This completes the proof.*

Observed that $R_{01}\leq 1$ *is equivalent to* $\beta\leq e^{d_2\tau_2}(d_3+d+\gamma+\frac{c}{b})[\mu d_1^{-1}f'(0)]^{-1}=\beta^*$ *, from theorem 2.2, it's not difficult to summarize that the disease will be eradicated if the infection rate* β *is less than some critical value.*

III. The locally asymptotically stability of the endemic equilibrium and Hopf bifurcation

In this section, we devoted to analysis the local asymptotic stability of the endemic equilibrium of system (2.2) and the Hopf bifurcations. For system (2.2), assume the following conditions hold,

$$(H_{31})(d_1+\beta)c+(d_3+d+\gamma)[b(d_1+\beta)+d_1]<\beta e^{-d_2\tau_2}(\mu+b\gamma e^{-d_4\tau_1}),$$

$$(H_{32})\ \Delta_1=a_2^2-4a_1a_3\geq 0.$$

then, there exists a positive endemic equilibrium $E^*(S^*,I^*)$.

where

$$S^*=[d_3+d+\gamma+c(b+I^*)^{-1}](1+I^*)\beta^{-1}e^{d_2\tau_2},$$

$$I^*=[-a_2+\sqrt{a_2^2-4a_1a_3}](2a)^{-1}.$$

where

$$a_1=(d_1+\beta)(d_3+d+\gamma)-\beta\gamma e^{-d_4\tau_1-d_2\tau_2},$$

$$a_2=(d_1+\beta)[b(d_3+d+\gamma)+c]+d_1(d_3+d+\gamma)-\beta e^{-d_2\tau_2}(\mu+b\gamma e^{-d_4\tau_1}),$$

$$a_3=d_1[b(d_3+d+\gamma)+c]-b\mu\beta e^{-d_2\tau_2}.$$

Redefine the basic reproduction number [14] as

$$R_{02}=\beta f'(I^*)S^*e^{-d_2\tau_2}[d_3+d+\gamma+bc(b+I^*)^{-2}]^{-1}.$$

Let $x(t)=S(t)-S^*$, $y(t)=I(t)-I^*$ and still denotes $S(t)$, $I(t)$ respectively. Substituting them into (2.2), by Taylor expansion,expanding it at the endemic equilibrium $E^*(S^*,I^*)$, then, the standard linearized differential equations at $(x,y)=(0,0)$ is :

$$\begin{cases}\dot x(t)=-[d_1+\beta f(I^*)]x(t)+[\gamma e^{-(d_4+\lambda)\tau_1}-\beta f'(I^*)S^*]y(t),\\ \dot y(t)=\beta f(I^*)e^{-(d_2+\lambda)\tau_2}x(t)-[(d_3+d+\gamma)+bc(b+I^*)^{-2}-\beta e^{-(d_2+\lambda)\tau_2}f'(I^*)S^*]y(t).\end{cases}$$

(3.1)

the corresponding characteristic equation of (3.1) is

$$\lambda^2+b_1\lambda+b_2+(m_1\lambda+m_2)e^{-\lambda\tau_2}+n_1e^{-\lambda(\tau_2+\tau_1)}=0$$

(3.2)

where

$$b_1=[d_1+\beta f(I^*)]+d_3+d+\gamma+bc(b+I^*)^{-2},$$

$$b_2=[d_1+\beta f(I^*)][d_3+d+\gamma+bc(b+I^*)^{-2}],$$

$$m_1=-\beta e^{-d_2\tau_2}f'(I^*)S^*,$$

$$m_2=-d_1\beta f'(I^*)S^*e^{-d_2\tau_2},$$

$$n_1=\beta\gamma f(I^*)e^{-(d_2\tau_2+d_4\tau_1)}.$$

Since system (3.1) has two delays : τ_1 and τ_2, Therefore, we usually investigate six cases : (1) $\tau_1=\tau_2=0$; (2) $\tau_1>0$, $\tau_2=0$; (3) $\tau_1=0$, $\tau_2>0$; (4) $\tau_1=\tau_2=\tau>0$; (5) $\tau_1>0$, $\tau_2\in[0,\tau_{20})$ and $\tau_1\neq\tau_2$; (6) $\tau_1\in[0,\tau_{10})$, $\tau_2>0$ and $\tau_1\neq\tau_2$.

Firstly, we consider case (1) : $\tau_1=\tau_2=0$. The basic reproduction number can be redefined as $R_{02}^{(1)}=\beta f'(I^*)S^*[d_3+d+\gamma+bc(b+I^*)^{-2}]^{-1}$, correspondingly,the transcendental equation of (3.2) can be reproduced as

$$\lambda^2+b_{11}\lambda+b_{12}=0$$

where

$$b_{11}=b_1+m_1, b_{12}=b_2+m_2+n_1.$$

If $R_{02}^{(1)}<1$, by the Routh-Hurwitz theorem, for $\Delta_1=b_{11}=[d_1+\beta f(I^*)]+[d_3+d+\gamma+bc\,(b+I^*)^{-2}](1-R_{02}^{(1)})>0$, we are able to sum up theorem 3.1 :

Theorem 3.1. When $\tau_1=\tau_2=0$, the endemic equilibrium $E^*(S^*,I^*)$ is asymptotically stable if $R_{02}^{(1)}<1$.

case (2) : $\tau_1>0$, $\tau_2=0$. The redefined basic reproduction is $R_{02}^{(2)}(R_{02}^{(2)}=R_{02}^{(1)})$, the characteristic equation of system (3.1) at $(0,0)$ is

$$\lambda^2+b_{21}(\tau_1)\lambda+b_{22}(\tau_1)+m_{21}(\tau_1)e^{-\lambda\tau_1}=0.$$

(3.3)

where

$$b_{21}(\tau_1)=b_1+m_1=[d_1+\beta f(I^*)]+[d_3+d+\gamma+bc(b+I^*)^{-2}](1-R_{02}^{(2)}),$$

$b_{22}(\tau_1)=b_2+m_2=[d_1(1-R_{02}^{(2)})+\beta f(I^*)]$
$[d_3+d+\gamma+bc(b+I^*)^{-2}]$,
$m_{21}(\tau_1)=-\beta\gamma f(I^*)e^{-d_4\tau_1}$.

Assume that $R_{02}^{(2)}<1$ is fulfilled for all $\tau_1\ge 0$. Let $\lambda=i\omega(\omega>0)$ be a root of equation (3.3), substituting it into (3.3) and separating the real and imaginary parts yield that

$$\sin\omega\tau_1=\frac{\omega b_{21}(\tau_1)}{m_{21}(\tau_1)},\cos\omega\tau_1=\frac{\omega^2-b_{22}(\tau_1)}{m_{21}(\tau_1)},$$

(3.4)

this leads to

$$g(\omega^2,\tau_1):=\omega^4+c_{21}(\tau_1)\omega^2+c_{22}(\tau_1)=0$$

(3.5)

where

$c_{21}(\tau_1)=b_{21}^2(\tau_1)-2b_{22}(\tau_1)$,
$c_{22}(\tau_1)=b_{22}^2(\tau_1)-m_{21}^2(\tau_1)$.

Let $v=\omega^2$, and then (3.5) becomes

$$g(v,\tau_1):=v^2+c_{21}(\tau_1)v+c_{22}(\tau_1)=0$$

(3.6)

Since $g(0,\tau_1)=c_{22}(\tau_1)$, $\lim_{v\to+\infty}g(v,\tau_1)=+\infty$, by discussing the roots of (3.6), we have the following theorem 3.2 :

Theorem 3.2. If (H_{33}) $\Delta=c_{21}^2-4c_{22}<0$ and $c_{21}\ge 0$ hold, the unary quadratic equation of (3.6) has no positive root; On the contrary, if (H_{34}) $\Delta=c_{21}^2-4c_{22}\ge 0$ and $c_{21}<0$ hold, then equation (3.6) has positive root.

Assume that Equation $g(v,\tau_1)=0$ has two positive real roots denoted by $v_1(\tau_1)$ and $v_2(\tau_1)$.

Accordingly, equation $g(\omega^2,\tau_1)=0$ will also has two positive real roots, which can be denoted by $\omega_1(\tau_1)=\sqrt{v_1(\tau_1)}$ and $\omega_2(\tau_1)=\sqrt{v_2(\tau_1)}$.

Define

$$I=\{\tau_1\ge 0: c_{21}(\tau_1)<0, c_{21}^2(\tau_1)-4c_{22}(\tau_1)\ge 0\}$$

and let

$$\tau_{1k}^j=\frac{1}{\omega_{1k}}\arccos\{\frac{(\omega_{1k}^2-b_2)}{b_3}\}+\frac{2j\pi}{\omega_{1k}}.$$

$k=1,2.\quad j=0,1,2,\cdots.$

Hence, the characteristic equation of (3.3) has pure imaginary roots $\pm i\omega_{1k}$ on interval I .Define $\tau_{10}=\min\{\tau_{11}^0,\tau_{12}^0\}$ and the related imaginary part is ω_{10}. Suppose that $\lambda(\tau_1)=\alpha(\tau_1)+i\omega(\tau_1)$ is the solution of (3.3) at $\tau_1=\tau_{10}$ satisfying $\alpha=0$ and $\omega=\omega_{10}$.

Before summing up the Hopf bifurcation, we can introducing the following lemma 3.1 :

Lemma 3.1. If $(H_{35})\cos\omega_{10}\tau_{10}+b_1\sin\omega_{10}\tau_{10}\ne 0$, then $\left(\frac{d(\mathrm{Re}\,\lambda)}{d\tau_1}\right)_{\lambda=i\omega_{10}}\ne 0$. In addition, the sign of $\left(\frac{d(\mathrm{Re}\,\lambda)}{d\tau_1}\right)_{\lambda=i\omega_{10}}$ and $\cos\omega_{10}\tau_{10}+b_1\sin\omega_{10}\tau_{10}$ is same.

Proof. In fact, substituting $\lambda(\tau_1)$ into (3.3) and differentiating it about τ_1, it concludes

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1}=\frac{2}{b_3e^{-\lambda\tau_1}}+\frac{b_1}{b_3\lambda e^{-\lambda\tau_1}}-\frac{\tau_1}{\lambda}$$

(3.7)

Substituting $\lambda=i\omega_{10}$ into (3.7), and separate the real part :

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau_1}\right)^{-1}=\frac{\cos\omega_{10}\tau_{10}+b_1\sin\omega_{10}\tau_{10}}{b_3}$$

(3.8)

As $\mathrm{Re}\left\{\left(\frac{d(\mathrm{Re}\,\lambda)}{d\tau_1}\right)^{-1}\right\}_{\lambda=i\omega_0}$ and $\left(\frac{d(\mathrm{Re}\,\lambda)}{d\tau_1}\right)_{\lambda=i\omega_{10}}$ have the same sign, hence

$$sign\left\{\left(\frac{d(\mathrm{Re}\,\lambda)}{d\tau_1}\right)\right\}_{\lambda=i\omega_{10}}=sign\left\{\mathrm{Re}\left(\frac{d(\mathrm{Re}\,\lambda)}{d\tau_1}\right)^{-1}\right\}$$
$$=\frac{\cos\omega_{10}\tau_{10}+b_1\sin\omega_{10}\tau_{10}}{b_3}$$
$$\ne 0.$$

This completes the proof.

Based on the theorem 3.2 and lemma 3.1, we can obtained the following theorem 3.3 by the Hopf bifurcation theorem.

Theorem 3.3. Suppose that (H_{32}) holds, When $\tau_2=0$, we can summarize that

if (H_{33}) is satisfied, then the endemic equilibrium E^* of (3.1) is asymptotically stable for $\tau_1\ge 0$;

if the condition of (H_{34}) is satisfied, then the endemic equilibrium E^* of (3.1) is asymptotically stable for $\tau_1\in[0,\tau_{10})$;

if (H_{33}) and (H_{35}) are satisfied, then the endemic equilibrium E^* of (3.1) is unstable for $\tau_1>\tau_{10}$ and goes with Hopf bifurcation for $\tau_1=\tau_{10}$.

case (3) : $\tau_1=0$, $\tau_2>0$. The basic reproduction is $R_{02}^{(3)}=R_{02}$, and the proving progress is similar to case (2), furthermore, we omit it and conclude the following theorem 3.4 :

Theorem 3.4. When $\tau_1=0$, the endemic equilibrium E^* of (3.1) is asymptotically stable for $\tau_2\in[0,\tau_{20})$ and unstable for $\tau_2>\tau_{20}$. In addition, for $\tau_2=\tau_{20}$, the endemic equilibrium E^* goes Hopf bifurcation, where τ_{20} represents the minimum critical value of time delay τ_2 for the occurrence of Hopf bifurcation when $\tau_1=0$.

case (4) : $\tau_1=\tau_2=\tau>0$. The redefined basic reproduction number is $R_{02}^{(4)}=\beta f'(I^*)S^*e^{-d_2\tau}[d_3+d+\gamma$

$+bc(b+I^*)^{-2}]^{-1}$, the proof procedure follows case (2), so, here, we omit it.

Theorem 3.5. When $\tau_1 = \tau_2 = \tau > 0$, the endemic equilibrium E^* of (3.1) is asymptotically stable for $\tau \in [0,\tau_0)$ and unstable for $\tau > \tau_0$. Further, the endemic equilibrium E^* goes Hopf bifurcation for $\tau = \tau_0$, where τ_0 represents the minimum critical value of time delay τ_2 for the appearance of Hopf bifurcation when $\tau_1 = \tau_2 = \tau > 0$.

case (5) : $\tau_1 > 0$, $\tau_2 \in [0,\tau_{20})$ and $\tau_1 \neq \tau_2$. We can regard τ_1 as a parameter and τ_2 in its stable interval. It is easy to see that $R_{02}^{(5)} = R_{02}$. Let $\lambda = i\omega$ is the root of system (3.1), substituting it into (3.1) and separating the imaginary and real parts, respectively. one can conceive

$$\begin{aligned} e_{51} &= e_{52}\sin\omega\tau_2 + e_{53}\cos\omega\tau_2 \\ e_{54} &= e_{53}\sin\omega\tau_2 - e_{52}\cos\omega\tau_2 \end{aligned} \qquad (3.9)$$

where

$$e_{51} = \omega^2 - b_2,\ e_{52} = m_1\omega - n_1\sin\omega\tau_1,$$

$$e_{53} = m_2 + n_1\cos\omega\tau_1,\ e_{54} = b_1\omega.$$

Reorganized that

$$\omega^4 + c_{51}\omega^2 + c_{52} + c_{53}\omega\sin\omega\tau_1 + c_{54}\cos\omega\tau_1 = 0$$

(3.10)

where

$$c_{51} = b_1^2 - m_1^2 + 2b_2,\ c_{52} = b_2^2 - n_1^2,$$

$$c_{53} = 2m_2n_1,\ c_{54} = -2m_1n_1.$$

Hypothesize that (H_{36}) hold, thus, equation (3.10) has four roots ω_{1k}, $k = 1,2,3,4$. Define

$$\tau_{1k}^j = \frac{1}{\omega_{1k}}\arccos\frac{A_{51}\omega_{1k}^2 + A_{52} + B_{51}\omega_{1k}\sin\omega_{1k}\tau_1 + (B_{52}\omega_{1k}^2 + B_{53})\cos\omega_{1k}\tau_1}{A_{53}\omega_{1k}^2 + A_{54} + B_{54}\omega_{1k}\sin\omega_{1k}\tau_1 + B_{55}\cos\omega_{1k}\tau_1} + \frac{2j\pi}{\omega_{1k}}.$$

(3.11)

where $k = 1,2,3,4.\quad j = 0,1,2,\cdots.$

where

$$A_{51} = m_2 - b_1m_1,\ A_{52} = -b_2m_2,\ A_{53} = m_1^2,$$

$$A_{54} = n_1^2 + m_2^2,\ B_{51} = b_1n_1,\ B_{52} = n_1,$$

$$B_{53} = -b_2n_1,\ B_{54} = -2m_1n_1,\ B_{55} = 2m_2n_1.$$

Let $\tau_{10}' = \min\{\tau_{1k}^j \mid k = 1,2,3,4;\ j = 0,1,2,\cdots.\}$, we can calculated that (3.10) has pure imaginary roots $\pm i\omega_{10}'$ when $\tau_1 = \tau_{10}'$, $\tau_2' \in [0,\tau_{20})$. Let $\lambda = \alpha + i\omega_5$ is the solution of (3.1) and satisfies $\alpha(\tau_{10}') = 0$, $\omega(\tau_{10}') = \omega_{10}'$. We can conclude the following transversal condition :

Lemma 3.2. If $(H_{37}) A_1C_1 + B_1D_1 \neq 0$, then $\left(\frac{d(\operatorname{Re}\lambda)}{d\tau_1}\right)_{\lambda = i\omega_{10}'} \neq 0$. Furthermore, the sign of $\left(\frac{d(\operatorname{Re}\lambda)}{d\tau_1}\right)_{\lambda = i\omega_{10}'}$ is same to $A_1C_1 + B_1D_1$.

Proof. In fact, substituting λ into (3.1), and taking the derivative of τ_1, there are the following differential function :

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1} = \frac{A_1 + B_1 i}{C_1 + D_1 i} + \frac{\tau_1 + \tau_2}{\omega_{10}'} i,$$

which leads to

$$\left\{\frac{d(\operatorname{Re}\lambda)}{d\tau_1}\right\}_{\lambda = i\omega_{10}'} = \operatorname{Re}\left\{\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\right\}_{\lambda = i\omega_{10}'} = \frac{A_1C_1 + B_1D_1}{C_1^2 + D_1^2} \neq 0.$$

at the value of $\lambda = i\omega_{10}'$, where

$$A_1 = b_1 + (m_1 - m_2)\cos\omega_{10}'\tau_2 - m_1\omega_{10}'\sin\omega_{10}'\tau_2,$$

$$B_1 = 2\omega_{10}' - m_1\omega_{10}'\cos\omega_{10}'\tau_2 + (m_1 - m_2)\sin\omega_{10}'\tau_2,$$

$$C_1 = \omega_{10}'n_1\sin\omega_{10}'(\tau_{10}' + \tau_2),$$

$$D_1 = \omega_{10}'n_1\cos\omega_{10}'(\tau_{10}' + \tau_2).$$

According to Hopf bifurcation theorem, we have the following theorem 3.6 :

Theorem 3.6. When $\tau_1 > 0$, $\tau_2 \in [0,\tau_{20})$ and $\tau_1 \neq \tau_2$, if (H_{36}) and (H_{37}) holds, then, the endemic equilibrium E^* of (3.1) is asymptotically stable for $\tau_1 \in [0,\tau_{10}')$ and $\tau_1 > \tau_{10}'$. Moreover, the endemic equilibrium E^* of (3.1) undergoes a Hopf bifurcation at $\tau_1 = \tau_{10}'$.

case (6) : $\tau_1 \in [0,\tau_{10})$, $\tau_2 > 0$ and $\tau_1 \neq \tau_2$. The investigation is homoplastic to case (5), so it can be elided too, whereas, the bifurcation theorem is given in the following :

Theorem 3.7. When $\tau_1 \in [0,\tau_{10})$, $\tau_2 > 0$ and $\tau_1 \neq \tau_2$. The endemic equilibrium E^* of (3.1) is asymptotically stable for $\tau_2 \in [0,\tau_{20}')$ and $\tau_2 > \tau_{20}'$. Besides, the endemic equilibrium E^* of (3.1) undergoes a Hopf bifurcation at $\tau_2 = \tau_{20}'$, where τ_{20}' represents the minimum critical value of time delay τ_2 for the occurrence of Hopf bifurcation when $\tau_1 \in [0,\tau_{10})$.

IV. Direction and Stability of Hopf Bifurcation

In section 3, we concluded some conditions in six cases, which makes (3.1) undergo Hopf bifurcation at the endemic equilibrium. In this section, by using the center manifold and the normal form theory in [25], we devoted to research the direction and stability of periodic solutions bifurcating from the endemic equilibrium when $\tau_2 = 0$.

Let $\omega^* = \omega(\tau_1^*)$ and $\tau_1 = \tau_1^* + v$, $v \in R$, $t = s\tau_1$, $x(s) = x(s\tau_1)$, $y(s) = y(s\tau_1)$ where $t =$ s. Then, it is not difficult to verify that $v = 0$ is a Hopf bifurcation value of (2.2), and system (2.2) can be rewritten as the following functional differential equation (FDE) :

$$\dot{u}(t) = L_v(u(t)) + G(v, u(t))$$

(4.1)

where $u(t) = (x(t), y(t))^T \in C$, $L_v : C \to R^2$ and $G : R \times C \to R^2$ where $C = C([-1,0], R_+^2)$ is the phase space. Define $L_v(\varphi) = B_1\varphi(0) + B_2\varphi(-1)$, $\varphi \in C$; $G(v, u(t)) = \tau_1(G_1, G_2)^T$.

where

$$B_1 = \tau_1 \begin{pmatrix} -d_1 - \beta f(I^*) & -\beta S^* f'(I^*) \\ \beta f(I^*) & \beta S^* f'(I^*) - (d_3 + d + \gamma + \frac{bc}{(b+I^*)^2}) \end{pmatrix},$$

$$B_2 = \tau_1 \begin{pmatrix} 0 & \gamma e^{-d_4\tau_1} \\ 0 & 0 \end{pmatrix},$$

$$G_1 = -\frac{\beta}{2}[2f'(I^*)x(t)y(t) + f''(I^*)S^* y^2(t)] - \frac{\beta}{6}[f'''(I^*)S^* y^3(t) + 3f''(I^*)x(t)y^2(t)] + O(4)$$

$$G_2 = \frac{\beta}{2}[2f'(I^*)x(t)y(t) + f''(I^*)S^* y^2(t)] + \frac{\beta}{6}[f'''(I^*)S^* y^3(t) + 3f''(I^*)x(t)y^2(t)] + bcy^2(t)(b+I^*)^{-2}(b + I^* + y(t))^{-1} + O(4).$$

According to the Riesz representation theorem, in $\theta \in [-1,0]$, there exists a 2×2 matrix $\eta(\theta, \upsilon)$ composed of bounded variation functions satisfying

$$L_\upsilon \varphi = \int_{-1}^{0} d\eta(\theta, \mu)\varphi(\theta),$$

(4.2)

In reality, it can select $\eta(\theta, \upsilon) = B_1\delta(\theta) - B_2\delta_{(\theta+1)}$, where

$$\delta(\theta) = \begin{cases} 1, & \theta = 0 \\ 0, & \theta \neq 0 \end{cases}$$

Define

$$A(\upsilon)\varphi = \begin{cases} \dot{\varphi}(\theta), & \theta \in [-1,0) \\ \int_{-1}^{0} d\eta(s, \upsilon)\varphi(s), & \theta = 0 \end{cases} \quad (4.3)$$

$$R(\upsilon)\varphi = \begin{cases} 0, & \theta \in [-1,0) \\ G(\upsilon, \varphi), & \theta = 0 \end{cases}$$

(4.4)

Let $u = (x, y)^T$, and rewrite the system (4.1) as

$$\dot{u}_t = A(\upsilon)u_t + R(\upsilon)u_t.$$

(4.5)

Define

$$A^*\psi(s) = \begin{cases} -\dot{\psi}(s), & s \in (0,1] \\ \int_{-1}^{0} d\eta^T(t, \tau^*)\psi(-t), & s = 0 \end{cases}$$

$$<\psi, \varphi> = \overline{\psi}(0)\varphi(0) - \int_{-1}^{0}\int_{0}^{\theta} \overline{\psi}(\xi - \theta)d\eta(\theta)\varphi(\xi)d\xi,$$

where $\psi \in C^1$, $\eta(\theta) = \eta(\theta, \tau_1^*)$, A^* and A are ad-joint operators. From Section 3, it arrives that $\pm i\omega^*\tau_1^*$ are eigenvalues of A, so, they are eigenvalues of A^* too.

By computations, we receive that $q_1(\theta) = (1, \beta f(I^*)[i\omega^* + d_3 + d + \gamma + bc(b + I^*)^{-2} - \beta f'(I^*)S^*]^{-1})e^{i\omega^*\tau_1^*\theta}$ is the eigenvector of A corresponding to $i\omega^*\tau_1^*$, and $q_2(\theta) = \overline{D}(1, [-i\omega^* + d_1 + \beta f(I^*)][\beta f(I^*)]^{-1})^T e^{i\omega^*\tau_1^*\theta}$ is the eigenvector of A^* corresponding to $-i\omega^*\tau_1^*$. Moreover, it satisfies $\langle q_2(\theta), q_1(\theta)\rangle = 1$ and $\langle q_2(\theta), \overline{q}_1(\theta)\rangle = 0$, where $D = 1 + [i\omega^* + d_1 + \beta f(I^*)(1 + \tau_1^* \gamma e^{-(d_4 + i\omega^*)\tau_1^*})][i\omega^* + d_3 + d + \gamma + bc(b + I^*)^{-2} - \beta f'(I^*)]$.

When $\tau_1 = \tau_1^*$, postulate that u_t be the solution of (4.1). Define that $z(t) = \langle q_2, u(t)\rangle$, so

$$\dot{z}(t) = \langle q_2, \dot{u}(t)\rangle = i\omega^*\tau_1^* z(t) + \overline{q}_2(0)\hat{G}(z, \overline{z}),$$

(4.7)

where

$$\hat{G} = G(\phi_0, W(z, \overline{z}) + 2\operatorname{Re}\{zq\}), W(z, \overline{z}) = u_t - 2\operatorname{Re}\{zq\},$$

$$W(z, \overline{z}) = W_{20}\frac{z^2}{2} + W_{11}z\overline{z} + W_{02}\frac{\overline{z}^2}{2} + \cdots$$

(4.8)

We know that if u_t is real, then $W(z, \overline{z})$ also real. Here, we only think over real solutions. For eq. (4.5) can be rewritten as $\dot{z}(t) = i\omega^*\tau_1^* z(t) + g(z, \overline{z})$, where

$$g(z, \overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots.$$

substituting (4.3) and (4.5) into the differential equation $\dot{W} = \dot{u}_t - \dot{z}q - \dot{\overline{z}}\overline{q}$, it has

$$\dot{W} = \begin{cases} AW - 2\operatorname{Re} P\{\overline{q}_2(0)\hat{f}q_1(\theta)\}, & \theta \in [-1,0) \\ AW - 2\operatorname{Re} P\{\overline{q}_2(0)\hat{f}q_1(\theta)\} + \hat{G}, & \theta = 0 \end{cases}$$

$$\overset{def}{=} AW + H(z, \overline{z}, \theta),$$

where

$$H(z, \overline{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + H_{21}(\theta)\frac{z^2\overline{z}}{2} + \cdots.$$

Expanding the above equation and comparing the factors, there are

$$(A - 2i\omega^*\tau_1^* I)W_{20}(\theta) = -H_{20}(\theta),$$

$$(A + 2i\omega^*\tau_1^* I)W_{02}(\theta) = -H_{02}(\theta),$$

$$AW_{11} = -H_{11}(\theta).$$

Further,

$$u_t = u(t + \theta) = W(z, \overline{z}, \theta) + zq(\theta) + \overline{z}\overline{q}(\theta),$$

therefore, it has $g(z, \overline{z}) = \overline{q}_2(0)\,\hat{G}(z, \overline{z})$, where

$$g_{20}=\tau_1^* D\{(i\omega^*+d_1)[\beta f(I^*)]^{-1}[2\beta f'(I^*)q_2(0)+\beta f''(I^*) S^* q_2^2(0)+bcW_{20}^{(2)}(0)(b+I^*)^{-2}]+bcW_{20}^{(2)}(0)(b+ I^*)^{-2}\},$$

$$g_{11}=\tau_1^* D\{(i\omega^*+d_1)[\beta f(I^*)]^{-1}[\beta f'(I^*)(q_2(0)+\bar{q}_2(0)) +\beta f''(I^*)S^* q_2(0)\bar{q}_2(0)+bcW_{11}^{(2)}(0)(b+I^*)^{-2}]+bc W_{11}^{(2)}(0)(b+I^*)^{-2}\},$$

$$g_{02}=\tau_1^* D\{i\omega^*+d_1[\beta f(I^*)]^{-1}[2\beta f'(I^*)\bar{q}_2(0)+\beta f''(I^*)S^*\bar{q}_2^2 (0)+bcW_{02}^{(2)}(0)(b+I^*)^{-2}]+bcW_{02}^{(2)}(0)(b+I^*)^{-2}\},$$

$$g_{21}=\tau_1^* D\{(i\omega^*+d_1)[\beta f(I^*)]^{-1}[2\beta f'(I^*)(W_{11}^{(2)}(0)+\frac{1}{2}W_{20}^{(2)}(0) +\frac{1}{2}W_{20}^{(1)}(0)\bar{q}_2(0)+W_{11}^{(1)}(0)q_2(0))+\beta f''(I^*)S^*(2q_2(0) W_{11}^{(2)}(0)+\bar{q}_2(0)W_{20}^{(2)}(0))]+\beta f''(I^*)(2q_2(0)\bar{q}_2(0)+q_2^2(0)) +\beta f'''(I^*)S^* q_2^2(0)\bar{q}_2(0)\}.$$

where

$$W_{20}(\theta)=i\tau_1^* D(\omega^*\tau_1^*)^{-1}e^{i\omega^*\tau_1^*\theta}\{(i\omega^*+d_1)[\beta f(I^*)]^{-1}[2\beta f'(I^*) q_2(0)+\beta f''(I^*)S^* q_2^2(0)+bcW_{20}^{(2)}(B+I^*)^{-2}]+bc W_{20}^{(2)}(b+I^*)^{-2}\}q(0)+i\tau_1^*\bar{D}(3\omega^*\tau_1^*)^{-1}e^{i\omega^*\tau_1^*\theta}\{(-i\omega^*+ d_1)[\beta f(I^*)]^{-1}[-D_1+bcW_{02}^{(2)}(B+I^*)^{-2}]+bcW_{02}^{(2)}(b +I^*)^{-2}\}\bar{q}(0)+e^{2i\omega^*\tau_1^*\theta}\tau_1^*|F_1|^{-1}(D_1[2i\omega^*+d_3+d+\gamma+ bc(b+I^*)^{-2}-\gamma e^{-(d_4+2i\omega^*)\tau_1^*}]+bc(b+I^*)^{-2}W_{20}^{(2)}[\gamma e^{-(d_4+2i\omega^*)\tau_1^*}-\beta f'(I^*)S^*],D_1(2i\omega^*+d_1)+bc(b +I^*)^{-2}W_{20}^{(2)}[2i\omega^*+d_1+\beta f(I^*)])^{-1},$$

$$W_{02}(\theta)=iD\omega^{*-1}e^{i\omega^*\tau_1^*\theta}\{(i\omega^*+d_1)[\beta f(I^*)]^{-1}[2\beta f'(I^*)\bar{q}_2(0)+ \beta f''(I^*)S^*\bar{q}_2^2(0)+bcW_{02}^{(2)}(b+I^*)^{-2}]+bcW_{02}^{(2)}(b+I^*)^{-2}\} q(0)+i\bar{D}(3\omega^*)^{-1}e^{-i\omega^*\tau_1^*\theta}\{(-i\omega^*+d_1)[\beta f(I^*)]^{-1}[-\bar{D}_1+ bc\bar{W}_{02}^{(2)}(b+I^*)^{-2}]+bc\bar{W}_{02}^{(2)}(b+I^*)^{-2}\}\bar{q}(0)+e^{2i\omega^*\tau_1^*\theta}\tau_1^* |F_2|^{-1}(\bar{D}_1[-2i\omega^*\tau_1^*+d_3+d+\gamma+bc(b+I^*)^{-2}-\beta f'(I^*)S^*] +[\gamma e^{-(d_4-2i\omega^*)\tau_1^*}-\beta f'(I^*)S^*](-\bar{D}_1+bc(b+I^*)^{-2}W_{02}^{(2)}),\bar{D}_1\beta f (I^*)+[-2i\omega^*\tau_1^*+d_1+\beta f(I^*)][-\bar{D}_1+bc(b+I^*)^{-2}W_{02}^{(2)}])^T$$

$$W_{11}(\theta)=-i\tau_1^* D(\omega^*\tau_1^*)^{-1}e^{i\omega^*\tau_1^*\theta}\{(i\omega^*+d_1)[\beta f(I^*)]^{-1}[\beta f'(I^*)(q_2(0) +\bar{q}_2(0))+\beta f''(I^*)S^* q_2(0)\bar{q}_2(0)+bcW_{11}^{(2)}(b+I^*)^{-2}]+ bcW_{11}^{(2)}(b+I^*)^{-2}\}q(0)+i\tau_1^*\bar{D}(\omega^*\tau_1^*)^{-1}e^{-i\omega^*\tau_1^*\theta}\{(-i\omega^* +d_1)[\beta f(I^*)]^{-1}[\beta f'(I^*)(\bar{q}_2(0)+q_2(0))+\beta f''(I^*) S^*\bar{q}_2(0)q_2(0)+bc\bar{W}_{11}^{(2)}(b+I^*)^{-2}]+bc\bar{W}_{11}^{(2)}(b+I^*)^{-2}\}\bar{q}_1(0) \bar{q}_1(0)-\tau_1^*|F_3|^{-1}(D_2[\beta f'(I^*)S^*-d_3-d-\gamma-bc(b+I^*)^{-2}] +[\beta f'(I^*)S^*-\gamma e^{-d_4\tau_1^*}](-D_2+bcW_{11}^{(2)}(b+I^*)^{-2}),-D_2\beta f (I^*)-[d_1+\beta f(I^*)](-D_2+bcW_{11}^{(2)}(b+I^*)^{-2}))^T.$$

where

$$D_1=-2\beta f'(I^*)q_1(0)q_2(0)-\beta f''(I^*)S^* q_2^2(0),$$

$$D_2=-\beta f'(I^*)(q_2(0)+\bar{q}_2(0))-\beta f''(I^*)S^* q_2(0)\bar{q}_2(0),$$

$$|F_1|=(2i\omega^*\tau_1^*+d)[2i\omega^*+d_3+d+\gamma+bc(b+I^*)^{-2}-\beta f'(I^*)S^*]+\beta f(I^*) (2i\omega^*+d_3+d+\gamma+bc(b+I^*)^{-2})-\gamma\beta f(I^*)e^{-(d_4+2i\omega^*)\tau_1^*},$$

$$|F_2|=[-2i\omega^*\tau_1^*+d_1+\beta f(I^*)][-2i\omega^*\tau_1^*+d_3+d+\gamma+bc(b+I^*)^{-2}-\beta f'(I^*)S^*]+\beta f(I^*)[\beta f'(I^*)S^*-\gamma e^{-(d_4-2i\omega^*)\tau_1^*}],$$

$$|F_3|=-d_1[\beta f'(I^*)S^*-d_3-d-\gamma-bc(b+I^*)^{-2}]+\beta f(I^*)[d_3+d+\gamma +bc(b+I^*)^{-2}]-\gamma\beta f(I^*)e^{-d_4\tau_1^*}.$$

Hence, the values of g_{20}, g_{11}, g_{02}, g_{21} can also be obtained. Moreover, we can compute the following parameters to estimate the direction and stability of periodic solutions bifurcating from the endemic equilibrium when $\tau_2=0$.

$$C_1(0)=\frac{i}{2\omega^*\tau_1^*}[g_{20}g_{11}-2|g_{11}|^2-\frac{1}{3}|g_{02}|^2]+\frac{g_{21}}{2},$$

$$\beta_2=2\,\mathrm{Re}\{C_1(0)\},\ \mu_2=-\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\,\lambda'(0)},$$

$$T_2=-\frac{\mathrm{Im}\{C_1(0)\}+\mu_2\,\mathrm{Im}\,\lambda'(0)}{\omega^*\tau_1^*}.$$

where μ_2 indicates the directions of the Hopf bifurcation, β_2 expresses the stability of the bifurcate periodic solutions, T_2 represents the period of the bifurcating periodic solutions.

By the value of these parameters above, we can get the following theorem 4.1 .

Theorem 4.1. If $\mu_2>0(<0)$, the Hopf bifurcation is super-critical (sub-critical); If $\beta_2<0(>0)$,

the bifurcation periodic solutions are trajectory asymptotically stable(unstable); And the period of the periodic solutions is increasing (decreasing) if $T_2>0(<0)$.

V. Numerical simulations

In this section, by different parameters, we present some numerical simulation results of system (2.1) to support these conclusions in this paper.

By choosing $f(I)=I(1+I)^{-1}$, $\mu=0.35$, $d_1=0.1$, $d_2=0.2$, $d_3=0.15$, $d_4=0.1$, $d=0.1$, $\gamma=1$, $\beta=2$, $b=0.01$, $c=0.55$, $\tau_1=1$, $\tau_2=0.1$, we can calculate the basic reproduction number as $R_{01}=0.12<1$, therefore when $\tau_1=1$ and $\tau_2=0.1$, the disease-free equilibrium is asymptotically stable if $R_{01}<1$ (see Fig.1) , which result is consistent with the analysis in section 3.

(b) While, if we choose $\mu=0.85$, $d_1=0.01$, $d_2=0.2$, $d_3=0.3$, $d_4=0.1$, $d=0.1$, $\gamma=8$, $\beta=10$, $b=1$, $c=30$, $\tau_1=1$, $\tau_2=0.1$, the basic reproduction is $R_{01}=21.7>1$. Hence, when $\tau_1=1$ and $\tau_2=0.1$, the endemic equilibrium is asymptotically stable if $R_{01}>1$(see Fig.2) .

(c) When choosing $\mu=0.83$, $d_1=0.1$, $d_2=0.2$, $d_3=0.3$, $d_4=0.1$, $d=0.1$, $\beta=12$, $\gamma=7$, $b=1$, $c=3$, $\tau_1=2$, $\tau_1=2$, then, $R_0=9.19>1$ and the periodic solution is asymptotically stable when $\tau_1=2$ and $\tau_2=0.23$ (see Fig.3, Fig.4) .

(d) For system (2.2), if we choose the same function of f, and let $\mu=0.75$, $d_1=0.1$, $d_2=0.2$, $d_3=0.15$, $d_4=0.1$, $d=0.1$, $\gamma=18$, $\beta=20$, $b=1$, $c=15$, $\tau_1=0.2<\tau_1^0=0.33$, $\tau_2=0$, thus, the endemic equilibrium is asymptotically stable when $0.2<\tau_1^0$ and $\tau_2=0$ (see Fig.5) .

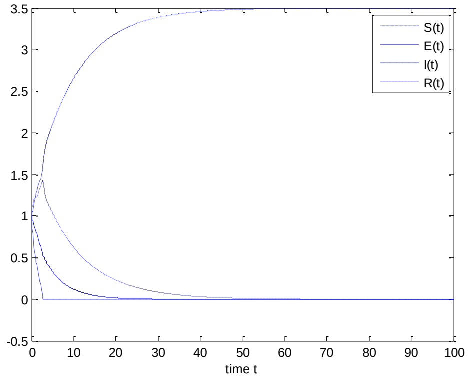


Fig. 1. *The disease-free equilibrium E_0 is asymptotically stable if $R_0 < 1$ when $\tau_1 = 1$ and $\tau_2 = 0.1$.*

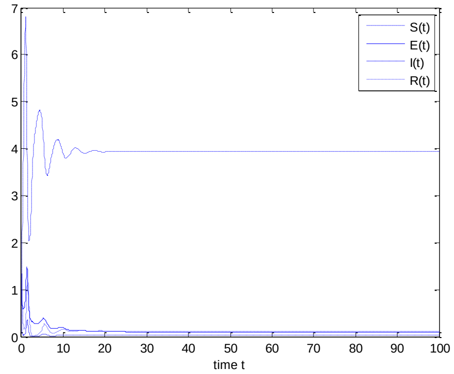


Fig. 2. *The endemic equilibrium is asymptotically stable if $R_0 > 1$ when $\tau_1 = 1$ and $\tau_2 = 0.1$.*

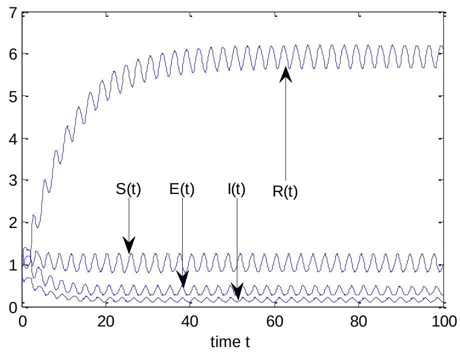


Fig. 3. *The endemic equilibrium is asymptotically stable if $R_0 > 1$ when $\tau_1 = 2$ and $\tau_2 = 0.23$.*

(e) In addition, based on (d), if $d_4 = 0.4$, $c = 21$, $\tau_1 = 26 > \tau_1^1 = 1.18$, we can also conclude that the endemic equilibrium is asymptotically stable when $\tau_1 = 26 > \tau_1^1$ and $\tau_2 = 0$ (see Fig.6) .

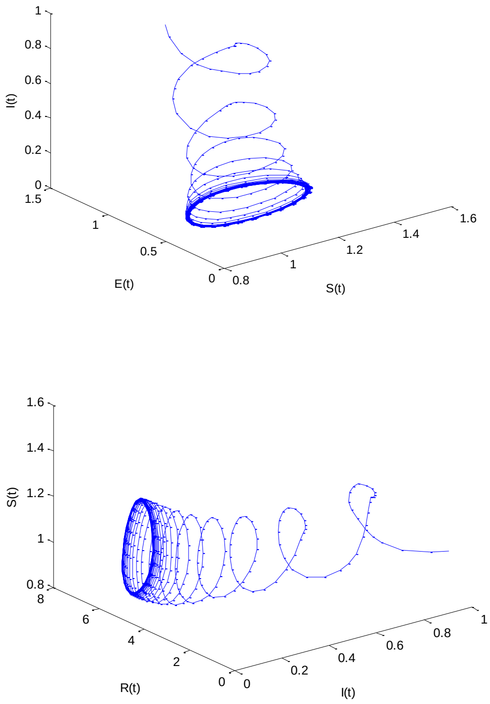


Fig. 4. *The* ../../../../../../../../Program Files/Youdao/Dict/7.0.1.0227/resultui/dict/?keyword=*phase* ../../../../../../../../Program Files/Youdao/Dict/7.0.1.0227/resultui/dict/?keyword=*diagram* $(S(t), E(t), I(t))$

$(I(t), R(t), S(t))$ *of system (2.1) when* $\tau_1 = 2$ *and*

$\tau_2 = 0.23$.

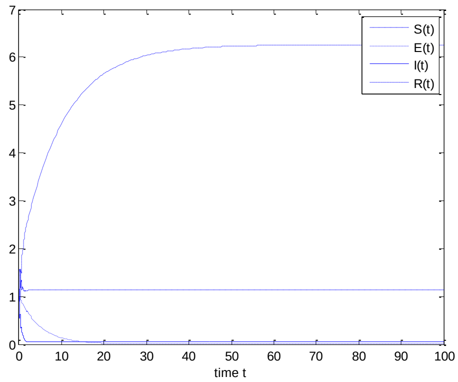


Fig. 5. *The endemic equilibrium is asymptotically stable when $\tau_1 = 0.2 < \tau_1^0 = 0.33$ and $\tau_2 = 0$.*

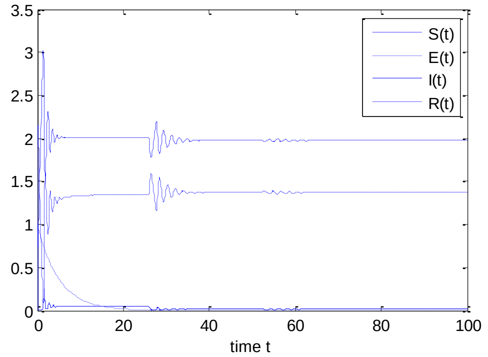


Fig. 6. *The endemic equilibrium is asymptotically stable when* $\tau_1 = 26 > \tau_1^1 = 1.18$ *and* $\tau_2 = 0$.

(f) Furthermore, let $\mu = 0.75$, $d_1 = 0.1$, $d_2 = 0.2$, $d_3 = 0.15$, $d_4 = 0.1$, $d = 0.1$, $\gamma = 18$, $\beta = 30$, $b = 1$, $c = 15$, $\tau_1 = 1 \in (\tau_1^0, \tau_1^1) = (0.33, 1.18)$, $\tau_2 = 0$, we can get the asymptotically stability of the periodic solution when $\tau_1 = 1 \in (\tau_1^0, \tau_1^1) = (0.33, 1.18)$ and $\tau_2 = 0$ (see Fig.7, Fig.8). It's concluded that the exposed is going to eradicate and the infected will be controlled in a small number.

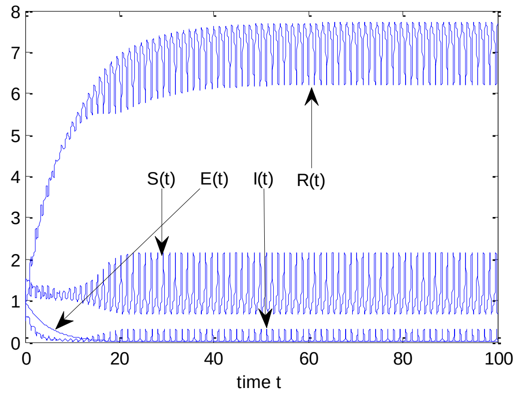
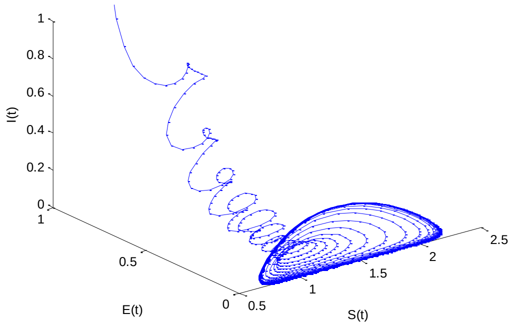


Fig. 7. *The periodic solutions is asymptotically stable when* $\tau_1 = 0.73 \in (\tau_1^0, \tau_1^1)$ *and* $\tau_2 = 0$.



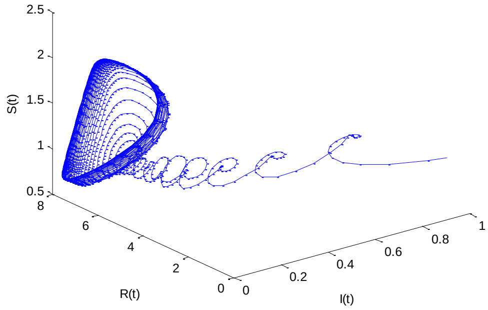


Fig. 8. *The* ../../../../../../../../Program Files/Youdao/Dict/7.0.1.0227/resultui/dict/?keyword=*phase* ../../../../../../../../Program Files/Youdao/Dict/7.0.1.0227/resultui/dict/?keyword=*diagram* $(S(t), E(t), I(t))$

$(I(t), R(t), S(t))$ *of system (2.1)when* $\tau_1 = 0.73 \in (\tau_1^0, \tau_1^1)$ *and* $\tau_2 = 0$.

(g) Eventually, it's obtained that $g_{20} = -77.42 + 0.694i$, $g_{02} = -64.9 + 321.28i$, $g_{11} = 0.42 - 0.78i$, $g_{21} = -3388.71 + 1730.4i$, $C_1(0) = -1700.4 - 2743.87i$, additionally, $\beta_2 = -3400.8 < 0$, $\mu_2 = -80.51 < 0$ and $T_2 = -403.71 < 0$. Consequently, the Hopf bifurcation is sub-critical, the bifurcation periodic solutions are asymptotically stable and the bifurcating period is decreasing .

VI.Conclusions

In this paper, we introduced three elements into a saturated recovery SEIR epidemiological system which are two delays, general nonlinear incidence rate and saturated recovery rate. Analyzing the basic reproduction number and Lyapunov function,we investigated the local and global stability of the disease-free equilibrium,concluded that the disease will die out, if the basic reproduction number can be controlled in the scope of $(0,1)$ well (see Fig.1). In addition, the locally asymptotically stability of the endemic equilibrium and the Hopf bifurcation were studied in six different cases by different basic reproduction number.

It is obvious that, after adding the saturated recovery rate to system (2.1), the infected is significantly closer to the origin comparing Fig.2 to Fig.9. In this paper, we only discussed a bifurcation at the positive endemic equilibrium of system (2.2), one can estimate the forward and backward bifurcation further.

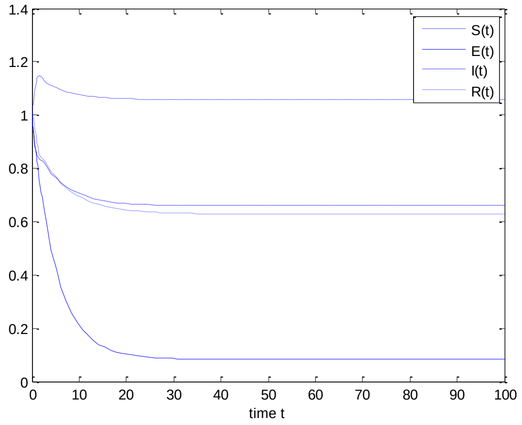


Fig. 9. *The endemic equilibrium is more closer to the origin in system (2.1) than in [14] when* $\tau_1 = 1$ *and* $\tau_2 = 0.1$.

What's more, we can see that the exposed is more stable and closer to the origin, when the latent period τ_2 is equal to 0 compared Fig.3 with Fig.7, so in this case, the latent delay can be neglected. And for that the function of $f(I) = 2I/(1+2I)$ is also adapted to system (2.1), hence, the applicability of general nonlinear incidence rate is wider than nonlinear incidence rate .

Based on the work in this paper, we can investigate the existence of global Hopf bifurcations by using the global Hopf bifurcation theorem [26] for system (4.1). Here, we omit it due to limited space of the paper.

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