Journal of Multidisciplinary Engineering Science and Technology (JMEST)
ISSN: 2458-9403
Vol. 4 Issue 7, July - 2017

A Study Of Periodic Structures Disturbed By Non-Periodic Boundary Conditions Or Parametric Modifications

Hammou AIT RIMOUCH[1]
Laboratory of Physics of Materials, FST, University Sultan Moulay Slimane
BP 523 Beni-Mellal, Morocco
Email: [1] aitrimouch_h@yahoo.fr

Abdelkhalak ELHAMI[2]
INSA Rouen
76800 Saint-Étienne-du-Rouvray-France.
Email: [2] abdelkhalak.elhami@insa-rouen.fr

***Abstract*—This article concerns the dynamic calculations of symmetric mechanical structures disturbed by arbitrary boundary conditions or parametric modifications. This type of configuration is frequently encountered in rotary machines, turbomachines, parabolic antennas, spatial structures...The objective of this article is to present a method which takes this repetitively into account in order to minimize the time and costs of predictive calculations. The proposed method is applicable to bidimensional and tridimensional problems. It is based on the theory of linear representations of finite groups combined with a reanalysis technique for modified mechanical structures.**

Keywords—finite group; dynamic; quasi-symmetric; structural modification; finite element; vibrations.

I. INTRODUCTION

The theory of representations of finite groups has been employed for many years in diverse scientific domains such as mathematics, chemistry, crystallography, and physics. However, it has been applied to structural calculations in the past years [12]. The main idea is to exploit the symmetry of the structures and symmetry groups in order to reduce the size of discretized problems. One problem of interest is the predictive calculation of symmetric mechanical structures with arbitrary boundary conditions. This technique is constituted of two steps.

- The first step consists in applying the theory of linear representations of finite groups which allows the dynamic problem to be divided in sub-problems on the basis cell, in order to extract the truncated modal basis (Y_1 , Λ_1) of the complete symmetric mechanical structure.
- The second step consists in predicting the dynamic behaviour of the preceding structure after modification of its boundary conditions, or after introducing parametric modifications. In order to accomplish this, a reanalysis technique for structural modifications is used based on the dynamic flexibility matrix.

The proposed procedure is demonstrated in the context of an industrial application, namely the analysis of a “clutch spring”.

II. PROPOSED SOLUTION METHODS

The proposed method is constituted of two steps.

A. First step (Dynamic analysis of periodique structures)

In the cylindrical reference frame fixed to the symmetry axis of the cell, the stiffness and mass matrix are the same for each cell. If it is assumed that a symmetric mesh is respected, then the following problem is solved:

$$(K_s^{(k)} - \omega^2 M_s^{(k)})\, Y_s^{(k)} = f_s^{(k)} \qquad (1)$$

$$K_s^{(1)} = K_s^{(2)} = \ldots = K_s^{(ns)} \;\; ; M_s^{(1)} = M_s^{(2)} = \ldots = M_s^{(ns)}$$

$$\left\{ \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} - \omega^2 \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \right\}^{(k)} \begin{bmatrix} Y_L \\ Y_R \\ Y_I \end{bmatrix}^{(k)} = \begin{bmatrix} f_L \\ f_R \\ f_I \end{bmatrix} \qquad (2)$$

The displacement and forces compatibility relations are applied to the base cell interfaces, yielding:

$$\begin{cases} Y_R^{(k)} = e^{ika} Y_L^{(k)}, a = 2\pi / ns, i^2 = -1 \\ f_{LR}^{(k)} = -e^{ika} f_{LL}^{(k)} \end{cases} \qquad (3)$$

The $Y_R^{(k)}$ are eliminated either by an energetic method employing the LAGRANGE equations or by a penalisation method.

In that follow, the direct elimination is considered. The system at iteration k becomes:

$$\begin{bmatrix} Z_{11} + Z_{22} + e^{ika} Z_{12} + e^{-ika} Z_{21} & Z_{13} + e^{-ika} Z_{23} \\ Z_{31} + e^{ika} Z_{32} & Z_{33} \end{bmatrix}^{(k)} \begin{bmatrix} Y_L \\ Y_I \end{bmatrix}^{(k)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (4)$$

with: $Z_{pq} = (K_{pq} - \omega^2 M_{pq})$

By writing:

$$Y_R = Y_R^r + i\, Y_R^i \;\; ; \quad Y_I = Y_I^r + i\, Y_i^i \;\; ; \quad \alpha = k.a$$

Due to the symmetry of the problem, only the (ns/2+1) subproblems need to be solved. For each α (the F_I are zero):

$$(\tilde{\tilde{K}}(\alpha) - \omega^2 \tilde{\tilde{M}}(\alpha))\, \tilde{\tilde{Y}}_s = 0 \qquad (5)$$

where: $\alpha \in \{0, 2\pi/ns, 4\pi/ns, \ldots, 2\pi n/ns\}$.

If *ns* is even then *n =ns/2*, if *ns* is odd *n=(ns–1)/2*

$$\tilde{\tilde{K}}(\alpha) = \begin{bmatrix} \tilde{\tilde{K}}_{11} & \tilde{\tilde{K}}_{12} \\ -\tilde{\tilde{K}}_{12}^{t} & \tilde{\tilde{K}}_{11} \end{bmatrix} \; ; \; \tilde{\tilde{M}}(\alpha) = \begin{bmatrix} \tilde{\tilde{M}}_{11} & \tilde{\tilde{M}}_{12} \\ -\tilde{\tilde{M}}_{12}^{t} & \tilde{\tilde{M}}_{11} \end{bmatrix} \quad (6)$$

$$\tilde{\tilde{K}}_{11} = \begin{bmatrix} K_{11}+K_{22}+(K_{12}+K_{21})\cos\alpha & K_{13}+K_{23}\cos\alpha \\ K_{31}+K_{32}\cos\alpha & K_{33} \end{bmatrix} \quad (7)$$

$$\tilde{\tilde{K}}_{12} = \begin{bmatrix} -(K_{12}-K_{21})\sin\alpha & K_{23}\sin\alpha \\ -K_{32}\sin\alpha & 0 \end{bmatrix} \quad (8)$$

$$\tilde{\tilde{M}}_{11} = \begin{bmatrix} M_{11}+M_{22}+(M_{12}+M_{21})\cos\alpha & M_{13}+M_{23}\cos\alpha \\ M_{31}+M_{32}\cos\alpha & M_{33} \end{bmatrix} \quad (9)$$

$$\tilde{\tilde{M}}_{12} = \begin{bmatrix} -\sin(\alpha)(M_{12}-M_{21}) & \sin(\alpha)M_{23} \\ -\sin(\alpha)M_{32} & 0 \end{bmatrix} \quad (10)$$

The modes of the global structure are expanded for a given calculated frequency, using the corresponding vector:

$$Y^T = Re\{\tilde{\tilde{Y}}_s, e^{ia}\tilde{\tilde{Y}}_s, e^{i2a}\tilde{\tilde{Y}}_s, ..., e^{i(ns-1)a}\tilde{\tilde{Y}}_s)^T \quad (11)$$

The demonstration of this expansion is given in [2], [5]. The advantages of this method are: reduced storage, only the stiffness matrix of the cell is required, and reduced calculation times. The validity of proposed procedure is illustrated in [5], [13] through numerical applications.

B. Second step (Structural Dynamic Modification)

The second step consists in applying a reanalysis technique for structural modifications based on the dynamic flexibility matrix.

1) Introduction

The purpose of this step is to apply a method, which gives a good approximation of the eigensolution of the modified structure. This will be performed with a limited number of modes of the original structure. The preceding formulations express the dynamic responses of the modified structure in the lower frequency band on a truncated modal base. This truncation, which was discussed by Elliott and Mitchell [10] and Ram and Braun [11], is a recognised source of error in the structural modification process. In order to reduce this effect, S. COGAN [6] proposes a procedure which completes the truncated modal base with additional Ritz vectors. Other authors [3,6,7] introduce the static residual effects in a form more or less approximate. H. Ait rimouch and al. [14] have developed a technique which takes into account, in a very significant manner, the contribution of the unknown modes.

The method proposed here uses the results of a modal analysis to transform a general problem of structural dynamic modification into an eigenvalue problem that takes partially accounts for the modes outside of the analysis band. The method is based on a dynamic flexibility formulation and leads to a linear eigenvalue problem. The manner which the formula is developed leads to a reduced and efficient linear eigenvalue problem. It is shown that, for a structure modified by the introduction of grounded degrees of freedom (DOF), the proposed technique allows its dynamic behaviour to be numerically calculated without adding stiffness and mass modification matrices.

2) The problem formulation

a) General case

Consider a conservative self-adjoint mechanical structure represented by mass and stiffness matrices *M* and $K \in R^{n,n}$ respectively, both symmetric and positive definite; *n* is the total number of DOFs of the discrete structural model. Its eigensolutions are regrouped in the modal $Y = [\cdots y_\nu \cdots] \in R^{n,n}$ and spectral $\Lambda = \text{Diag}\{\lambda_\nu\} \in R^{n,n}$ matrices, $\nu=1,\ldots, n$, which satisfy the bi-orthogonal relations

$$Y^T M Y = I_n \quad \text{and} \quad Y^T K Y = \Lambda \quad (12)$$

where I_n is the identity matrix. The eigenvalues are ordered by magnitude ($\lambda_1 \leq \ldots \leq \lambda_n$).

The modified structure is characterised by the mass $\hat{M} = M+\Delta M \in R^{n,n}$ and the stiffness $\hat{K} = K+\Delta K \in R^{n,n}$ matrices, where ΔM and ΔK are the mass and stiffness modification ones, respectively. Its eigenvalue problem, in physical coordinates, is given by

$$[\hat{K} - \hat{\lambda}_\nu \hat{M}]\,\hat{y}_\nu = 0\,, \qquad \nu=1, \ldots, n, \quad (13)$$

where $\hat{y}_\nu \in R^{n,1}$ and $\hat{\lambda}_\nu \in R$ are the ν^{th} eigenvector and eigenvalue of the modified structure, respectively.

It is assumed that the modifications do not alter the order of the system. The equation of motion of the modified structure can be written:

$$\left[I_n - \hat{\lambda}_\nu \hat{M}\hat{K}^{-1}\right] z_\nu = 0\,, \quad (14)$$

where:

$$z_\nu = \hat{M}\hat{y}_\nu \quad (15)$$

Usually, the matrices ΔK and ΔM are sparse and assumed to be rank *r* ($r \leq q$). They can be factorises as follows:

$$\Delta K = K_q U \quad \text{and} \quad \Delta M = M_q U \quad (16)$$

$$K_q = \begin{bmatrix} \Delta K_{qq} \\ 0 \end{bmatrix} \; ; \; M_q = \begin{bmatrix} \Delta M_{qq} \\ 0 \end{bmatrix} \in R^{n,q} \; ; \; U = \begin{bmatrix} I_q & 0 \end{bmatrix} \tag{17}$$

The Sherman-Morison relation can be used to express $(K+\Delta K)^{-1}$ in the form:

$$\begin{aligned} (K+\Delta K)^{-1} &= (K+K_q U)^{-1} \\ &= K^{-1}\left(I_n - QK^{-1}\right) \end{aligned} \tag{18}$$

$$Q = K_q\left(I_q + UK^{-1}K_q\right)^{-1}U \tag{19}$$

$$K^{-1} = Y\Lambda^{-1}Y^T \tag{20}$$

Substituting (18) and (19) into (14) and premultiplying by Y^T, leads to:

$$\left[I_n - \hat{\lambda}_\nu (I_n + Y^T \Delta M\, Y)(I_n - \Lambda^{-1}Y^T Q\, Y)\Lambda^{-1}\right]c_\nu = 0 \tag{21}$$

with

$$c_\nu = Y^T z_\nu \tag{22}$$

The solutions of the previous equation are exact if all the modes of the original structure are used. However this condition is seldom checked and the solutions can be only approximate. Hence, the first advantage of the modal representation is that the dynamics of the original structure can be adequately represented by relatively few of its fundamental (lowest frequencies) modes. Moreover, it can be applied in a large variety of experimental and analytical cases.

Therefore, our goal is to evaluate the eigensolutions of the modified structure without recourse to an exact and, thus, costly reanalysis. In other words, we would like to search the best approximate solutions of (21), by using only the identified modal parameters of the original structure.

To distinguish between the identified (subscript 1) and unknown (subscript 2) portions of the original structure eigenbasis, we partition the full matrices Y, Λ and c_ν as

$$Y = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} \; ; \; \Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \; ; \; c_\nu = \begin{bmatrix} c_{1\nu} \\ c_{2\nu} \end{bmatrix}; \tag{23}$$

where $Y_1 \in \mathrm{R}^{n,m}$ and $\Lambda_1 = \mathrm{diag}\{\lambda_\nu,\ \nu=1,\ldots,m\}$ are assumed to be known, $Y_2 \in \mathrm{R}^{n,n-m}$ and $\Lambda_2 = \mathrm{diag}\{\lambda_\nu, \nu = m+1,\ldots,n\}$ are unknown, $c_{1\nu} \in R^{m,1}$ and $c_{2\nu} \in R^{n-m,1}$, m is the number of known modes ($m<n$).

After introduction of the above matrix partitions into (21), we obtain

$$\left[I_m - \hat{\lambda}_\nu A_1\right]c_{1\nu} + \hat{\lambda}_\nu B_1 c_{2\nu} = 0 \tag{24}$$

$$\hat{\lambda}_\nu A_2 c_{1\nu} + \left[I_{n-m} - \hat{\lambda}_\nu B_2\right]c_{2\nu} = 0 \tag{25}$$

with

$$A_1 = \left[(I_m + \delta M_{11})\left(I_m - \Lambda_1^{-1}Y_1^T QY_1\right) - Y_1^T \Delta M R_S QY_1\right]\Lambda_1^{-1} \;,$$

$$B_1 = \left[(I_m + \delta M_{11})\Lambda_1^{-1}Y_1^T QY_2 - \delta M_{12}\left(I_{\hat{n}} - \Lambda_2^{-1}Y_2^T QY_2\right)\right]\Lambda_2^{-1} \;,$$

$$A_2 = \left[(I_{\hat{n}} + \delta M_{22})\Lambda_2^{-1}Y_2^T QY_1 - \delta M_{21}\left(I_m - \Lambda_1^{-1}Y_1^T QY_1\right)\right]\Lambda_1^{-1},$$

$$B_2 = \left[(I_{\hat{n}} + \delta M_{22})\left(I_{\hat{n}} - \Lambda_2^{-1}Y_2^T QY_2\right) - \delta M_{21}\Lambda_1^{-1}Y_1^T QY_2\right]\Lambda_2^{-1},$$

$$R_S = Y_2\Lambda_2^{-1}Y_2^T = K^{-1} - Y_1\Lambda_1^{-1}Y_1^T,$$

$$\delta M_{ij} = Y_i^T \Delta M Y_j,\ i,j = 1,2,$$

$$\hat{n} = n - m.$$

We note that, in (24), the contribution of the matrix B_1 can be neglected, because of the multiplication by the inverse of Λ_2 matrix. This latter contains truncated higher eigenvalues of the original structure. So, a first approximation consists in replacing (24) by the following equation:.

$$\left[I_m - \hat{\lambda}_\nu A_1\right]c_{1\nu} = 0 \tag{26}$$

Considering the fact that ΔM and Q are dominated by zero elements, equation (26) can be reduced to:

$$\left[I_m - \hat{\lambda}_\nu \hat{A}_1\right]c_{1\nu} = 0 \tag{27}$$

where:

$$\hat{A}_1 = \left[(I_m + \delta m)\left(I_m - \Lambda_1^{-1}\delta Q\right) - Y_{1q}^T \Delta M_{qq}\tilde{R}_S Q_{qq}Y_{1q}\right]\Lambda_1^{-1},$$

$$Q_{qq} = \Delta K_q\left(I_q + Y_q\Lambda^{-1}Y_q^T\Delta K_q\right)^{-1},$$

$$\delta m = Y_{1q}^T \Delta M_{qq} Y_{1q} \text{ and } \delta Q = Y_{1q}^T Q_{qq} Y_{1q}.$$

b) Case of grounding DOF:

In this case, it is supposed that:

$$\Delta M_q = 0 \; ; \; \Delta K_q = kI_q \quad [\, k \text{ real positive scalar }] \tag{28}$$

Substituting (28) into (27) and letting k go to infinity, the latter is written:

$$\left[I_m - \hat{\lambda}_\nu\left(I_m - \Lambda_1^{-1T}Y_{1q}H_{qq}Y_{1q}\right)\Lambda_1^{-1}\right]c_{1\nu} = 0; \nu = 1,\ldots,m \tag{29}$$

where: $H_{qq} = Y_q\,\Lambda^{-1}\,Y_q^T$

The eigenvalues $\hat{\lambda}_\nu$ of the modified structure are determined by the solutions of (27) in the case a) and by (29) in case b). The associated eigenvectors $\hat{y}_\upsilon$ are obtained from the following equation:

$$\hat{y}_\nu = Y\left[I_n + Y_{qc}^T \Delta M_{qq} Y_q\right]^{-1}c_\nu \;,\ \nu = 1,\ldots,m \tag{30}$$

III. NUMERICAL SIMULATION

A truncated conical shell with flanges (Fig.1) can represent the example studied here. The base (1/12 of the global structure) cell in the first step is modelled by plate elements with 6 DOF per node and 106 nodes including 11 interface nodes. A mesh containing 732 nodes is represented (Fig. 1).

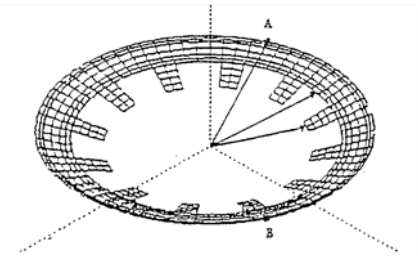

Fig. 1. *Clutch spring mesh*

For each step, the notation "*δf/f* %**"**, in table I and II, represent the relative error, in %, on the approximate eigenfrequencies with respect to this of the FEM one.

a) First step: the boundary conditions are symmetrical and the plate is Clamped-Free.

Table I gives the eigenvalues calculated respectively by:

1) FEM (ANSYS)

2) The proposed method (FIRST STEP)

The maximal error obtained in this step is equal to 3 % for calculated the frequencies.

b) Second step: we take the eigenmodes and eigenfrequencies of the first step and we apply the reanalysis technique for structural modifications.

Table II gives the eigenvalues calculated respectively by:

1) FEM (ANSYS)

2) The proposed method (SECONDE STEP)

In (Fig. 1) the nodes A and B are grounded.

The maximal error obtained in this step is equal to 0.4 % for the calculated frequencies.

Remark

Use of the two techniques helps to study periodic structures disturbed by non-periodic boundary conditions in an approximate way. The quality of simulated results is encouraging. This way of doing helps us to not modelise the global structure, and thus reduces storage place for the matrices and computing time.

TABLE I. FIRST EIGENFREQUENCIES OF CLUTCH SPRING OBTAINED BY FIRST STEP AND THEIR COMPARISON WITH THOSE OF THE FEM.

Number of nodal diameters	***eigenvalues calculated by FEM***	***eigenvalues estimated at the FIRST STEP***	***δf/f %***
0	7511	7528	0.2
1	5196	5354	3.0
2	4308	4403	2.2
3	4691	4739	1.1
4	5961	5993	0.5
5	7988	8027	0.4
6	10703	10720	0.1
CPU	1h37mn	595s	

TABLE II. FIRST EIGENFREQUENCIES OF THE ATTACHED CLUTCH SPRING OBTAINED BY SECOND STEP AND THEIR COMPARISON WITH THOSE OF THE FEM.

Number of mode	***eigenvalues calculated by FEM***	***eigenvalues estimated at the SECOND STEP***	***δf/f %***
1	4629	4629	0.00
2	4635	4635	0.00
3	5066	5066	0.00
4	5154	5154	0.00
5	5800	5800	0.00
6	6025	6027	0.04
7	6854	6864	0.15
8	7163	7165	0.02
9	7821	7821	0.00
10	8628	8639	0.13
11	9882	9891	0.09
12	11403	11421	0.17
13	12827	12874	0.37
14	14710	14752	0.28

IV. CONCLUSION

This article presented some calculation methods for structural vibrations in the case of cyclically symmetric structures with arbitrary boundary conditions. It contributes essentially to the development of methods based on linear representations of finite groups which lead to a decomposition of the initial problems which represent to a basis cell and on a reanalysis technique for structural modifications.

This work has led to the development of a finite element code that can be incorporated into many finite element codes that can be incorporated into many finite element software packages. Finally a numerical simulation is presented.

REFERENCES

[1] A. El Hami A., B. Radi, "Stochastique dynamique of structures, Wiley & Son ISTE 2016.

[2] H. Changwu, A. El Hami, B. Radi, " Metamodel-based Inverse Method for Parameter Identification: Elastic-Plastic Damage Model", Accepted, Engineering Optimization Volume: 49 Issue: 4, p. 633-653 Published: APR 2017.

[3] H. Ait Rimouch, G. Lallement, S. Cogan "Analyse de Structures mécaniques modifiées", Revue du GAMI : Mécanique Matériaux Electricité Vol N°445, 1992, p.36-39.

[4] B. Haisy, W. Springer, "Assimplified method for extracting rotational degrees-of-freedom Information from Modal Test Data" International Journal of modal Analysis, Vol N°3, 1986, pp. 35-39.

[5] A. Guerine, A. El Hami, L. Walha, T. Fakhfakh, M. Haddart, "Dynamic response of wind turbine gear system with uncertain-but-bounded parameters using interval analysis method", Renewable Energy, Volume 113, 2017, Pages 679–687.

[6] S. Cogan, "Réanalyse des structures modifiées à partir des données expérimentales. Rigidification et substitution" Ph.D. of University of Franche-Comté France,1990.

[7] E. François, "Synthese par sous-structuration dynamique issue de l'identification modal" Docteur-Ingénieur de l'Université de Franche-Comté France ,1988.

[8] H. Belhadj, G. Lallement, A. El Hami, A. Perriot, "Iterative numerical methods applied to repetitive systems" Journal of GAMI, Mécanique Matériaux Electricité Vol N°445, Sep. 1992.

[9] A. Mohsine, A. El Hami, "A Robust Study of Reliability-Based Optimisation Methods under Eigen-frequency", International Journal of Computer Methods in Applied Mechanics and Engineering, Vol:199, No.17-20, pp1006-1018, 2010.

[10] K.B. ELLIOTT, L.D. MITCHELL L.D., "The effect of modal truncation on modal modification", Proceedings of the 5th International Modal Analysis Conference, Imperial College-SEM, London, England 1987, p. 72-78.

[11] Y. RAM, S. BRAUN, "Predicting the effect of structural modifications: upper and lower bounds due to modal truncation", International Journal of Analytical and Experimental Modal analysis 6, 1991, pp. 201-213.

[12] A. Bossavit, "Symmetry groups and the boundary value problems". Comp. Meth. Mech. Engng 56, 1986, pp. 167-215.

[13] D.L. Thomas, "*Dynamic of rotationally periodic structures*". International Journal of Methods in Engineering, 4, 1979, p. 81-102.

[14] H. Ait rimouch, A. Marfouk, "An efficient method for reducing the effects of modal truncation for dynamic reanalysis problems". JOURNAL OF MECHANICAL ENGINEERING. Strojn. Cas., Vol. 60, No. 5-6, 2009, pp. 257-275 .

[15] J. F. IMBERT : "Analyse des Structures par Éléments Finis" 1979, Cepadues Édition.

[16] A. S. Householder, 3The Theory of Matrices in Numerical Analysis", Blaisdell, New York, 1964. Dover edition, 1975.

APPENDIX: PRINCIPAL NOTATIONS

ns , *nc* : number of substructures and number of DOF of cell
nl, nr, ni : number of DOF of left, right and interior of cell
K_s , $M_s \in R^{nc,\,nc}$: stiffness and mass matrices of cell
$\overset{\approx}{K}(\alpha)$, $\overset{\approx}{M}(\alpha) \in R^{nc,\,nc}$: matrices of the eigensystems
$Y_s \in R^{nc,\,nc}$: displacement vector of the global structure
$Y_L^{(k)} \in R^{nl,\,nl}$: left displacement vectors of k^{th} cell
$Y_R^{(k)} \in R^{nr,\,nr}$: right displacement vectors of k^{th} cell
$Y_I^{(k)} \in R^{ni,\,ni}$: interior displacement vectors of k^{th} cell
f_{LL}, f_{LR} : interface forces of left and right sides of k^{th} cell
f_L^{ext}, f_R^{ext}, f_I^{ext} : components of decomposition of the left, right and interior external forces applied to the k^{th} cell
$K, M \in R^{n,n}$: stiffness and mass matrices, both are symmetric and positive definite,
$\Delta K, \Delta M \in R^{n,n}$: stiffness and mass modification matrices, both are symmetric
q : number of modified DOF,
m : number of identified modes
n : number of DOF the discret model;
$Y \in R^{n,n}$, $\Lambda \in R^{n,n}$: modal and spectral matrices of the initial structure
$y_\nu \in R^{n,1}$: ν^{th} eigenvector of the initial structure
λ_ν : ν^{th} eigenvalue of the initial structure
$Y_1 \in R^{n,m}$, $\Lambda_1 \in R^{m,m}$: known modal sub-basis of initial structure
$Y_2 \in R^{n,\,n-m}$, $\Lambda_2 \in R^{n-m,\,n-m}$: unknown modal sub-basis of initial structure
$\hat{Y} \in R^{n,n}$, $\hat{\Lambda} \in R^{n,n}$: approximate modal and spectral matrices of the modified structure
$\hat{y}_\nu \in R^{n,1}$: ν^{th} eigenvector of the modified structure
$\hat{\lambda}_\nu$: ν^{th} eigenvalue of the modified structure
I_n : unit matrix of order *n*