A new local fractional α -integral transform for solving the initial value problems with local fractional derivative

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***Abstract*—we present a new local fractional α-integral transform ,its inverse , some of its properties and transformation for some fractional functions. In this paper are considered the Initial value problems with local fractional derivative. Analytical solutions for the homogeneous and nonhomogeneous local fractional differential equations are discussed by using this new α -integral transform**

Keywords— new α-integral transform ; local fractional function ; local fractional differential equations;

I. Introduction

Calculus transform has played an important role in areas randing from fundamental scieces to engineering in the past years and has been applied to a wide class of functions[1,2]. This new α-transform is focused firstly on some previous knowledge of known integral transformations which mention Fourier transform Laplace Sumudu [3], Elzaki [4,5] Yang-Laplace[1] and a new transform integral [6]. The ordinary and partial differential equations have found applications in many problems in mathematical physics.[7,8] Initial value problems for ordinary and partial differential equations have been developed by some authors [9,10,11]. There are analytical methods and numerical methods for solving the differential equations,
such as the finite element method[12] , the Adomian decomposition method[13,14] ,the variational itheration method[15] ,and other methods . In this paper, our aim is to use the new α – integral transform to solve initial value problems with local fractional derivative

The paper is organized as follows :

In Section 2 , we introduce the notions of local fractional calculus theory used in this paper.
In Section 3, we give the definition of the α – integral transform some properties
Section 4, is devoted to the solutions for the homogeneous and nonhomogeneous Initial value problems with local fractional derivative.
In Section 5 , are gived our conclusions.

II. Matematical Fundamentals

Local Fractional Calculus

Definition 1 : The function f(x) is called local fractional continuous at $x = x_0$ if there is the relation $|f(x) - f(x_0)| < \varepsilon^{\alpha}$,$0 < \alpha \le 1$. with $|x - x_0| < \delta$ for ε > 0 , δ > 0 and ε,δ ϵ R .
It is denoted by $\lim_{x\to x_0} f(x) = f(x_0)$.

Definition 2 : The function f(x) is called local fractional continuous on the interval (a,b) if for ε, δ > 0 and ε,δ ϵ R satisfies the relation $|f(x) - f(x_0)| < \varepsilon^{\alpha}$, $0<\alpha\le 1$.
It is denoted by $f(x)\epsilon C_{\propto}(a,b)$.

Definition 3 : In Fractal space let $f(x)\epsilon C_{\propto}(a,b)$;
Local fractional derivative of f(x) of order α at the point $x=x_0$ is given by [1,2,16 – 21]

$$D_x^{(\propto)} f(x_0) = \frac{d^{\propto}}{dx^{\propto}} f(x)|_{x=x_0} = f^{(\propto)}(x_0) = \lim_{x\to x_0} \frac{\Delta^{\alpha}\,(f(x) - f(x_0))}{(x - x_0)^{\propto}} , \quad (6)$$

Were $\Delta^{\alpha}\,(f(x) - f(x_0)) \cong \Gamma(\propto +1)(f(x) - f(x_0))$.
The formulas of local Fractional derivatives of special functions used in the paper are as follows

$$D_x^{(\propto)}\, a\, g(x) = a\, D_x^{(\propto)}\, g(x) \quad (7)$$

$$\frac{d^{\propto}}{dx^{\propto}}\left(\frac{x^{n\propto}}{\Gamma(1+n\propto)}\right) = \frac{x^{(n-1)\propto}}{\Gamma(1+(n-1)\propto)} \quad n\epsilon N . \quad (8)$$

Definition 4 : A partition of the interval $[a, b]$ is denoted by (t_j , t_{j+1}) për j= 0,1,…,N-1 , $t_0=a$ and $t_N=b$ with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = max\{\Delta t_{0,}\, t_1 , \dots\}$.

Definition 5 : Local fractional integral of f(x) in the interval $[a, b]$ is given by [1,2,21]

$${}_aI_b^{(\propto)} f(x) = \frac{1}{\Gamma(\propto+1)} \int_a^b f(t)(dt)^{\propto}$$

$= \lim_{\Delta t\to 0}\sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^{\propto}$ (9)

The formulas of local fractional integrals of some special functions used in this work are as follows :

$${}_0I_x^{(\propto)}\, a\, g(x) = a\ {}_0I_x^{(\propto)}\, g(x) \quad (10)$$

$${}_0I_t^{(\propto)}\left(\frac{x^{n\propto}}{\Gamma(1+n\propto)}\right) = \frac{x^{(n+1)\propto}}{\Gamma(1+(n+1)\propto)} \quad n \epsilon N \ . \quad (11)$$

III . A NEW LOCAL FRACTIONAL INTEGRAL TRANSFORM AND ITS INVERSE FORMULA

A new Local Fractional α- integral transform and its inverse formula .

Definition 6 : Let $\frac{1}{\Gamma(\propto+1)}\int_0^{+\infty}|f(t)|(dt)^{\propto} < K < \infty$. The α- integral transform f(x) is given by

$$K_{\propto}\{f(t)\} = A_{\propto,f}(v) = \frac{1}{\Gamma(\propto+1)}\frac{1}{v^{\propto}}\int_0^{+\infty} E_{\propto}(-(\tfrac{t}{v^2})^{\propto}) f(t)(dt)^{\propto} \qquad 0<\alpha\leq 1 \quad (12)$$

were the integral converges and $v^{\alpha} \epsilon R^{\alpha}$

Definition 7 : The inverse formula of the α- integral transform is given by

$$K_{\propto}^{-1}\{A_{\propto,f}(v)\} = \frac{1}{(2\pi i)^{\propto}}\int_{\beta-i\omega}^{\beta+i\omega} E_{\alpha}((vt)^{\propto})\, A_{\alpha}(\tfrac{1}{\sqrt{v}^{\alpha}})\frac{(dv)^{\propto}}{\sqrt{v}^{\propto}} \quad (13)$$

Where $v^{\alpha} = \beta^{\alpha} + i^{\alpha}\omega^{\alpha}$; here i^{α} is fractal imaginary unit of v and Re(v) = β > 0 .

Some properties the α – integral transform are presented as follows

$$K_{\alpha}\{E_{\alpha}(x^{\alpha})\} = \frac{v^{\alpha}}{1-v^{2\alpha}} \quad (14)$$

$$K_{\alpha}\{sin_{\alpha}(c^{\alpha}x^{\alpha})\} = \frac{c^{\alpha}v^{3\alpha}}{1+c^{2\alpha}v^{4\alpha}} \quad (15)$$

$$K_{\alpha}\{cos_{\alpha}(c^{\alpha}x^{\alpha})\} = \frac{v^{\alpha}}{1+c^{2\alpha}v^{4\alpha}} \quad (16)$$

$$K_{\alpha}\{x^{k\alpha}\} = v^{\alpha(2k+1)}\Gamma(\alpha k+1) \quad (17)$$

$$K_{\alpha}\{x^{k\alpha}E_{\alpha}(c^{\alpha}k^{\alpha})\} = \frac{v^{\alpha(2k+1)}\Gamma(\alpha k+1)}{(1-c^2v^2)^{(2k+1)\alpha}} \quad (18)$$

$$K_{\propto}\{E_{\propto}(c^{\propto}t^{\propto})f(t)\} = A_{\propto,f}(1-v^2\, c) \quad (19)$$

$$K_{\propto}\{(af(t)+bg(t)\} = aK_{\propto}\{f(t)\} + bK_{\propto}\{g(t)\} \quad (20)$$

$$K_{\alpha}\{f^{(n\alpha)}(t)\} = \frac{A_{\alpha,f}(v)}{v^{2\alpha n}} - \sum_{k=0}^{n-1}\frac{f^{(k\alpha)}(0)}{v^{(2(n-k)-1)\alpha}} \quad (21)$$

III INITIAL VALUE PROBLEMS WHITH LOCAL FRACTIONAL DERIVATES

In this section we handle the homogeneous and nonhomogeneous initial value problems with local fractional derivative.

Examle 1:

Let us consider the homogeneous Initial value 1problems with local fractional derivative in the form

$$\frac{d^{4\alpha}y}{d^{4\alpha}x} - y = 0 \quad (22)$$

With initial boundary conditions

$$y(0) = 0 \ , \qquad y^{(\alpha)}(0) = 0$$
$$y^{(2\alpha)}(0) = 0 \ , \qquad y^{(3\alpha)}(0) = 1 \ . \quad (23)$$

From (21) we have

$$K_{\alpha}\{y^{(4\alpha)}(x)\} = \frac{K_{\alpha}\{y(x)\}}{v^{8\alpha}} + \frac{y(0)}{v^{7\alpha}} + \frac{y^{(\alpha)}(0)}{v^{5\alpha}} + \frac{y^{(2\alpha)}(0)}{v^{3\alpha}} + \frac{y^{(3\alpha)}(0)}{v^{\alpha}} = \frac{K_{\alpha}\{y(x)\}}{v^{8\alpha}} + \frac{1}{v^{\alpha}} \quad (24)$$

Hence (22) can be writen

$$\frac{K_{\alpha}\{y(x)\}}{v^{8\alpha}} + \frac{1}{v^{\alpha}} - K_{\alpha}\{y(x)\} = 0 \quad (25)$$

Which leads to

$$K_{\alpha}\{y(x)\}\left[\frac{1}{v^{8\alpha}} - 1\right] = \frac{1}{v^{\alpha}} \quad (26)$$

So that

$$K_{\alpha}\{y(x)\} = \frac{v^{7\alpha}}{1-v^{8\alpha}} \ . \quad (27)$$

Therefore , we get

$$y(x) = K_{\alpha}^{-1}\left\{\frac{1}{4}\frac{v^{\alpha}}{1-v^{2\alpha}} - \frac{1}{4}\frac{v^{\alpha}}{1+v^{2\alpha}} - \frac{1}{2}\frac{v^{3\alpha}}{1-v^{4\alpha}}\right\} \quad (28)$$
$$= \frac{1}{4}E_{\alpha}(-x^{\alpha}) - \frac{1}{4}E_{\alpha}(x^{\alpha}) - \frac{1}{2}sin_{\alpha}(x^{\alpha}) \ .$$

Examle 2:

Let us consider the homogeneous Initial value problems with local fractional derivative in the form

$$\frac{d^{2\alpha}y}{d^{2\alpha}x} - \frac{d^{\alpha}y}{d^{\alpha}x} + 2y = 0 \quad (29)$$

With initial boundary conditions

$$y(0) = 1 \ , \qquad y^{(\alpha)}(0) = 0 \ . (30)$$

From (21) we have

$$K_{\alpha}\{y^{(2\alpha)}(x)\} = \frac{K_{\alpha}\{y(x)\}}{v^{4\alpha}} + \frac{y(0)}{v^{3\alpha}} + \frac{y^{(\alpha)}(0)}{v^{\alpha}} \quad (31)$$

$$K_\alpha\{y^{(\alpha)}(x)\} = \frac{K_\alpha\{y(x)\}}{v^{2\alpha}} + \frac{y(0)}{v^{2\alpha}}$$

Hence (29) can be written

$$\frac{K_\alpha\{y(x)\}}{v^{4\alpha}} + \frac{y(0)}{v^{3\alpha}} + \frac{y^{(\alpha)}(0)}{v^{\alpha}} - \frac{K_\alpha\{y(x)\}}{v^{2\alpha}} - \frac{y(0)}{v^{2\alpha}} - 2K_\alpha\{y(x)\} = 0 \quad (32)$$

Hence , making use of initial boundary conditions we obtain

$$K_\alpha\{y(x)\}\left[\frac{1}{v^{4\alpha}} - \frac{1}{v^{2\alpha}} + 2\right] = \frac{1}{v^{3\alpha}} - \frac{1}{v^{\alpha}} \quad (33)$$

So that

$$K_\alpha\{y(x)\} = \frac{v^{\alpha}}{1-v^{2\alpha}} \ . \quad (34)$$

Therefore , we get

$$y(x) = K_\alpha^{-1}\left\{\frac{v^{\alpha}}{1-v^{2\alpha}}\right\} = E_\alpha(-x^{\alpha}) \quad (35)$$

Examle 3:

Let us consider the non - homogeneous Initial value problems with local fractional derivative

$$\frac{d^{2\alpha}y}{d^{2\alpha}x} - y = sin_\alpha(x^{\alpha}) \quad (36)$$

With initial boundary conditions

$$y(0) = 0 \ , \qquad y^{(\alpha)}(0) = 1 \ . \quad (37)$$

From (21) we have

$$K_\alpha\{y^{(2\alpha)}(x)\} = \frac{K_\alpha\{y(x)\}}{v^{4\alpha}} + \frac{y(0)}{v^{3\alpha}} + \frac{y^{(\alpha)}(0)}{v^{\alpha}} \quad (38)$$

By using(15) and (38) , (36) can be written

$$\frac{K_\alpha\{y(x)\}}{v^{4\alpha}} + -2K_\alpha\{y(x)\} = \frac{v^{3\alpha}}{1+v^{4\alpha}} \quad (39)$$

Hence , making use of initial boundary conditions we obtain

$$K_\alpha\{y(x)\}\left[\frac{1}{v^{4\alpha}} - 1\right] = \frac{v^{3\alpha}}{1+v^{4\alpha}} - \frac{1}{v^{\alpha}} \quad (40)$$

So that

$$K_\alpha\{y(x)\} = \frac{3}{4}\frac{v^{\alpha}}{1-v^{2\alpha}} - \frac{3}{4}\frac{v^{\alpha}}{1+v^{2\alpha}} - \frac{1}{2}\frac{v^{3\alpha}}{1+v^{4\alpha}} \quad (41)$$

Therefore we get

$$y(x) = K_\alpha^{-1}\left\{\frac{3}{4}\frac{v^{\alpha}}{1-v^{2\alpha}} - \frac{3}{4}\frac{v^{\alpha}}{1+v^{2\alpha}} - \frac{1}{2}\frac{v^{3\alpha}}{1+v^{4\alpha}}\right\} \quad (42)$$

$$\frac{3}{4}E_\alpha(-x^{\alpha}) - \frac{3}{4}E_\alpha(x^{\alpha}) - \frac{1}{2}sin_\alpha(x^{\alpha}) \ .$$

Examle 4:

Let us consider the non - homogeneous Initial value problems with local fractional derivative

$$\frac{d^{2\alpha}y}{d^{2\alpha}x} + y = E_\alpha(x^{\alpha}) \quad (43)$$

With initial boundary conditions

$$y(0) = 0 \ , \qquad y^{(\alpha)}(0) = 1 \ . \quad (44)$$

From (20) we have

$$K_\alpha\{y^{(2\alpha)}(x)\} = \frac{K_\alpha\{y(x)\}}{v^{4\alpha}} + \frac{y(0)}{v^{3\alpha}} + \frac{y^{(\alpha)}(0)}{v^{\alpha}} \quad (45)$$

By using(14) and (45) , (43) can be written

$$\frac{K_\alpha\{y(x)\}}{v^{4\alpha}} - \frac{1}{v^{\alpha}} + K_\alpha\{y(x)\} = \frac{v^{\alpha}}{1-v^{2\alpha}} \quad (46)$$

Hence , making use of initial boundary conditions we obtain

$$K_\alpha\{y(x)\}\left[\frac{1}{v^{4\alpha}} + 1\right] = \frac{v^{\alpha}}{1-v^{2\alpha}} + \frac{1}{v^{\alpha}} \quad (47)$$

So that

$$K_\alpha\{y(x)\} = \frac{1}{2}\frac{v^{3\alpha}}{1+v^{4\alpha}} - \frac{1}{2}\frac{v^{\alpha}}{1+v^{4\alpha}} + \frac{1}{2}\frac{v^{\alpha}}{1-v^{2\alpha}} \quad (48)$$

Therefore we get

$$y(x) = K_\alpha^{-1}\left\{\frac{1}{2}\frac{v^{3\alpha}}{1+v^{4\alpha}} - \frac{1}{2}\frac{v^{\alpha}}{1+v^{4\alpha}} + \frac{1}{2}\frac{v^{\alpha}}{1-v^{2\alpha}}\right\} \quad (49)$$

$$= \frac{1}{2}sin_\alpha(x^{\alpha}) - \frac{1}{2}cos_\alpha(x^{\alpha}) + \frac{1}{2}E_\alpha(x^{\alpha}) \ .$$

IV . CONCLUSIONS

In this work we have used the local fractional α-integral transform to handle the homogeneous and non-homogeneous initial value problems with local fractional derivative. Some illustrative examples of approximate solutions for local fractional initial value problems s are discussed. The obtained results illustrate that the local fractional α- integral transform is an efficient mathematical tool to solve the homogeneous and non-homogeneous initial value problems with local fractional derivative.

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