Dynamic vibration characteristics of non-homogenous beam-model MEMS

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Abstract—In this paper a Timoshenko micro-beam model based on the modified couple stress theory is established to capture the size effects on the vibrational and dynamical behavior of the system. Mechanical properties of the micro-beam are supposed to vary through thickness based on the power law. Governing equations are derived using the Hamilton’s principle. For free vibration analysis, a closed-form solution approach is presented. The verification of the model is accomplished by comparing the obtained results with benchmark results existing in the literature. A detailed consideration for the effects of material length scale parameter, power index and ratio of beam length to beam thickness upon the vibrational behavior of the model are reported. It is observed that these parameters have substantial role in the dynamic behavior of micro-structures.

Keywords—Micro-beam; Modified couple stress theory; Vibration Analysis; Timoshenko Beam Theory, Nondimensional Frequency

I. INTRODUCTION

With the advance of Micro/Nano-technology, devices with thickness dimensions at the order of microns and sub microns are of high interest. They have some benefits in comparison to macro devices such like: they are light, often inexpensive and more durable. The most used geometrical shapes for the simulations of such devises are beams and plates. Since the controlled experiments in micro structures are difficult and cost a lot, affording the decent mathematical models is vital [1], [2].

Functionally graded materials (FGMs) belong to a class of graded composites. These non-homogenous materials are a mixture of two components with diverse mechanical properties which are being designed to exploit the benefits of the both constituents. An intentionally smooth variation of the materials in one specific direction is used to make graded materials capable of possessing different properties. Functionally graded materials usually consist of metal and ceramic phases. Ceramic part of the material delivers high temperature resistance and metal part impedes fracture due to thermal stresses in few cycles of loading. Material gradation causes smaller stress impositions. Moreover these stresses can be located in desired positions of the structure.

Ceramic parts, because of their low thermal conduction coefficients can bear high temperature gradients. The ductile metal phase also increases the potential to withstand under rough dynamic loadings. Facing high shear stresses at the location of jointing two different components of composites is a serious problem which can be obviated by using material variant rules, existing in functionally graded materials. Thus ameliorated stress distribution, enhanced thermal gradient resistance and increased toughness, are all salient characters of functionally graded materials that make such mechanical devices applicable in biomechanics, optoelectronics, Micro/Nano-technologies. With simultaneous growth of micro structures and material technologies, functionally graded materials are extensively used in micro-scaled devices such as thin films in the form of shape memory alloys, micro-switches, micro-actuators, Micro/Nano-electro-mechanical systems (MEMS and NEMS) [3] through [8].

Some researchers have published different works regarding non-homogenous micro-structures. Studies in [9] through [12] are experimentally validated. They show that when the dimensions of a structure are scaled down, it is crucial to capture the size-dependent response of the system which is not afforded by classical continuum mechanics theories; as a result, researches tried to present non-classical theories in order to consider the small scale effects in micro and Nano-structures. Strain gradient theory, couple stress theory and modified couple stress theory are of the most prevalent theories used. Strain gradient theory proposed in [13] accounts for three intrinsic material length scale parameters. The mentioned parameters above, relate curvature tensor, deviator tensor and dilatation tensor to the stress tensor. Couple stress theory presented in [14], [15] and [16] uses four material parameters, including two classical and two additional parameters. Based on this theory, a group of researchers in [17] proposed that three equilibrium equations should be considered for the material element; classical equilibrium equations of forces and moments of forces and an additional equation for the equilibrium of moments of couples. They made an inference that this additional equation implies the symmetry of the couple stress tensor. So they improved the constitutive equations and presented the modified couple stress theory. Besides the continuum theories, beam theories play significant role in the prediction of static and dynamic response...
of the structures. Euler-Bernoulli beam theory is simple in comparison to the other theories since the shear characteristics and deformations are neglected. This is a decent theory when the ratio of length to the thickness is larger than 20. However, to prognosticate more accurate behavior, especially when the beam is thicker, some restrictive assumptions should be omitted and shear characteristics should be accounted. In this case Timoshenko and other high order beam theories are suitable.

Utilizing modified couple stress theory, the mechanical properties of an Euler–Bernoulli beam in the static condition in [18] is studied. In [19] size-dependent free vibration behavior of carbon reinforced polymer micro-cantilevers based on the modified couple stress theory is carried out, in which used Euler-Bernoulli beam model and the axially-graded material in in consideration. In another research, free vibration behavior of micro-scaled structures using the modified couple stress theory and three different beam theories: Euler-Bernoulli, Timoshenko and third order shear deformation theories is accomplished [20]. In [21] functionally graded Timoshenko Nano-beam model for free vibration analysis is modeled, and non-local elasticity theory and principle of minimum potential energy for obtaining the governing equations are utilized. In [22], researchers have studied dynamic response of a functionally graded micro beam based on the strain gradient theory. They modeled thick beams using Timoshenko beam theory. Another group of researchers in [23] presented Timoshenko size-dependent model, using the nonlocal elasticity theory. Free vibration analysis of Euler-Bernoulli micro beams using an approximate method is carried out in [24], which is based on the modified couple stress theory. In another research paper, vibration analysis of a temperature-dependent micro-beam is reported. This study is based on the modified couple stress theory and the thermo-mechanical properties of the system are varying according to temperature shifts and thermal stresses play a major role in the characteristic determination [25].

In this paper, free lateral vibration response of functionally graded Timoshenko micro beam is presented. The beam is graded in the direction of the thickness. The modified couple stress theory in addition to the Hamilton’s principle is used to obtain the governing equations, boundary and initial conditions. The effects of material length scale parameter upon dimensionless natural frequencies of a simply-supported micro beam are reported. Also, for first time, higher order modes of vibration influenced by gradient index and wide range of slenderness ratios are obtained. The other noble case of this paper is to use different distribution rule for functionally graded materials and inertia-based process for calculating the dimensionless frequencies.

II. MATHEMATICAL MODELING

a. Modified couple stress theory

With the modified couple stress theory, strain energy function depends on both strain and rotation gradients. In the Cartesian coordinates, \( u_x, u_y \) and \( u_z \) are defined as the displacement field vectors in the directions of the beam’s length, width and thickness. The displacement gradient tensor \( u_{ij} \) can be decomposed into symmetric and anti-symmetric parts as the tensors of strain and rotation, respectively:

\[
\begin{align*}
    u_{ij} &= \varepsilon_{ij} + \omega_{ij} \\
    \text{where} \\
    \varepsilon_{ij} &= \frac{1}{2} \left( u_{ij} + u_{ji} \right) \\
    \omega_{ij} &= \frac{1}{2} \left( u_{ij} - u_{ji} \right)
\end{align*}
\]

Using the alternator \( \varepsilon \) rotation vector dual to the rotation tensor can be defined as:

\[
\theta_j = \frac{1}{2} \omega_{jk} \varepsilon_{jkl} \tag{4}
\]

Now gradient of rotation is defined as:

\[
\kappa_{ij} = \theta_j = \frac{1}{2} \omega_{jk} \varepsilon_{kij} \tag{5}
\]

Where \( \kappa \) denotes the curvature tensor.

Constitutive equations of the modified couple stress theory are defined by the strain energy density function \( e_s \) [23]:

\[
e_s = \frac{1}{2} \lambda \varepsilon_{ii} \varepsilon_{jj} + G \varepsilon_{ij} \varepsilon_{ij} + 2G \kappa_{ij} \kappa_{ij} \tag{6}
\]

That \( \lambda \) and \( G \) are two classical Lame constants. \( I \) is Lame-type non-classical material parameter which introduces the couple stress effects.

Eq. (1) leads to the constitutive equations as:

\[
\begin{align*}
    \sigma_{ij} &= \frac{\partial e_s}{\partial \varepsilon_{ij}} = 2G \varepsilon_{ij} + \lambda \delta_{ij} \varepsilon_{kk} \\
    \varrho_{ij} &= \frac{\partial e_s}{\partial \omega_{ij}} = 4G \kappa_{ij} \kappa_{ij}
\end{align*}
\]

In which \( \sigma_{ij} \) and \( \varrho_{ij} \) denote the classical stress and couple stress tensors.

The strain energy of the deformed linear elastic body based on the classical strain and curvature tensors is associated to the symmetric classical stress and deviatoric couple stress tensor. This energy for a body occupying volume \( V \) is defined as:

\[
U_s = \frac{1}{2} \int_V \left( \sigma_{ij} \varepsilon_{ij} + \varrho_{ij} \kappa_{ij} \right) \, dV \tag{9}
\]
b. Fuctionally graded materials

In Figure 1. a functionally graded micro beam of length $L$, width $b$ and thickness $h$ is shown. We supposed that the micro beam is made up of two dissimilar materials and the effective mechanical properties of the beam vary through the thickness direction.

Using decent medley rule for micro structures, one can describe the effective mechanical properties $P$ based on the rule of mixture as:

$$ P = P_a V_a + P_s V_s $$

(10)

Where $P_a$ and $P_s$ are the effective mechanical properties of the constituents, $V_a$ and $V_s$ are the volume fractions, restricted by the following equation:

$$ V_a + V_s = 1 $$

(11)

We used power-law form to define the mechanical properties of the micro structure. The volume fraction of the second material is defined by:

$$ V_s = \left(\frac{z}{h} + \frac{1}{2}\right)^k $$

(12)

In which $k$ denotes the power-law exponent that specifies the material variation contour through the thickness direction. The three requisite properties for delineation are Young’s modulus, shear modulus and density which can be replaced with $P$ as follows:

$$ E(z) = (E_a - E_s) \left(\frac{z}{h} + \frac{1}{2}\right) + E_s $$

(13)

$$ G(z) = (G_a - G_s) \left(\frac{z}{h} + \frac{1}{2}\right) + G_s $$

(14)

$$ \rho(z) = (\rho_a - \rho_s) \left(\frac{z}{h} + \frac{1}{2}\right) + \rho_s $$

(15)

It can be easily depicted that the micro functionally graded beam reaches pure material properties at the top and bottom surfaces.

c. Timoshenko beam theory

In the Timoshenko beams, shear strain due to distortion is not being neglected; hence, rotation angle is not equal to derivation of the lateral displacement. According to this theory, axial displacement, $u$, lateral displacement, $w$, of any point on the neutral axis is expressed as follows [24, 25]:

$$ u(x,y,z) = u(x,t) - z\theta(x,t) $$

(16)

$$ u_y(x,y,z) = 0 $$

(17)

$$ u_k(x,y,z) = w(x,t) $$

(18)

Where $\theta$, shows the total angle of rotation of the cross section.

Nonzero strains are obtained as:

$$ \varepsilon_{xx} = \frac{\partial u}{\partial x} - z \frac{\partial \theta}{\partial x} $$

(19)

$$ \varepsilon_{xy} = \frac{\partial w}{\partial x} + \frac{1}{2} \left( \frac{\partial u}{\partial y} - \theta \right) $$

(20)

$$ \kappa_{xy} = \kappa_{yx} = \frac{-1}{4} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial u}{\partial x} \right) $$

(21)

d. Governing equations of motion

Governing equations expressing the vibrational motions are obtained using the Hamilton’s principle.

$$ \delta \left[ \int_{t_1}^{t_2} (T - Us) \, dt \right] = 0 $$

(22)

In which $T$ is kinetic energy and $Us$ is potential energy.

Exerting the Eqs. (13)-(21) into Eq. (24) gives the potential energy. Also kinetic energy for a Timoshenko beam is obtained using Eq. (19):

$$ U_s = \frac{1}{2} \int_V \left( 2G(z) + \lambda(z) \right) \varepsilon_{xx}^2 + 4G(z) \varepsilon_{yy}^2 + 4G(z) \varepsilon_{yy}^2 $$

(23)

$$ T = \frac{1}{2} \int_V \rho(z) \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \, dV $$

(24)

Using variation calculus and the detail by detail method, final form of the energy terms can be expressed as following:

$$ \int_{t_1}^{t_2} \left[ \delta \left( \frac{\partial T}{\partial u} \right) \delta u - \left( D_1 \frac{\partial^2 u}{\partial x^2} + D_2 \frac{\partial^2 u}{\partial x \partial t} + D_3 \frac{\partial^2 u}{\partial t^2} \right) \delta u \right] \, dx \, dt $$

(25)

$$ \int_{t_1}^{t_2} \left[ \delta \left( \frac{\partial T}{\partial w} \right) \delta w - \left( F_1 \frac{\partial^2 w}{\partial x^2} + F_2 \frac{\partial^2 w}{\partial x \partial t} + F_3 \frac{\partial^2 w}{\partial t^2} \right) \delta w \right] \, dx \, dt $$

(26)

Coefficients used in Eqs. (20) and (21) are defined in the following:

$$ S_2 = \frac{1}{4} l^2 \int_A G(z) \, dA $$

(27)

Neglecting the shear effects, some of the coefficients are obtained based on the modulus of elasticity.

$$ S_4 = \int_A E(z)(-z) \, dA $$

(28)

$$ S_5 = \int_A E(z)(z)^2 \, dA + \frac{1}{2} l^2 \int_A G(z) \, dA $$

(29)

$$ S_7 = k_1 \int_A G(z) \, dA $$

(30)

$$ D_1 = \int_A E(z) \, dA $$

(31)

$$ D_7 = \int_A \rho(z)(-z) \, dA $$

(32)

$$ F_1 = \int_A \rho(z) \, dA $$

(33)

$$ S_4, S_5, D_4, D_5, D_6, D_7, F_2 = 0 $$

(34)

$$ D_3 = S_4, F_3 = D_7 $$

(35)

Using Eqs. (27)-(35) into the Eq. (22), gives a system of coupled partial differential equations, known as the governing equations.

III. Solution procedure

For the governing equations, related to free vibration of a simply-supported FG micro beam, an analytical solution based on the Navier method is

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presented. Navier method as a kind of discretizing approach which expands the displacement and rotation functions as products of undetermined coefficients and specific trigonometric functions which satisfy the boundary conditions, at \( t = 0, L \). The functions are defined in the following form:

\[
\begin{align*}
\psi(x, t) &= \sum_{n=1}^{\infty} a_n \sin(n \pi x) e^{i \omega_n t}, \quad \alpha = (n \pi) / L \\
b(x, t) &= \sum_{n=1}^{\infty} b_n \cos(n \pi x) e^{i \omega_n t} \\
c(x, t) &= \sum_{n=1}^{\infty} c_n \cos(n \pi x) e^{i \omega_n t}
\end{align*}
\]

(36) \quad (37) \quad (38)

In which \( a_n, b_n, c_n \) are the coefficients to be determined for each \( n \). Boundary conditions of a simply-supported beam are as follows:

\[
\psi|_{x=0, L}, \quad \frac{\partial^2 \psi}{\partial x^2}|_{x=0, L} = 0
\]

(39)

Substituting Eqs. (36)-(38) into Eqs. (25)-(26) leads to the following system of equations:

\[
\begin{align*}
(F_4 \omega_n^4 - S_2 \alpha_n^4 - S_3 \alpha_n^2) a_n + (-S_4 \alpha_n^3 + S_5 \alpha_n^2) b_n &= 0 \\
(-S_4 \alpha_n^3 + S_5 \alpha_n^2) a_n + (D_3 \omega_n^2 - S_6 \alpha_n^2 - S_7) b_n + (D_5 \omega_n^2 - S_6 \alpha_n^2) c_n &= 0 \\
(F_2 \omega_n^2 - D_2 \alpha_n^2) b_n + (F_1 \omega_n^2 - D_4 \alpha_n^2) c_n &= 0
\end{align*}
\]

(40) \quad (41) \quad (42)

Determinant of the coefficient matrix of the above equations, gives frequency equation in the form of polynomial for each \( n \). Setting this polynomial equal to zero gives the frequency of each mode.

IV. Results and discussion

Steel and alumina (\( Al_xodela \)) are the two constituents of the simply-supported functionally graded micro beam investigated in this study. The mechanical properties vary through the thickness direction based on a power-law, Table 1 shows these mechanical properties of pure steel and alumina. The beam length and width are equal to \( L = 10000, b = 1000 \) micro meters. Natural frequencies are non-dimensionalized using the following equation:

\[
\nonumber \omega_0 = \omega L^2 \sqrt{\rho/\mu} \frac{A}{E L^2}, \quad \text{where} \quad L = \frac{bh^3}{12}, \quad \text{is the second moment of inertia of the beam cross section and} \quad A \text{ is area of the cross section.}
\]

The dimensionless frequencies are accompanied with slenderness ratio, material distribution for three modes of vibration. According to experimental tests reported by Lame, non-classical parameter \( l \) is taken 15 micro meters. \( k_s \) represents the shear correction factor which for rectangular cross sections is equal to 5/6. For verification and check validity of the present analysis, the results are compared with those of Euler-Bernoulli FG Nano beam (\( [21] \)) in Tables 2-3 and Timoshenko FG Nano beam (\( [23] \)) in Tables 4-5.

Putting the material length scale parameter equal to zero gives the equations which the corresponding solution procedure culminates in dimensionless frequencies of classical theory shown in Tables 2-5. Tables 4-5 show that, generally the modified couple stress theory overestimates classical frequencies rather than those of the non-local elasticity theory. Dimensionless frequencies based on the modified couple stress theory for the first three modes of vibration are shown in Tables 6-7; results are shown in the division of different slenderness ratios and diverse power indexes. According to Tables 6-7, as the power index increases the frequency decreases and as the slenderness ratio increases the frequency increases.

Figs. 1-3 demonstrate the variation of dimensionless frequencies with varying material distribution (power index) and specific slenderness ratios for the first, second and three mode of vibration. It can be observed that there is a sharp gradient when the power index is in small range \( 0 < k < 2 \); rate of changes reduce at \( 2 < k < 5 \) and finally, the frequency has the least variation with power index for \( 5 < k < 10 \). To make an inference for so large power index values, we expect an independent behavior of the micro structure. Figs. 1-3 also substantiate the reduction of frequencies by increasing power index and decreasing slenderness ratio concluded from Tables 6-7.

Fig. 4 demonstrates the gradation of dimensionless frequency with slenderness ratios. As it can be seen, for all power indexes, frequency increases as the slenderness ratio takes larger quantities. Also it seems that the rates of changes with slenderness ratio are similar to each other for different material distributions.

V. Conclusion

The free vibration analysis of functionally graded micro beam based on the modified couple stress theory is presented in this study. The simply-supported micro beam is modeled according to the Timoshenko beam theories. The non-classical constitutive equations are formed due to assumptions of the modified couple stress theory. The Hamilton’s principle is used to derive the governing equations, initial and boundary conditions. Based on the Navier method, an analytical solution is proposed. Numerical results express the effect of the material length scale parameter upon the vibration behavior of the FG micro beam. As a result it seems crucial to exert the non-classical parameter, mentioned above in the vibration analysis of micro structures. The other incisive case playing a tangible role in the analysis is the power index. Due to the results, it is feasible to reach the specific frequency by selecting the appropriate power quantity. Effects of the slenderness ratios and comparative dimensions of the micro beam are characterized such that smooth gradation of frequency is obtained by different beam length to thickness ratio. Consequently, a fastidious design with respect to the material length scale parameter, decent power index value and slenderness ratio results in the desired and predictable dynamic behavior of the FG micro beams.
### TABLE 1. Mechanical Properties of FGM Constituents

<table>
<thead>
<tr>
<th>Properties</th>
<th>Steel</th>
<th>Alumina</th>
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<td>3960 (Kg/m$^3$)</td>
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<td>$E$</td>
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<td>390 (GPa)</td>
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<td>$\nu$</td>
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### TABLE 2. Comparison of Non-dimensional Natural Frequencies With [21] (2012). ($k=0.0,1.0,2.0,0.5$)

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<th>$L/h$</th>
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### TABLE 3. Comparison of Non-dimensional Natural Frequencies With [21] (2012). ($k=1.2,5,10$)

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### TABLE 4. Comparison of Non-dimensional Natural Frequencies With [23] (2013). ($k=0.0,1.0,2.0,0.5$)

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### Table 5. Comparison of Non-dimensional Natural Frequencies With [23] (2013). (k=1,2,5,10)

<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>20</td>
<td>6.9830</td>
<td>6.9676</td>
<td>5.9289</td>
<td>5.9172</td>
<td>6.2286</td>
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<tr>
<td>50</td>
<td>7.0876</td>
<td>6.9917</td>
<td>6.0118</td>
<td>5.9389</td>
<td>5.7417</td>
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<tr>
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<td>7.3713</td>
<td>6.9952</td>
<td>6.2286</td>
<td>5.9421</td>
<td>5.9464</td>
</tr>
</tbody>
</table>

### Table 6. Non-dimensional Natural Frequencies Based On The Modified Couple Stress Theory. (k=0,0.1,0.2,0.5)

<table>
<thead>
<tr>
<th>$L/h$</th>
<th>Mode of vibration</th>
<th>$k = 0$</th>
<th>$k = 0.1$</th>
<th>$k = 0.2$</th>
<th>$k = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$n = 1$</td>
<td>9.7183</td>
<td>9.0576</td>
<td>8.5622</td>
<td>7.6276</td>
</tr>
<tr>
<td></td>
<td>$n = 2$</td>
<td>37.1922</td>
<td>34.6661</td>
<td>32.7707</td>
<td>29.1903</td>
</tr>
<tr>
<td></td>
<td>$n = 3$</td>
<td>78.5071</td>
<td>73.1835</td>
<td>69.1854</td>
<td>61.6187</td>
</tr>
<tr>
<td>30</td>
<td>$n = 1$</td>
<td>9.8999</td>
<td>9.2272</td>
<td>8.7227</td>
<td>7.7712</td>
</tr>
<tr>
<td></td>
<td>$n = 2$</td>
<td>39.3869</td>
<td>36.7108</td>
<td>34.7039</td>
<td>30.9176</td>
</tr>
<tr>
<td></td>
<td>$n = 3$</td>
<td>87.8443</td>
<td>81.8767</td>
<td>77.4009</td>
<td>68.9546</td>
</tr>
<tr>
<td>90</td>
<td>$n = 1$</td>
<td>10.2935</td>
<td>9.5979</td>
<td>9.0757</td>
<td>8.0882</td>
</tr>
<tr>
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<td>$n = 2$</td>
<td>41.1491</td>
<td>38.3683</td>
<td>36.2810</td>
<td>32.3332</td>
</tr>
<tr>
<td></td>
<td>$n = 3$</td>
<td>92.4923</td>
<td>86.2419</td>
<td>81.5502</td>
<td>72.6765</td>
</tr>
</tbody>
</table>

### Table 7. Non-dimensional natural frequencies based on the modified couple stress theory. (k=1,2,5,10)

<table>
<thead>
<tr>
<th>$L/h$</th>
<th>Mode of vibration</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 5$</th>
<th>$k = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$n = 1$</td>
<td>6.8877</td>
<td>6.3210</td>
<td>5.8448</td>
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<tr>
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<td>$n = 2$</td>
<td>26.3458</td>
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<td>$n = 3$</td>
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<td>50.8746</td>
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<tr>
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<td>$n = 1$</td>
<td>7.0182</td>
<td>6.4418</td>
<td>5.9579</td>
<td>5.6907</td>
</tr>
<tr>
<td></td>
<td>$n = 2$</td>
<td>27.9199</td>
<td>25.6235</td>
<td>23.6940</td>
<td>22.6310</td>
</tr>
<tr>
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<td>$n = 3$</td>
<td>62.2624</td>
<td>57.1284</td>
<td>52.8103</td>
<td>50.4393</td>
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<tr>
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<td>$n = 1$</td>
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<td>6.6909</td>
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<td>5.8957</td>
</tr>
<tr>
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<td>$n = 2$</td>
<td>29.1864</td>
<td>26.7470</td>
<td>24.6835</td>
<td>23.5675</td>
</tr>
<tr>
<td></td>
<td>$n = 3$</td>
<td>65.6023</td>
<td>60.1177</td>
<td>55.4778</td>
<td>52.9693</td>
</tr>
</tbody>
</table>
Fig 1. Variation of non-dimensional natural frequencies of the first mode with power index.

Fig 2. Variation of non-dimensional natural frequencies of the second mode with power index.

Fig 3. Variation of non-dimensional natural frequencies of the third mode with power index.

Fig 4. Gradation of non-dimensional natural frequencies with slenderness ratios.
VI. References


