ASYMMETRIC NONLINEAR INSTABILITY OF A LIQUID JET

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Abstract–Nonlinear asymmetric breakup of a circular capillary jet when an asymmetric periodic initial disturbance is given at the surface is presented. It is shown that the complex amplitude of the wave can be described by nonlinear Schrödinger equation. Numerical examples are presented for various values of asymmetry. Also included the analysis and numerical works when the surrounding gas is absent.

Key words-method of multiple scales;nonlinear asymmetric jet;Schrödinger equation.

1. Introduction

When a dense fluid is ejected into a less dense fluid from a hole, a jet formes. The fluid jet is inherently unstable and breakup easily. The capillary force is shown with linear theory to be responsible for the onset of instability in the presence or absence of fluid viscosities. Subsequent to the onset, the amplitude of disturbances grows rapidly and the neglected nonlinear terms in the linear theory are no longer negligible. Thus the nonlinear evolution of disturbances that lead to the eventual pinching off of drops from a liquid jet can only be explained with nonlinear theories. The neglected nonlinear convective acceleration term in the linear theory can induce higher harmonics which appear to lead to the formation of satellites between the main drops.

The breakup of a liquid jet into droplets has been of great interest because of the numerous applications in various fields. In some modern applications of instability of jets, it is advantageous to hasten the breakup, but in other applications suppression of the breakup is essential.

Recent applications include internal combustion engines, spray drying, ink jet printing, film coating, nuclear safety curtain formation, agricultural sprays, fiber and sheet drawing, powered milk processing, powder metallurgy, toxic material removal, and encapsulation of biomedical material. The stability and the disintegration of liquid jets have been investigated by linear and nonlinear instability theories and computational methods.

These analyses are mostly concerned with the axisymmetric jets, but asymmetric analyses are scant, moreover the nonlinear work is rare.

This assumption of axisymmetric initial disturbance and subsequent growth has the serious weaknesses. This assumption is true only for low-speed jets known as capillary jets when the breakup mechanism is due to the growth of axisymmetric oscillation induced by the competition between cohesive and disruptive forces on the surface of the jet.

Although the axisymmetric mode appears in the low speed jets, under high speed conditions substantially different jet instability is noticed in the experiments. When the difference between the velocity of the jet and that of the ambient gas is large, asymmetric instability is observed. With increase in the relative velocity between the liquid and the gas, the mechanism of the jet instability is characterized by the first transverse mode(m=1) in the experiments reported in[1].

It was Lord Rayleigh [2] who first made a detailed analysis concerning the capillary waves on a liquid column of an ideal fluid in the absence of the surrounding gas. He showed by means of a first order perturbation calculation that the only unstable disturbances must be axisymmetric, and that their wave lengths must be longer than the circumference of the jet. By extending Rayleigh's theory, Weber [3] included the effects of both the liquid viscosity and the pressure of the surrounding gas on the stability of a columnar jet. He showed that the viscous effect does not alter the value of the cut-off wave number predicted by the inviscid theory and that the influence of the ambient air is not very significant so far as the speed of the columnar jet is not too large.

Liquid breakup and formation of droplets are nonlinear phenomena. The inadequate linear theory has led the development of the nonlinear analysis to investigate the shape of the waves on the jet surface, as well as to estimate the volume of main and satellite drops formed during the breakup.

Yuen [4] developed a third-order nonlinear theory for the capillary instability of a inviscid liquid jet, neglecting the effect of surrounding air using the method of straining coordinates. He found that the cut-off wave number and the fundamental frequency of the wave for a given k are different from linearized theory. Wang [5], Nayfeh [6] , and Lafrance [7] have also carried out nonlinear perturbation analyses of a capillary of an inviscid liquid jet of circular cross section in the absence of the surrounding gas.

Only a limited number of works is found that is concerned with the asymmetric analysis of a liquid jet. Yang [1] worked on the linear nonaxisymmetric instability of an invscid liquid jet in the presence of an injected coaxial gas. He derived a dispersion equation that accounts for the growth of asymmetric waves. He showed that there exists a critical Weber number which is defined as the ratio of surface tension force to the inertial force, below which the nonaxisymmetric disturbance becomes unstable, while Ibrahim and Jog [8] investigated a nonlinear asymmetric breakup of a liquid jet exposed to a swirling gas stream by a perturbation expansion technique with the initial amplitude of the disturbance as the perturbation parameter. The have neglected liquid and gas viscosities. They obtained solutions upto the second order, and the breakup time of the jet is obtained numerically until the deepest trough of the wave profile coincides with the centerline of the jet, whereas we carried out the solutions upto the third order to obtain a nonlinear Schrödinger equation to determine the cut-off wave number k, and the region of stability. The problem of nonlinear breakup of an asymmetric electrohydrodynamic jet was investigated by the present author[9,10] using the method of straining of coordinates and multiple scales.

In this presentation, a nonlinear problem is considered, in which a jet of invscid fluid having a circular cylindrical geometry with or without the surrounding gas is given a nonaxisymmetric initial disturbance at the surface. In this paper, by the method of multiple scales, we have developed a third order asymmetric nonlinear theory on the propagation of waves over the surface of circular jet. The basic equations with the accompanying boundary conditions are given in Sec.2. The first order theory and the linear dispersion relation are obtained in Sec.3. In Sec.4 we have derived second order solutions. In Sec.5 third order problem is considered. Sec.6 discusses the case when there is no surrounding gas. Finally Sec.7 is devoted to some numerical examples and discussions.

2. Formulation

We consider an incompressible, inviscid fluid jet whose density is ρ_1 and whose radius is a is injected with a uniform velocity U_1 along with a coaxial gas of density ρ_2 at a uniform velocity U_2.

We use the cylindrical polar coordinates (r, θ, z) with $z-$ axis taken along the axis of the jet.

The interface is defined by a function of θ, z and time. Let $\eta(\theta, z, t)$ denote the elevation of the free surface measured from the unperturbed level. Now, a periodic initial disturbance is given at the surface of the jet.

If $\mathbf{u}_1$ and $\mathbf{u}_2$ denote velocity fields , at any time t, then equations are as follows:

$$\nabla \cdot \boldsymbol{u}_1 = 0, \tag{2.1}$$

$$\nabla \cdot \boldsymbol{u}_2 = 0, \tag{2.2}$$

If $\phi^{()}$ and $\phi^{(2)}$ denote velocity potentials of the liquid and the gas, respectively, so that $\mathbf{u}_j = \nabla\phi^{(j)}, (j = 1, 2)$, then the equations for $\phi^{(j)}$ are given by

$$\nabla^2\phi^{(j)} = 0, \qquad (j = 1, 2) \tag{2.3}$$

for $r \le a + \eta(\theta, z, t)$,

The unit normal $\mathbf{n}$ to the surface is given by

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \left(-\mathbf{e}_r + \frac{1}{r}\frac{\partial \eta}{\partial \theta}\mathbf{e}_\theta + \frac{\partial \eta}{\partial z}\mathbf{e}_z\right)\left\{1 + \left(\frac{\partial \eta}{r\partial \theta}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2\right\}^{-\frac{1}{2}}, \tag{2.4}$$

where $F = 0$ is the equation of the surface of jet.

The condition that the interface is moving with the fluid leads to

$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi^{(j)}}{\partial r} + \frac{1}{r^2}\frac{\partial \phi^{(j)}}{\partial \theta}\frac{\partial \eta}{\partial \theta} + \frac{\partial \phi^{(j)}}{\partial z}\frac{\partial \eta}{\partial z} = 0 \quad \text{at} \quad r = a + \eta, \quad (j = 1, 2) \tag{2.5}$$

Now the boundary condition at the free surface is,

$$\left[\!\left[\rho\left(-\frac{\partial \phi}{\partial t} - \frac{1}{2}\left\{\left(\frac{\partial \phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right\}\right)\right]\!\right] - \frac{T}{r|\nabla F|}\left\{1 + \left(\frac{1}{r}\frac{\partial \eta}{\partial \theta}\right)^2 \frac{2}{|\nabla F|^2}\right\}$$

$$+ \frac{T}{|\nabla F|^3}\left[\frac{\partial^2 \eta}{\partial z^2}\left\{1 + \left(\frac{1}{r}\frac{\partial \eta}{\partial \theta}\right)^2\right\} - \frac{2}{r^2}\frac{\partial \eta}{\partial \theta}\frac{\partial^2 \eta}{\partial \theta \partial z}\frac{\partial \eta}{\partial z} + \frac{1}{r^2}\frac{\partial^2 \eta}{\partial \theta^2}\left\{1 + \left(\frac{\partial \eta}{\partial z}\right)^2\right\}\right] = C, \tag{2.6}$$

where C is a constant and $[\![h]\!]$ represents the difference in a quantity as we cross the interface,*i.e.*, $[\![h]\!] = h^{(1)} - h^{(2)}$ and T is the surface tension and F is given by

$$F = r - \eta(\theta, z, t) - a.$$

To investigate the nonlinear effect on the stability of the jet, we employ the method of multiple scales. Introducing ϵ as a small parameter, we assume the following expansion of the variables:

$$\phi^{(1)}(r, \theta, z, t) = \sum_{n=0}^{3} \epsilon^n \phi_n^{(1)}(r, \theta_0, \theta_1, \theta_2, z_0, z_1, z_2; t_0, t_1, t_2) + O(\epsilon^4), \tag{2.8}$$

$$\phi^{(2)}(r, \theta, z, t) = \sum_{n=0}^{3} \epsilon^n \phi_n^{(2)}(r, \theta_0, \theta_1, \theta_2, z_0, z_1, z_2; t_0, t_1, t_2) + O(\epsilon^4), \tag{2.9}$$

and

$$\eta(\theta, z, t) = \sum_{n=1}^{3} \epsilon^n \eta_n(\theta_0, \theta_1, \theta_2, z_0, z_1, z_2; t_0, t_1, t_2) + O(\epsilon^4). \tag{2.10}$$

The multiple scales $z_n(\equiv \epsilon^n z)$,$\theta_n(\equiv \epsilon^n \theta$ and $t_n(\equiv \epsilon^n t)$ are assumed to satisfy the following derivative expansions:

$$\frac{\partial}{\partial z} = \sum_{n=0}^{2} \epsilon^n \frac{\partial}{\partial z_n} + O(\epsilon^3), \tag{2.11}$$

$$\frac{\partial}{\partial \theta} = \sum_{n=0}^{2} \epsilon^n \frac{\partial}{\partial \theta_n} + O(\epsilon^3), \tag{2.11}$$

$$\frac{\partial}{\partial t} = \sum_{n=0}^{2} \epsilon^n \frac{\partial}{\partial t_n} + O(\epsilon^3), \tag{2.12}$$

If we substitute (2.8)-(2.10) into (2.5)-(2.6), boundary conditions for various orders are obtained. A Maclaurin series expansion of the boundary conditions at $r = a$ provides successive orders of approximation to these conditions which are then used to specify the problem in those orders.

3. Linear theory

We substitute the expressions (2.8), (2.9) and (2.10) for $\phi^{(1)}, \phi^{(2)}$ and η, respectively into the field equations (2.3) and the boundary conditions (2.5)-(2.6). Equating the coefficient of first power of ϵ leads to

$$\nabla_0^2 \phi_1^{(1)} = 0, \tag{3.1}$$

$$\nabla_0^2 \phi_1^{(2)} = 0, \tag{3.2}$$

where

$$\nabla_0^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta_0^2} + \frac{\partial^2}{\partial z_0^2}.$$

We take $\phi_0^{(j)} = U_j z_0, \quad (j = 1, 2)$. The various boundary conditions at the interface are, (at $r = a$)

$$\frac{\partial \eta_1}{\partial t_0} + U_j \frac{\partial \eta_1}{\partial z_0} - \frac{\partial \phi_1^{(j)}}{\partial r} = 0, \qquad (j = 1, 2) \tag{3.3}$$

$$-\left[\!\left[\rho\left(\frac{\partial \phi_1}{\partial t_0} + U\frac{\partial \phi_1}{\partial z_0}\right)\right]\!\right] + T\left\{\frac{\eta_1}{a^2} + \frac{\partial^2 \eta_1}{a^2 \partial \theta_0^2} + \frac{\partial^2 \eta_1}{\partial z_0^2}\right\} = 0, \tag{3.4}$$

The solutions to (3.1)-(3.3) are given by

$$\eta_1 = A(z_1, z_2, \theta_1, \theta_2; t_1, t_2)e^{i\vartheta} + \bar{A}(z_1, z_2, \theta_1, \theta_2; t_1, t_2)e^{-i\vartheta}, \tag{3.5}$$

$$\phi_1^{(1)} = i(U_1 k - \omega)A\frac{I_m(kr)}{I'_m(ka)}e^{i\vartheta} + c.c. + B_1(z_1, z_2, \theta_1, \theta_2; t_1, t_2), \tag{3.6}$$

$$\phi_1^{(2)} = i(U_2 k - \omega)A\frac{K_m(kr)}{K'_m(ka)}e^{i\vartheta} + c.c. + B_2(z_1, z_2, \theta_1, \theta_2; t_1, t_2), \tag{3.7}$$

where $\bar{A}$ is the complex conjugate of A and $\vartheta = kz_0 + m\theta_0 - \omega t_0$, $I_m(kr)$ and $K_m(kr)$ are the modified Bessel functions of the first and second kind, respectively and $I'_m(ka) = \frac{dI_m(kr)}{dr}|_{r=a}$, $K'_m(ka) = \frac{dK_m(kr)}{dr}|_{r=a}$. Substituting (3.5)-(3.7) into (3.4), we obtain following dispersion relation

$$D(\omega, k, m) = \rho_1(\omega - U_1 k)^2\gamma_m^{(1)} - \rho_2(\omega - U_2 k)^2\gamma_m^{(2)} + \frac{T}{a^2}\{1 - m^2 - (ka)^2\} = 0, \tag{3.8}$$

where

$$\gamma_m^{(1)} = \frac{I_m(ka)}{I'_m(ka)}, \quad \gamma_m^{(2)} = \frac{K_m(ka)}{K'_m(ka)} \tag{3.9}$$

Equation(3.8) is in agreement with the result obtained by Yang [1].

4.Second order solutions

The second order problem is governed by

$$\nabla_0^2\phi_2^{(j)} = -2\frac{\partial^2\phi_1^{(j)}}{\partial z_0\partial z_1} - \frac{2}{r^2}\frac{\partial^2\phi_1^{(j)}}{\partial\theta_0\partial\theta_1}, \quad (j=1,2) \tag{4.1}$$

The boundary conditions at $r=a$ are:

$$L(\eta_2,\phi_2^{(j)}) = -\frac{\partial\eta_1}{\partial t_1} + \frac{\partial^2\phi_1^{(j)}}{\partial r^2}\eta_1 - \frac{\partial\eta_1}{\partial z_0}\frac{\partial\phi_1^{(j)}}{\partial z_0} - \frac{\partial\eta_1}{\partial z_1}\frac{\partial\phi_0^{(j)}}{\partial z_0} - \frac{1}{a^2}\frac{\partial\eta_1}{\partial\theta_0}\frac{\partial\phi_1^{(j)}}{\partial\theta_0}, \quad (j=1,2), \tag{4.2}$$

$$N(\eta_2,\phi_2^{(1)},\phi_2^{(2)}) = \left[\!\!\left[\rho\left(\frac{\partial\phi_1}{\partial t_1}+\frac{\partial^2\phi_1}{\partial t_0\partial r}\eta_1+\frac{1}{2}\left\{\left(\frac{\partial\phi_1}{\partial r}\right)^2+\left(\frac{1}{a}\frac{\partial\phi_1}{\partial\theta_0}\right)^2+\left(\frac{\partial\phi_1}{\partial z_0}\right)^2\right\}+U\left\{\frac{\partial\phi_1}{\partial z_1}+\frac{\partial^2\phi_1}{\partial z_0\partial r}\eta_1\right\}\right)\right]\!\!\right]$$
$$+\frac{T}{a}\left\{\frac{\eta_1^2}{a^2}-\frac{1}{2}\left(\frac{\partial\eta_1}{\partial z_0}\right)^2+\frac{3}{2}\left(\frac{1}{a}\frac{\partial\eta_1}{\partial\theta_0}\right)^2\right\}+T\left\{\frac{2}{a^2}\eta_1\frac{\partial^2\eta_1}{\partial\theta_0^2}-2\frac{\partial^2\eta_1}{\partial z_0\partial z_1}-\frac{2}{a^2}\frac{\partial^2\eta_1}{\partial\theta_0\partial\theta_1}\right\}, \tag{4.3}$$

where $L(\eta_2,\phi_2^{(j)})$, and $N(\eta_2,\phi_2^{(1)},\phi_2^{(2)})$ denote the left-hand sides of (3.3), and (3.4) with $\eta_1,\phi_1^{(j)}$ being replaced by η_2, and $\phi_2^{(j)},(j=1,2)$, respectively.

Substituting the first order solutions in (4.2)-(4.3), we obtain following equations:

$$L(\eta_2,\phi_2^{(j)}) = -\left(\frac{\partial A}{\partial t_1}+U_j\frac{\partial A}{\partial z_1}\right)e^{i\vartheta}+i(U_jk-\omega)\left\{2\left(k^2+\frac{m^2}{a^2}\right)\gamma_m^{(j)}-\frac{1}{a}\right\}A^2e^{i2\vartheta}+c.c., \quad (j=1,2) \tag{4.4}$$

$$N(\eta_2,\phi_2^{(1)},\phi_2^{(2)}) = \left[\!\!\left[\rho i(Uk-\omega)\gamma_m\left[\frac{\partial A}{\partial t_1}+U\frac{\partial A}{\partial z_1}\right]e^{i\vartheta}\right]\!\!\right]-2Ti\left(k\frac{\partial A}{\partial z_1}+\frac{m}{a}\frac{\partial A}{\partial\theta_1}\right)e^{i\vartheta}$$
$$+\left[\left[\!\!\left[\rho\frac{(Uk-\omega)^2}{2}\left\{-3+\left(\frac{m^2}{a^2}+k^2\right)\gamma_m^2\right\}\right]\!\!\right]+\frac{T}{a^3}\{\frac{1}{2}(m^2+k^2a^2)+1-4m^2\}\right]A^2e^{2i\vartheta}$$
$$+c.c.+\left[\left[\!\!\left[\rho(Uk-\omega)^2\left\{-1+\left(\frac{m^2}{a^2}+k^2\right)\gamma_m^2\right\}\right]\!\!\right]+\frac{T}{a^3}\{2-m^2-k^2a^2\}\right]A\bar{A}+\frac{\partial B_1}{\partial t_1}, \tag{4.5}$$

Equations(4.4)-(4.5) furnish following second order solutions:

$$\eta_2 = N_2e^{i\vartheta}+q_1A^2e^{i2\vartheta}+c.c.+\left(|A|^2q_2+\frac{\partial B_1}{\partial t_1}\rho_1-\frac{\partial B_2}{\partial t_1}\rho_2\right)\frac{a^2}{T}, \tag{4.6}$$

where

$$N_2 = \frac{1}{i}\left\{\frac{\partial A}{\partial z_1}\left(-\frac{m}{k}+\left\{\frac{m^2}{ka}+ka\right\}\gamma_m^{(1)}-\frac{U_1}{U_1k-\omega}\right)-\frac{\partial A}{\partial t_1}\frac{1}{U_1k-\omega}\right\}$$

$$\phi_2^{(1)} = (U_1k-\omega)\left\{\frac{rI_{m+1}(kr)}{I'_m(ka)}\frac{\partial A}{\partial z_1}+\frac{\partial}{\partial m}\left(\frac{I_m(kr)}{I'_m(ka)}\right)\frac{\partial A}{\partial\theta_1}\right\}e^{i\vartheta}$$
$$-i(U_1k-\omega)\left\{2\left(\frac{m^2}{a^2}+k^2\right)\gamma_m^{(1)}-\frac{1}{a}-2q_1\right\}\frac{I_{2m}(2kr)}{I'_{2m}(2ka)}A^2e^{i2\vartheta}+c.c.$$
$$+B_3(z_1,z_2;t_1,t_2), \tag{4.7}$$

$$\phi_2^{(2)} = (U_2k-\omega)\Big\{-\frac{rK_{m+1}(kr)}{K_m'(ka)}\frac{\partial A}{\partial z_1} + \frac{\partial}{\partial m}\Big(\frac{K_m(kr)}{K_m'(ka)}\Big)\frac{\partial A}{\partial \theta_1}\Big\}e^{i\vartheta}$$
$$+(U_2k-\omega)\frac{K_m(kr)}{K_m'(ka)}\Big\{\frac{\partial A}{\partial z_1}\Big[\Big(\frac{m^2}{ka}+ka\Big)(\gamma_m^{(1)}-\gamma_m^{(2)}) + \frac{U_2}{U_2k-\omega} - \frac{U_1}{U_1k-\omega}\Big]$$
$$+\frac{\partial A}{\partial t_1}\Big[\frac{1}{U_2k-\omega}-\frac{1}{U_1k-\omega}\Big]\Big\}e^{i\vartheta}$$
$$-i(U_2k-\omega)\Big\{2\Big(\frac{m^2}{a^2}+k^2\Big)\gamma_m^{(2)} - \frac{1}{a} - 2q_1\Big\}\frac{K_{2m}(2kr)}{K_{2m}'(2ka)}A^2e^{i2\vartheta} + c.c. + B_4(z_1,z_2;t_1,t_2), \tag{4.8}$$

where

$$q_1 = \Big[\Big[\!\Big[\rho\frac{(Uk-\omega)^2}{2}\Big\{-3+\Big(\frac{m^2}{a^2}+k^2\Big)\gamma_m^2\Big\} + 2\rho(Uk-\omega)^2\Big\{2\gamma_m\Big(\frac{m^2}{a^2}+k^2\Big)-\frac{1}{a}\Big\}\gamma_{2m}\Big]\!\Big]$$
$$+\frac{T}{a^3}\{\frac{1}{2}(m^2+k^2a^2)+1-4m^2\}\Big]\frac{1}{D(2\omega,2k,2n)}, \tag{4.9}$$

$$q_2 = \Big[\!\Big[\rho(Uk-\omega)^2\Big\{-1+\Big(\frac{m^2}{a^2}+k^2\Big)\gamma_m^2\Big\}\Big]\!\Big] + \frac{T}{a^3}\{2-m^2-k^2a^2\} \tag{4.10}$$

with

$$\gamma_{2m}^{(1)} = \frac{I_{2m}(2ka)}{I_{2m}'(2ka)}, \qquad \gamma_{2m}^{(2)} = \frac{K_{2m}(2ka)}{K_{2m}'(2ka)}.$$
$$I_{2m}'(2ka) = \frac{dI_{2m}(2kr)}{dr}\Big|_{r=a}, \quad K_{2m}'(2ka) = \frac{dK_{2m}(2kr)}{dr}\Big|_{r=a}$$
$$\frac{\partial I_m(kr)}{\partial m} = \lim_{\nu\to m}\frac{\partial I_\nu(kr)}{\partial \nu} = I_m(kr)\ln(\frac{1}{2}kr) - \sum_{n=0}^{\infty}\frac{(\frac{1}{2}kr)^{m+2n}}{n!(n+m)!}\psi(n+m+1)$$
$$\psi(n+1) = \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}-\gamma$$

where γ denotes Euler's constant, 0.57751....[11, p.377]. $\partial K_m(kr)/\partial m$ is similarly defined.

Here $D(2\omega, 2k, 2m)$ is obtained from (3.8) by replacing $\gamma_m^{(1)}, \gamma_m^{(2)}$ by $\gamma_{2m}^{(1)}, \gamma_{2m}^{(2)}$ and ω, k and m by $2\omega, 2k$ and $2m$ respectively, and B_3 and B_4 are arbitrary constants. Furthermore, we assume that $D(2\omega, 2k, 2m) \neq 0$. The case $D(2\omega, 2k, 2m) = 0$ corresponds to the second harmonic resonance.

The vanishing of coefficient of $e^{i\vartheta}$ in (4.5) gives the secular equation

$$\frac{\partial A}{\partial t_1} + V_k\frac{\partial A}{\partial z_1} + V_m\frac{\partial A}{\partial \theta_1} = 0, \tag{4.11}$$

where V_k and V_n are the group velocities expressed as

$$V_k = \frac{\partial \omega}{\partial k}, \tag{4.12}$$

$$V_m = \frac{\partial \omega}{\partial m}, \tag{4.13}$$

5.Third order problem

We proceed now to the third order problem.

$$\nabla_0^2\phi_3^{(j)} = -2\frac{\partial^2\phi_2^{(j)}}{\partial z_0\partial z_1} - \frac{\partial^2\phi_1^{(j)}}{\partial z_1^2} - 2\frac{\partial^2\phi_1^{(j)}}{\partial z_0\partial z_2} - \frac{2}{r^2}\frac{\partial^2\phi_2^{(j)}}{\partial\theta_0\partial\theta_1} - \frac{1}{r^2}\frac{\partial^2\phi_1^{(j)}}{\partial\theta_1^2} - \frac{2}{r^2}\frac{\partial^2\phi_1^{(j)}}{\partial\theta_0\partial\theta_2}, (j=1,2) \tag{5.1}$$

The solution to (5.1), $\phi_3^{(1)}$ takes the following form

$$\begin{aligned}\phi_3^{(1)} = &-(U_1k-\omega)\left[\frac{i}{2}\left\{r^2 I_m(kr) - \frac{r}{k}I_{m+1}(kr)(1+2m)\right\}\frac{\partial^2 A}{\partial z_1^2} - rI_{m+1}(kr)\frac{\partial A}{\partial z_2}\right]\frac{e^{i\vartheta}}{I'_m(ka)}\\ &-(U_1k-\omega)i\left[\frac{\partial^2\alpha(r)}{\partial k\partial m} - \frac{\partial\alpha(r)}{\partial m}\left\{\frac{m}{k} - \gamma_m^{(1)}\left(ka+\frac{m^2}{ka}\right)\right\}\right]\frac{\partial^2 A}{\partial z_1\partial\theta_1}e^{i\vartheta}\\ &+(U_1k-\omega)\left[\frac{\partial\alpha(r)}{\partial m}\frac{\partial A}{\partial\theta_2} - \frac{i}{2}\frac{\partial^2\alpha(r)}{\partial m^2}\frac{\partial^2 A}{\partial\theta_1^2}\right]e^{i\vartheta} + c.c.\\ &-\frac{1}{4}\frac{\partial^2 B_1}{\partial z_1^2}r^2 + \frac{1}{2}\frac{\partial^2 B_1}{\partial\theta_1^2}\theta_0(2\pi-\theta_0)+\cdots,\end{aligned} \tag{5.2}$$

where

$$\alpha(r) = \frac{I_m(kr)}{I'_m(ka)}$$

$$\eta_3 = \frac{1}{i}N_3 e^{i\vartheta} + c.c + \cdots, \tag{5.3}$$

where

$$\begin{aligned}N_3 = &-\frac{i}{\omega-U_1k}\frac{\partial^2 A}{\partial t_1\partial z_1}\left(-\frac{m}{k}+\left\{\frac{m^2}{ka}+ka\right\}\gamma_m^{(1)} - \frac{2U_1}{U_1k-\omega}\right) - \frac{i}{(U_1k-\omega)^2}\frac{\partial^2 A}{\partial t_1^2} + \frac{1}{(\omega-U_1k)}\frac{\partial A}{\partial t_2}\\ &-\frac{\partial A}{\partial z_2}\left(\frac{m}{k}\left\{1-\frac{m}{a}\gamma_m^{(1)}\right\} - ak\gamma_m^{(1)} + \frac{U_1}{U_1k-\omega}\right) - i\frac{\partial^2 A}{\partial z_1^2}\left[-(2m+1)\left(-\frac{m}{k}\left\{1-\frac{m}{a}\gamma_m^{(1)}\right\}+ak\gamma_m^{(1)}\right)\frac{1}{2k}\right.\\ &\left.+a\gamma_m^{(1)} + \frac{a^2}{2} - \left(-\frac{m}{k}+\left(\frac{m^2}{ka}+ka\right)\gamma_m^{(1)} - \frac{U_1}{U_1k-\omega}\right)\frac{U_1}{U_1k-\omega}\right]\end{aligned}$$

where $\cdots$ denotes nonsecular terms.

$$\phi_3^{(2)} = (U_2k-\omega)p_3e^{i\vartheta} + c.c. - \frac{1}{4}\frac{\partial^2 B_2}{\partial z_1^2}r^2 + \frac{1}{2}\frac{\partial^2 B_2}{\partial\theta_1^2}\theta_0(2\pi-\theta_0)+\cdots, \tag{5.4}$$

where p_3 can be found in the Appendix.

Using (5.2) and (5.4) and eliminating η_3 in (2.5) and (2.6), we obtain following nonsecularity condition:

$$i\left(\frac{\partial A}{\partial t_2} + V_k\frac{\partial A}{\partial z_2} + V_m\frac{\partial A}{\partial\theta_2}\right) + R_1\frac{\partial^2 A}{\partial z_1^2} + 2R_2\frac{\partial^2 A}{\partial z_1\partial\theta_1} + R_3\frac{\partial^2 A}{\partial\theta_1^2} = QA^2\bar{A} + RA, \tag{5.5}$$

where

$$R_1 = \frac{1}{2}\frac{\partial^2 \omega}{\partial k^2}, \quad R_2 = \frac{1}{2}\frac{\partial^2 \omega}{\partial k \partial m}, \qquad R_3 = \frac{1}{2}\frac{\partial^2 \omega}{\partial m^2}$$

$$Q = \frac{S_1}{2[\![\rho(\omega - Uk)\gamma_m]\!]}$$

$$R = \frac{S_2}{2[\![\rho(\omega - Uk)\gamma_m]\!]}$$

S_1 and S_2 are found in the Appendix.

Introducing the transformation

$$\tau = t_2 = \epsilon^2 t = \epsilon t_1, \quad \xi = (z_2 - V_k t_2)\frac{1}{\epsilon} = (z_1 - V_k t_1), \tag{5.6}$$

$$\zeta = \left\{\theta_2 - \frac{R_2}{R_1} z_2 - \left(V_n - \frac{R_2}{R_1} V_k\right) t_2\right\}\frac{1}{\epsilon} = \left\{\theta_1 - \frac{R_2}{R_1} z_1 - \left(V_n - \frac{R_2}{R_1} V_k\right) t_1\right\}. \tag{5.7}$$

Equation (5.5) governing the evolution of the amplitude of the wave packet reduces to

$$i\frac{\partial A}{\partial \tau} + P_1 \frac{\partial^2 A}{\partial \xi^2} + P_2 \frac{\partial^2 A}{\partial \zeta^2} = QA^2 \bar{A} + RA, \tag{5.8}$$

where

$$P_1 = R_1, \qquad P_2 = R_3 - \frac{R_2^2}{R_1}$$

Now that ϕ_3 is expressed as (5.2), the boundary condition (2.5) in $O(\epsilon^3)$ becomes

$$\frac{\partial \eta_3}{\partial t_0} = \left(-q_2 \frac{\partial \bar{A}A}{\partial t_1} - \frac{\partial^2 B_1}{\partial t_1^2}\rho_1 + \frac{\partial^2 B_2}{\partial t_1^2}\rho_2\right)\frac{a^2}{T} - \frac{1}{a}\frac{\partial \bar{A}A}{\partial t_1} - \frac{1}{2}\frac{\partial^2 B_j}{\partial z_1^2} a$$
$$+(U_j k - \omega)\left\{\left(2k\gamma_m^{(j)} - \frac{U_j}{a(U_j k - \omega)}\right)\frac{\partial \bar{A}A}{\partial z_1} + \gamma_m^{(j)} \frac{2m}{a^2}\frac{\partial \bar{A}A}{\partial \theta_1}\right\} + \cdots, (j = 1, 2)$$

where $\cdots$ again indicates the non-secular terms. The non-secular condition for η_3 with respect to t_0 gives

$$\left(-q_2 \frac{\partial \bar{A}A}{\partial t_1} - \frac{\partial^2 B_1}{\partial t_1^2}\rho_1 + \frac{\partial^2 B_2}{\partial t_1^2}\rho_2\right)\frac{a^2}{T} - \frac{1}{a}\frac{\partial \bar{A}A}{\partial t_1} - \frac{1}{2}\frac{\partial^2 B_j}{\partial z_1^2} a$$
$$+(U_j k - \omega)\left\{\left(2k\gamma_m^{(j)} - \frac{U_j}{a(U_j k - \omega)}\right)\frac{\partial \bar{A}A}{\partial z_1} + \gamma_m^{(j)} \frac{2m}{a^2}\frac{\partial \bar{A}A}{\partial \theta_1}\right\} = 0, (j = 1, 2) \tag{5.9}$$

If we assume that B_1 depends on the slower scales only through the amplitude A, (5.9) reduces to

$$\frac{a^2}{T}\left[\!\!\left[\rho\left(\frac{\partial^2 B}{\partial z_1^2} V_k^2 + 2\frac{\partial^2 B}{\partial z_1 \partial \theta_1} V_k V_m + \frac{\partial^2 B}{\partial \theta_1^2} V_m^2\right)\right]\!\!\right] + \frac{1}{2}\frac{\partial^2 B_j}{\partial z_1^2} a = T_1^{(j)} \frac{\partial \bar{A}A}{\partial z_1} + T_2^{(j)} \frac{\partial \bar{A}A}{\partial \theta_1}, \tag{5.10}$$

where

$$T_1^{(j)} = \left\{\left(q_2 \frac{a^2}{T} + \frac{1}{a}\right) V_k + (U_j k - \omega)\left(2k\gamma_m^{(j)} - \frac{U_j}{a(U_j k - \omega)}\right)\right\}\frac{\partial \bar{A}A}{\partial z_1},$$

$$T_2^{(j)} = \left\{ \left(q_2 \frac{a^2}{T} + \frac{1}{a} \right) V_m + (U_j k - \omega) \gamma_m^{(j)} \frac{2m}{a^2} \right\} \frac{\partial \bar{A} A}{\partial \theta_1}, \qquad (j = 1, 2)$$

Using the transformation (5.6)-(5.7), the above equation can be put into the form

$$\frac{a^2}{T} \left[\!\left[\rho \left(M_1 \frac{\partial^2 B}{\partial \xi^2} + M_2 \frac{\partial^2 B}{\partial \xi \partial \zeta} + M_3 \frac{\partial^2 B}{\partial \zeta^2} \right) \right]\!\right] + \frac{a}{2} \left(\frac{\partial}{\partial \xi} - \frac{R_2}{R_1} \frac{\partial}{\partial \zeta} \right)^2 B_j$$
$$= T_1^{(j)} \frac{\partial |A|^2}{\partial \xi} + \left(T_2^{(j)} - T_1^{(j)} \frac{R_2}{R_1} \right) \frac{\partial |A|^2}{\partial \zeta}, \quad (j = 1, 2)$$

Assume A, B_1 and B_2 are functions of $\chi = \ell \xi + n \zeta$ and τ only [12], then the above equation yields

$$\frac{a^2}{T} (M_1 \ell^2 + M_2 \ell n + M_3 n^2) \left[\!\left[\rho \frac{\partial^2 B}{\partial \chi^2} \right]\!\right] + \frac{a}{2} \left(\ell - \frac{R_2}{R_1} n \right)^2 \frac{\partial^2 B_j}{\partial \chi^2}$$
$$= \left\{ T_1^{(j)} \ell + \left(T_2^{(j)} - T_1^{(j)} \frac{R_2}{R_1} \right) n \right\} \frac{\partial |A|^2}{\partial \chi}, \quad (j = 1, 2) \qquad (5.11)$$

Integrating (5.11) , R can be expressed in the form

$$R = G A \bar{A} + C(\tau)$$

where $C(\tau)$ is a constant of integration.

And (5.8) can be written as

$$i \frac{\partial A}{\partial \tau} + P_3 \frac{\partial^2 A}{\partial \chi^2} = Q_1 A^2 \bar{A} + R^* A, \qquad (5.12)$$

which is a nonlinear Schrödinger equation, and

$$P_3 = P_1 \ell^2 + P_2 n^2, \quad Q_1 = Q + G, \quad R^* = C(\tau)$$

We note that R^* may be eliminated from(5.12) by an appropriate frequency shift in A. The stability of the solution of (5.12) is subject to the same criterion as that found by Hasimoto and Ono [13],

$$P_3 Q_1 > 0, \qquad (5.13)$$

In the case when the propagation is along the z-axis, namely, when A is dependent on the slower scales z_1, z_2, (5.8) reduces to

$$i \frac{\partial A}{\partial \tau} + P_1 \frac{\partial^2 A}{\partial \xi^2} = Q A^2 \bar{A} + \frac{S_2}{2 [\![\rho (\omega - U k) \gamma_m]\!]} A, \qquad (5.14)$$

$$S_2 = - \left[\!\left[\rho (U k - \omega) \gamma_m \left(2k \frac{\partial B}{\partial z_1} + V_k (U k - \omega) \left\{ \gamma_m \left(\frac{m^2}{a^2} + k^2 \right) - \frac{1}{a} \right\} \left[\!\left[\rho \frac{\partial B}{\partial z_1} \right]\!\right] \frac{a^2}{T} \right) \right]\!\right]$$
$$- V_k \left[\!\left[\rho \frac{\partial B}{\partial z_1} \right]\!\right] \frac{a^2}{T} \left(\frac{2T}{a^2} (1 - m^2) - [\![(U k - \omega)^2]\!] \right), \qquad (5.15)$$

$$\frac{\partial B_j}{\partial z_1} = \left(\frac{\rho_1\rho_2}{\rho_j}\frac{a^2}{T}V_k^2\left[\!\left[2k(Uk-\omega)\gamma_m - \frac{U}{a}\right]\!\right]\right.$$
$$\left. + \frac{a}{2}\left\{\left(q_2\frac{a^2}{T}+\frac{1}{a}\right)V_k + (U_jk-\omega)(2k\gamma_m^{(j)} - U_j)\right\}\right)\frac{A\bar{A}}{\Delta} + C_j(t_2, z_2), \quad (j=1,2) \tag{5.16}$$

where $\Delta = \frac{a^3}{2T}(\rho_1-\rho_2)V_k^2 + \frac{a^2}{4}$.

So (5.14) is of the form of (5.12) and the stability condition is

$$P_1Q_1 > 0$$

.

6. Liquid jet without the surrounding gas.

In this case, it is convenient to normalize the physical variables by using the radius of the unperturbed jet a for the characteristic length, and $\sqrt{T/\rho_1 a}$ for the characteristic speed and use the moving frame of reference with a uniform speed U_1. The resulting equations can be obtained from the previous equations by setting $T = a = 1$ and $U_1 = 0$, and omitting all terms associated with the surrounding gas. We also use the notation

$$\gamma_m^{(1)} = \frac{I_m(k)}{I'_m(k)k} \triangleq \frac{I_a}{k}, \qquad \gamma_{2m}^{(1)} = \frac{I_{2m}(2k)}{I'_{2m}(2k)2k} \triangleq \frac{I_b}{2k}$$

where prime denotes the differentiation with respect to the argument. Thus S_2 in (5.15) reduces to

$$S_2 = \left[\frac{\omega I_a}{k}\left\{2k - \omega V_k\left(\frac{I_a}{k}(m^2+k^2)-1\right) - V_k\{2(1-m^2)-\omega^2\}\right]\frac{\partial B_1}{\partial z_1}$$
$$= \left[2\omega I_a + V_k\frac{\omega^2 I_a}{k}\left\{kI_a^{-1} - \frac{k^2+m^2}{k}I_a + 1 - \frac{2(1-m^2)k}{\omega^2 I_a}\right\}\right]\frac{\partial B_1}{\partial z_1}$$

And $\frac{\partial B_1}{\partial z_1}$ in (5.16) reduces to

$$\frac{\partial B_1}{\partial z_1} = \left(\frac{1}{2}+V_k^2\right)^{-1}\{V_k(q_2+1) - 2\omega I_a\}A\bar{A} + C_1(z_2, t_2)$$

Thus (5.14) becomes

$$i\frac{\partial A}{\partial \tau} + P_1\frac{\partial^2 A}{\partial \xi^2} = Q_1A^2\bar{A} + R_1^*A$$

where

$$Q_1 = Q + \left(\frac{1}{2}+V_k^2\right)^{-1}\{V_k(q_2+1) - 2\omega I_a\}\left[k + \frac{\omega V_k}{2}\left\{kI_a^{-1} - \frac{k^2+m^2}{k}I_a + 1 - \frac{2(1-m^2)}{k^2+n^2-1}\right\}\right]$$
$$R_1^* = \left[k + \frac{\omega V_k}{2}\left\{kI_a^{-1} - \frac{k^2+m^2}{k}I_a + 1 - \frac{2(1-m^2)}{k^2+m^2-1}\right\}\right]C_1$$

with

$$Q = \left\{ \omega^2 \left[4(k^2+m^2)\frac{I_a}{k} + \frac{1}{2} + (2q_1 - 2(m^2+k^2)k^{-1}I_a + 1)\{2(1+\frac{m^2}{k^2})I_aI_b - 1\} + \frac{3I_a^2}{2k^2}(k^2-3m^2) - (q_1+q_2)\left\{1+\frac{I_a}{k} - I_a^2\left(1+\frac{m^2}{k^2}\right)\right\}\right] + \frac{1}{2}(k^2+9m^2-6) + 2(q_1+q_2)(1-m^2) - \frac{3}{2}(m^2+k^2)^2 - 2(m^2+k^2)q_1 \right\} \frac{k}{2\omega I_a}$$

When $m = 0$, these equations completely agree with the results of Kakutani *et al.* [14].

7. Numerical examples and discussion

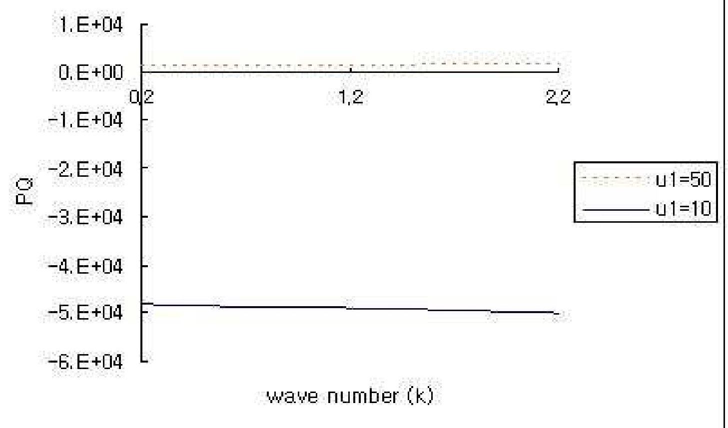

Fig.1. Variation of PQ with resect to k(m=2)

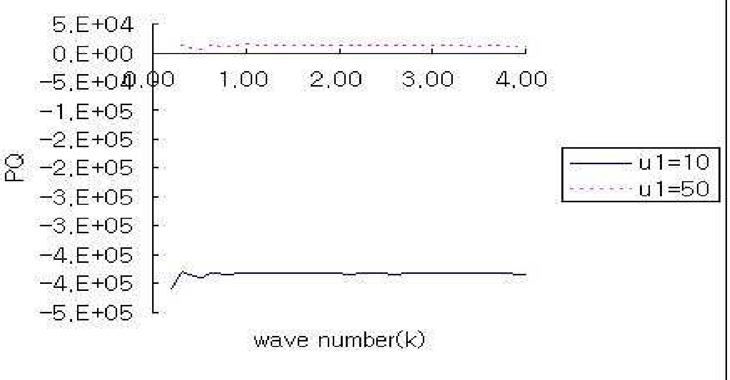

Fig.2. Variation of PQ with respect to k(m=3)

When the gas is present outside the liquid jet, values of PQ versus wave number k have been plotted for different modes in Fig.1-2. When m=2 the figure for PQ is displayed in Fig.1. It is known that by linear theory, the wave is always stable, however by the nonlinear theory, it is unstable .

Here, the inner liquid is water, and the outer gas is air. $U_2 = 48.5m/sec$, $U_1 = 10m/sec$ and $a = 0.5$ cm are chosen.

However when the speed of the liquid U_1 is 50 m/sec, the situation is quite different. It is stable.

The case for $m = 3$ is shown Figure 2. The situation is very similar to the case when $m = 2$. When U_1=10 m/sec, the jet is unstable . When U_1=50 m/sec,it is stable . By linear theory, in both cases, the jet is stable.

In Figures 3-5, we display the values of PQ, when there is no surrounding gas, for various wave number k for different modes. When m=0 the figure for PQ is displayed in Fig.3. It is known that by linear theory the cut-off wave number is k=1, namely the wave is stable when $k > 1$, however by the nonlinear theory, it is stable only for $1 < k < 1.28$ but unstable for all other values.

When $m = 1$ the jet is always unstable.

The cases for $m = 2$ and $m = 3$ are shown Figure 4 and Figure 5, respectively.

Only for the values of k where PQ is positive, the wave is stable. Therefore as we can notice from these figures the values of k where the wave is stable are quite limited. When m=3, second harmonic resonance takes place at k=7.357.. . At that point PQ is infinite, thus the second harmonics blow up.

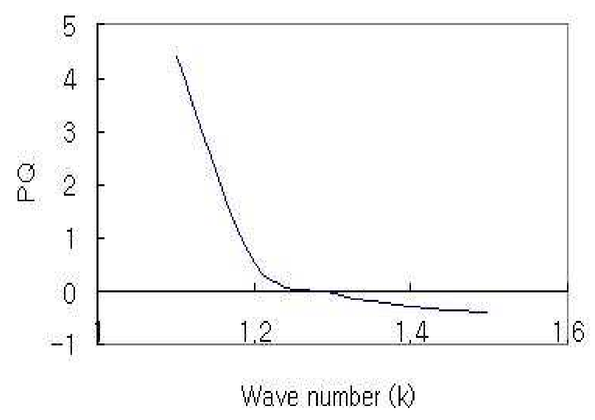

Fig.3. Variation of PQ w.r.t. k(m=0)

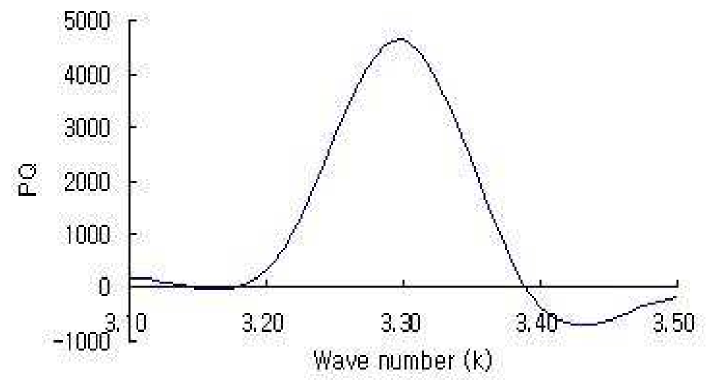

Fig.4. Variation of PQw.r.t. k (m=2)

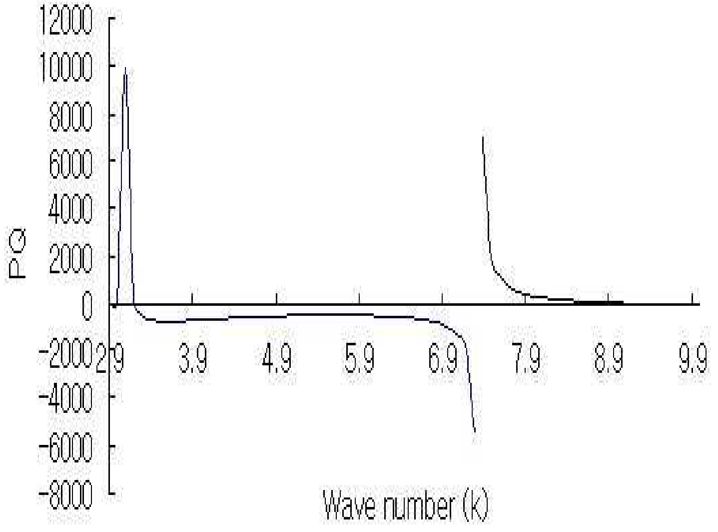

Fig.5. Variation of PQw.r.t. k (m=3)

7. Conclusions

In this paper the problem nonaxisymmetric nonlinear instability of a capillary jet is considered and the results of the investigation may be summarized as follows:

1. By the linear theory, the present nonaxisymmetric circular jet is always stable, however, by a nonlinear theory the values of wave number for which the jet is stable are rather limited .

2. Instability takes place by two cases, when the wave number takes a imaginary number, which will lead to the exponential growth of disturbance or the amplitude A grows large, even though the initial disturbance remains periodic. It is the latter case with nonlinear asymmetric analysis.

When m=0, as the present solution agrees with that of Kakutani *et. al.*[12], it is believed that the present solution is correct.

Appendix.

$$\begin{aligned}
p_3 = &\Bigg[2k\frac{\partial A}{\partial z_2} - i\frac{\partial^2 A}{\partial z_1^2} - 2ik\frac{\partial^2 A}{\partial z_1^2}\Bigg[\Big\{ka + \frac{m^2}{ka}\Big\}(\gamma_m^{(1)} - \gamma_m^{(2)}) + \frac{U_2}{U_2k-\omega} - \frac{U_1}{U_1k-\omega}\Bigg] \\
&-2ik\frac{\partial^2 A}{\partial z_1 \partial t_1}\left(\frac{1}{U_2k-\omega} - \frac{1}{U_1k-\omega}\right)\Bigg]\Bigg[\frac{-rK_{m+1}(kr)}{2kK_m'(ka)} + \frac{\beta(r)}{2k}\Big\{\frac{m}{k} - \Big(ka + \frac{m^2}{ka}\Big)\gamma_m^{(2)}\Big\}\Bigg] \\
&+ik\frac{\partial^2 A}{\partial z_1^2}\Bigg[-\frac{\beta(r)}{2k}(r^2 + a^2 + 2a\gamma_m^{(2)}) - \frac{m+1}{k^2}\Big\{\frac{rK_{m+1}(kr)}{K_m'(ka)} + \Big\{\frac{m}{k} - \Big(ka + \frac{m^2}{ka}\Big)\gamma_m^{(2)}\Big\}\beta(r)\Big\}\Bigg] \\
&-i\Bigg[\frac{\partial^2\beta(r)}{\partial k \partial m} - \frac{\partial \beta(r)}{\partial m}\Big\{\frac{m}{k} - \gamma_m^{(1)}\Big(ka + \frac{m^2}{ka}\Big) - \frac{U_2}{(U_2k-\omega)} + \frac{U_1}{(U_1k-\omega)}\Big\}\Bigg]\frac{\partial^2 A}{\partial z_1 \partial \theta_1} \\
&+\frac{\partial^2 A}{\partial \theta_1 \partial t_1}\left(\frac{1}{U_2k-\omega} - \frac{1}{U_1k-\omega}\right)\frac{\partial \beta(r)}{\partial m} - \frac{i}{2}\frac{\partial^2 A}{\partial \theta_1^2}\frac{\partial^2 \beta(r)}{\partial m^2} + \frac{\partial A}{\partial \theta_2}\frac{\partial \beta(r)}{\partial m} \\
&+\frac{\beta(r)}{U_2k-\omega)}\Bigg[\frac{\partial A}{\partial t_2} + \frac{1}{i}\frac{\partial^2 A}{\partial t_1 \partial z_1}\Big\{-\frac{m}{k} + \gamma_m^{(1)}\Big(ka + \frac{m^2}{ka}\Big) - \frac{U_1}{U_1k-\omega}\Big\} \\
&-\frac{1}{i}\frac{\partial^2 A}{\partial t_1^2}\frac{1}{U_1k-\omega} + N_3 + U_2\left(\frac{\partial A}{\partial z_2} + \frac{\partial N_2}{\partial z_1}\right)\Bigg]
\end{aligned}$$

with

$$\beta(r) = \frac{K_m(kr)}{K_m'(ka)}$$

$$\begin{aligned}
S_1 = &\Big[\!\Big[\rho(Uk-\omega)^2\Big\{4\Big(k^2 + \frac{m^2}{a^2}\Big)\gamma_m + \frac{1}{2a} + \Big\{2q_1 - 2\Big(\frac{m^2}{a^2} + k^2\Big)\gamma_m + \frac{1}{a}\Big\}\Big\{4\Big(k^2 + \frac{m^2}{a^2}\Big)\gamma_m\gamma_{2m} - 1\Big\} \\
&+\frac{3\gamma_m^2}{2a}\Big(k^2 - 3\frac{m^2}{a^2}\Big) - \Big(q_1 + q_2\frac{a^2}{T}\Big)\Big[1 + \Big\{\frac{1}{a} - \Big(\frac{m^2}{a^2} + k^2\Big)\gamma_m\Big\}\gamma_m\Big]\Big\}\Big]\!\Big] \\
&+\frac{T}{2a^2}\Big(k^2 + 9\frac{m^2}{a^2} - \frac{6}{a^2}\Big) + \frac{2T}{a^3}\Big(q_1 + q_2\frac{a^2}{T}\Big)(1 - m^2) - \frac{3T}{2}\Big(\frac{m^2}{a^2} + k^2\Big)^2 - \frac{2T}{a}\Big(\frac{m^2}{a^2} + k^2\Big)q_1,
\end{aligned}$$

$$\begin{aligned}
S_2 = -&\Big[\!\Big[\rho(Uk-\omega)\gamma_m\Big(\frac{\partial B}{\partial z_1}2k + \frac{1}{a^2}\frac{\partial B}{\partial \theta_1}2m - \Big[\!\Big[\rho\frac{\partial B}{\partial t_1}\Big]\!\Big]\frac{a^2}{T}(Uk-\omega)\Big\{\gamma_m\Big(\frac{m^2}{a^2} + k^2\Big) - \frac{1}{a}\Big\}\Big)\Big]\!\Big] \\
&+\Big[\!\Big[\rho\frac{\partial B}{\partial t_1}\Big]\!\Big]\frac{a^2}{T}\Big\{\frac{2T}{a^2}(1 - m^2) - [\![(Uk-\omega)^2]\!]\Big\}.
\end{aligned}$$

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