Exact Analysis of Laminated Plates with Anisotropic Plies

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***Abstract*—In continuation of the present author's recent development of Poisson's theory in resolving sixteen-decade-old problem of Poisson-Kirchhoff's boundary conditions paradox and its extended Poisson theory (EPT), a sequence of two dimensional problems converging to three dimensional problems within small deformation theory of elasticity is presented here in the analysis of laminated plates with anisotropic plies. Extended Poisson theory along with its modifications unlike higher order theories based on energy principles is shown to be the most convenient and simple procedure for obtaining exact solutions of problems within the applicable range of small deformation theory. Preliminary solutions for transverse stresses in primary problems of bending, extension, and associated torsion problems are exact solutions of 3-D problems. Determination of displacements and in-plane stresses consists of a sequence of solutions of uncoupled 2-D problems defined from appropriate Fourier series expansion of transverse stresses.**

Keywords—Elasticity, Plates, Laminates, Bending, Torsion, Extension

I. INTRODUCTION

Kirchhoff's theory [1] and first-order shear deformation theory based on Hencky's work [2] abbreviated as FSDT of plates in bending are simple theories and continuously used to obtain design information. In Kirchhoff's theory, $w_0(x, y)$ is governed by a fourth-order equation associated with two edge conditions instead of three edge conditions required in a 3-D problem. Consequence of this lacuna is the well known Poisson-Kirchhoff boundary conditions paradox (vide Reissner's article [3]). Assumption of zero transverse shear strains is discarded in FSDT forming a three-variable model. Reactive (statically equivalent) transverse shears are combined with in-plane shear resulting in approximation of associated torsion problem instead of flexure problem. It is necessary to use zero rotation $\omega_z = (v_{,x} - u_{,y})$ about the vertical axis to decouple bending and associated torsion problems but not sufficient condition. Kirchhoff's theory is in a way 0[th] order shear deformation theory though zero ω_z is satisfied. Reactive transverse shear stresses [τ_{xz} τ_{yz}] and thickness-wise linear strain ε_z from constitutive relation form the basis for resolving the paradox. The theory thus developed is designated as "Poisson's theory of plates in bending" [4]. $w_0(x, y)$ is a domain variable in FSDT and indirectly in Reissner's theory. These theories are intended for rectification of lacuna in Kirchhoff's theory. It is not proper, for this purpose, to use St.Venant's torsion problem in which normal strains are zero to justify these theories. Associated torsion problem in the presence of bending loads is different from St. Venant's torsion problem.

Most of the investigations reported in the literature on the analysis of laminated composite plates are based on energy principles with vertical displacement as domain variable. Ply element equations in layer-wise theories are coupled and not convenient if the stacking sequence contains large number of plies (Moreover, there is a need to incorporate proper modifications in these theories in the analysis of unsymmetrical laminates). These theories are definitely not useful in the generation of a proper sequence of 2-D problems converging to 3-D problems. It does not serve much purpose to compare results from these theories with those from the sequence of 2-D problems reported in the present author's earlier investigations [5, 6].

In the Extended Poisson's Theory (EPT) [5, 6], ply analysis is independent of lamination and continuity of displacements and transverse stresses across interfaces is through solution of a supplementary problem in the face ply along with recurrence relations. It is shown that the expansion of displacements in polynomials of thickness coordinate z is not adequate for proper estimation of face and neutral plane deflections. This fact is overlooked in the analysis of even isotropic homogeneous plates through widely used FSDT and other shear deformation theories. Solution of a supplementary problem based on Levy's work [7] is required for obtaining neutral plane deflection which is higher than face deflection. It is, however, observed that an error in the estimation of face deflection is much higher than that of neutral plane deflection. But it is desirable to provide uniform approximation to non-zero displacements and stresses in the neutral plane (mid-plane) or face planes along each vertical normal of the plate. This is particularly necessary in the analysis of laminates embedded with piezoelectric actuators so as to describe a proper electric field due to actuators.

Here, one may recall Jemielita's inspiring article [8], 'On the winding paths of the theory of plates' with the following relevant observations: Facts in the development of plate theories have proved that one is supposed to study the previous works before creating a new theory. A significant observation in the referred treatise by Toudhunter and completed after his death by Pearson (1886) is that 'the would-be researcher either wastes much time in learning the history of his subject, or else works away regardless of earlier investigations'. One could think that Pearson's words written on 23rd June 1886 became out of date in times of a stormy progress of communication but, unfortunately, it is not the case.

One should note that analysis of plates with different geometries and material properties under different kinematic and loading conditions does not provide much scope for development of new theories other than those with the analysis of primary problems of a square plate.

II. PRIMARY PROBLEMS

For simplicity in presentation, a symmetric laminate bounded by $0 \le X, Y \le a$ and $Z = \pm h_n$ planes with interfaces $Z = h_k$ in the Cartesian coordinate system (X, Y, Z) is considered. For convenience, coordinates X, Y, and Z and displacements U, V, and W in non-dimensional form $x = X/L$, $y = Y/L$, $z = Z/h_n$, $u = U/h_n$, $v = V/h_n$, $w = W/h_n$ and half-thickness ratio $\alpha = h_n/L$ with reference to a characteristic length L [mod(x, y) $\le$ 1] are utilized. The material of each ply is homogeneous and anisotropic with monoclinic symmetry. Interfaces are given by $z = \alpha_k = h_k/h_n$ (k = 1, 2…., n-1) in the upper-half of the laminate.

With the above notation, equilibrium equations in terms of stress components are:

$$\alpha\,(\sigma_{x,x} + \tau_{xy,y}) + \tau_{xz,z} = 0 \quad (1a)$$
$$\alpha\,(\sigma_{y,y} + \tau_{xy,x}) + \tau_{yz,z} = 0 \quad (1b)$$
$$\alpha\,(\tau_{xz,x} + \tau_{yz,y}) + \sigma_{z,z} = 0 \quad (2)$$

in which suffix after ',' denotes partial derivative operator.

Here, it is convenient to denote displacements [u, v] as $[u_i]$, (i = 1, 2), in-plane stresses $[\sigma_x, \sigma_y, \tau_{xy}]$ and transverse stresses $[\tau_{xz}, \tau_{yz}, \sigma_z]$ as $[\sigma_i]$, $[\sigma_{3+i}]$, (i = 1, 2, 3), respectively. With the corresponding notation for strains, strain-displacement relations are

$$[\varepsilon_1, \varepsilon_2, \varepsilon_3] = \alpha\,[u_{,x}, v_{,y}, u_{,y} + v_{,x}] \quad (3)$$
$$[\varepsilon_4, \varepsilon_5, \varepsilon_6] = [u_{,z} + \alpha\, w_{,x}, v_{,z} + \alpha\, w_{,y}, w_{,z}] \quad (4)$$

Strain-stress and semi-inverted stress-strain relations within small deformation theory with the usual summation convention of repeated suffix denoting summation over specified integer values are:

$$\varepsilon_i = S_{ij}\,\sigma_j \quad (i, j = 1, 2, 3, 6) \quad (5)$$
$$\varepsilon_r = S_{rs}\,\sigma_s \quad (r, s = 4, 5) \quad (6)$$
$$\sigma_i = Q_{ij}[\varepsilon_j - S_{j6}\,\sigma_z] \quad (i, j = 1, 2. 3) \quad (7)$$
$$\sigma_r = Q_{rs}\,\varepsilon_s \quad (r, s = 4, 5) \quad (8)$$

With σ_i in equations (7), in-plane equilibrium equations (1) become

$$\alpha\,[Q_{1j}(\varepsilon_j - S_{j6}\,\sigma_z)_{,x} + Q_{3j}(\varepsilon_j - S_{j6}\,\sigma_z)_{,y}] + \tau_{xz,z} = 0 \quad (9a)$$

$$\alpha\,[Q_{2j}(\varepsilon_j - S_{j6}\,\sigma_z)_{,y} + Q_{3j}(\varepsilon_j - S_{j6}\,\sigma_z)_{,x}] + \tau_{yz,z} = 0 \quad (9b)$$

Upper face values of displacements $[u, v, w]^u$ and transverse stresses $[\tau_{xz}, \tau_{yz}, \sigma_z]^u$ in a ply are related to its lower face values $[u, v, w]^b$ and $[\tau_{xz}, \tau_{yz}, \sigma_z]^b$, respectively, through the solution of equations (1, 2) together with the following three conditions at each of constant x (and y) edges.

A. Edge conditions in primary problems

In EPT of primary plate problems, in-plane displacements [u, v] in each ply require two term representation in extension problems and one term representation in bending (or associated torsion) problems. Prescribed conditions with subscripts $_n$ equal to $_0$ and $_1$ in extension and bending problems, respectfully, at each of x = constant edges (with analog conditions along y = constant edges) in the primary problems are

$$u = \tilde{u}_n(y) \text{ or } \sigma_{xn}(y) = T_{xn}(y) \quad (10a)$$
$$v = \tilde{v}_n(y) \text{ or } \tau_{xyn}(y) = T_{xyn}(y) \quad (10b)$$
$$w_n(y) = 0 \text{ or } \tau_{xzn}(y) = T_{xzn}(y) \quad (11)$$

Third edge condition (11) is required only in the face ply since interface continuity of displacements and transverse stresses is through solution of a supplementary problem defined in the face ply and recurrence relations across interfaces.

It is to be noted that vertical displacement is a face (and interface) variable but domain variable in the associated torsion problems (note that the condition $w_1(y) = 0$ cannot be imposed in the extension problems). Prescribed transverse stresses along z = ± 1 faces of the plate are $[T_{xz}(x, y), T_{yz}(x, y), T_z(x, y)]_n$. Due to odd and even z-distribution, however, $[T_{xz1}, T_{kyz1}, w_1, T_{z0}]$ correspond to extension problems and vice versa in bending problems (contradiction between zero face shear conditions and prescribed transverse shears along wall of the plate in the bending problem is resolved earlier [5]).

B $f_n(z)$ functions and their use

In reducing 3-D problems into a sequence of 2-D problems, it is found convenient to generate a complete set of co-ordinate functions $f_n(z)$, (n = 0,1,2,3,...) with associated 2-D variables such that f_{2n+1} and f_{2n} are odd and even functions of z with reference to mid- plane (z = 0). They are formulated with $f_0 = 1$ from recurrence relations, $f_{2n+1,zz} = f_{2n,z} = -f_{2n-1}$ (n ≥ 1) such that $f_{2n}(\pm\alpha_k) = 0$ so that they are extremely useful in obtaining preliminary solutions of primary problems.

Displacements, strains and stresses are expressed in the form (with sum n = 0, 1, 2,...)

$$w = f_n(z)\, w_n(x, y)$$
$$[u,\ v] = f_n[u, v]_n \quad (12)$$
$$[\varepsilon_x, \varepsilon_y, \gamma_{xy}, \varepsilon_z] = f_n[\varepsilon_x, \varepsilon_y, \gamma_{xy}, \varepsilon_z]_n \quad (13)$$
$$[\sigma_x, \sigma_y, \tau_{xy}, \sigma_z] = f_n[\sigma_x, \sigma_y, \tau_{xy}, \sigma_z]_n \quad (14)$$
$$[\gamma_{xz}, \gamma_{yz}, \tau_{xz}, \tau_{yz}] = f_n[\gamma_{xz}, \gamma_{yz}, \tau_{xz}, \tau_{yz}]_n \quad (15)$$

In order to maintain continuity of a 3-D variable across interfaces and keep the associated 2-D variable as a free variable, it is necessary to replace f_{2n+1} by f^*_{2n+1} given by

$$f^*_{2n+1} = f_{2n+1} - \beta_{2n-1} f_{2n-1}\ ,\ n = 1, 2, ----$$

In the above equation, $\beta_{2n-1}\alpha_k^2 = [f_{2n+1}(\alpha_k) / f_{2n-1}(\alpha_k)]$ so that $f^*_{2n+1}(\alpha_k) = 0$. With the above replacement of odd f_n functions, transverse stresses and the corresponding displacements become continuous across the interfaces if the variables associated with f_0 and f_1 are continuous across interfaces.

Prescribed upper and bottom face conditions along with edge conditions can be modified such that even functions $f_{2n}(z)$ and odd functions $f_{2n+1}(z)$ in the z-distribution of in-plane displacements are for analysis of extension and bending problems, respectively. Correspondingly, vertical displacement w(x, y, z) is odd and even in the extension and bending problems, respectively, due to transverse shear strain-displacement relations. In displacement based models, classical theories of plates deal with determination of basic variables $[u, v, w]_0$. In the present work, role of linear thickness-wise distribution of each one of three displacements, six strains and six stress components in the analysis is considered. In the preliminary analysis of primary problems, it is found that $f_n(z)$ functions up to n = 5 are necessary and adequate to generate proper sequence 2-D problems. They are

$$[f_0, f_1, f_2, f_3] = [1, z, \tfrac{1}{2}(\alpha_k^2 - z^2), \tfrac{1}{2}(\alpha_k^2 z - z^3/3)] \quad (16)$$
$$f_4 = [(5\alpha_k^4 - 6\alpha_k^2 z^2 + z^4)/24 \quad (17a)$$
$$f_5 = z\,(25\alpha_k^4 - 10\alpha_k^2 z^2 + z^4)/120] \quad (17b)$$

III. PRELIMINARY ANALYSIS OF PRIMARY PROBLEMS

In auxiliary problems in EPT, transverse stresses are expressed as $[\tau_{xz}, \tau_{yz}]_n = -\alpha\,[\psi_{n,x}, \psi_{n,y}]$. In bending problem with $\sigma_z = z\,\sigma_{z1}$, one gets from static equation (2) $\alpha^2\Delta\psi_0 = \sigma_{z1}$ ($\sigma_{z1} = q_1/2$) and the transverse shear stresses are independent of elastic deformations. In extension problem, the plate is subjected to normal stress $\sigma_{z0} = q_0(x, y)/2$, asymmetric shear stresses $[\tau_{xz1}, \tau_{yz1}] = \pm [T_{xz1}(x, y), T_{yz1}(x, y)]$ along top and bottom faces of the plate. Here, $\sigma_{z0} = q_0/2$ satisfying face condition does not participate in static equation (2) and the corresponding applied face shears $[T_{xz1}, T_{yz1}]$ are gradients of a given harmonic function $\tilde{\psi}_1$ so that $[T_{xz}, T_{yz}] = -\alpha\,[\tilde{\psi}_{1,x}, \tilde{\psi}_{1,y}]$. Transverse shear stresses and normal stress satisfying face conditions are $[\tau_{xz}, \tau_{yz}] = -\alpha\, z\,[\tilde{\psi}_{1,x}, \tilde{\psi}_{1,y}]$ and $\sigma_{z0} = q_0(x, y)/2$.With $\sigma_z = f_2\,\sigma_{z2}$, one gets $\alpha^2\Delta\psi_1 = \sigma_{z2}$ with σ_{z2} dependent on elastic deformations due to in-plane displacements $f_2[u, v]_2$.

In EPT, one should note that the error in the analysis with reference to the exact solution of 3-D problem is due to participation of $w_0(x, y)$ in the bending problem and $w_1(x, y)$ from $w = z\,w_1(x, y)$ in extension problem in the transverse strain-displacement relations (w_0 and w_1 are from z-integration of ε_z from constitutive relations). Determination of 2-D variables $[u, v]_n$ is independent of w_0 and w_1.

A. Initial solutions of primary bending problem

A brief description of preliminary solutions of primary bending problem from [6] is presented here necessary for development of sequence of uncoupled 2-D problems. Transverse stresses are in the form

$$[\tau_{xz}, \tau_{yz}] = [\tau_{xz0}, \tau_{yz0}] + [f_2\,\tau_{xz2}, f_2\,\tau_{yz2}]^k$$
$$\sigma_z = z\sigma_{z1} + [f_3\,\sigma_{z3}]^k \quad (18)$$

Transverse stresses $[\tau_{xz0}, \tau_{yz0}, z\sigma_{z1}]$ are independent of lamination and material constants. Second expression in (18) consists of reactive stresses in the ply which are also independent of lamination due to the chosen $f_k(z)$ functions. Universal solution for ψ_0 is governed by $\alpha^2\,\Delta\psi_0 = q_1/2$.

Face deflection w_{0f} from strain-displacement relations is given by

$$w_{0f}(x, y) = \int [\gamma_{xz0}\,dx + \gamma_{yz0}\,dy] - \int [u_1\,dx + v_1\,dy] \quad (19)$$

Transverse shear stresses $[\tau_{xz}, \tau_{yz}]$ and normal stress σ_z along with $(z\,q_1/2)$ in EPT are

$$[\tau_{xz}, \tau_{yz}] = [\tau_{xz}, \tau_{yz}]_0 + f_2(z)\,[\tau_{xz}, \tau_{yz}]_2 \quad (20)$$
$$\sigma_z = z\,[(1/2)q_1 - \beta_1\,\sigma_{z3}] + f_3\,\sigma_{z3} \quad (21)$$

In the above equations,

$$\tau_{xz2} = Q_{44}u_1 + Q_{45}\,v_1 + \tau_{xz0}$$
$$\tau_{yz2} = Q_{55}v_1 + Q_{45}\,u_1 + \tau_{yz0} \quad (22)$$

One equation governing $[u, v]_1$ from equilibrium equations is

$\alpha[(Q_{44}\, u_1+Q_{45}\, v_1)_{,x}+(Q_{54}\, u_1+Q_{55}\, v_1)_{,y}=\beta_1\sigma_{z3}$ (23)

Modified displacements $[u_1, v_1]^*$ for the purpose of satisfying integrated equilibrium equations (1) are

$$u^*_1 = (u_1 + \gamma_{xz0} - \alpha\, w_{0,x})$$
$$v^*_1 = (v_1 + \gamma_{yz0} - \alpha w_{0,y}) \quad (24)$$

Contributions of ψ_1 and w_0 in $[u^*, v^*]_1$ are one and the same in giving corrections to w(x, y, z) and transverse stresses. Hence, w_0 in $[u^*, v^*]_1$ is replaced by ψ_1 so that $[u^*, v^*]_1$ are

$$u^*_1 = -\alpha\,(2\psi_{1,x}+\varphi_{1,y}) + \gamma_{xz0}$$
$$v^*_1 = -\alpha\,(2\psi_{1,y}-\varphi_{1,x}) + \gamma_{yz0} \quad (25)$$

Correspondingly, in-plane strains $[\varepsilon_x^*, \varepsilon_y^*, \gamma_{xy}^*]_1$ with $[\tilde{\varepsilon}_{x1}, \tilde{\varepsilon}_{y1}, \tilde{\gamma}_{xy1}] = -\alpha^2\, [(2\psi_{1,xx}+ \varphi_{1,xy}), (2\psi_{1,yy}- \varphi_{1,xy}), (4\psi_{1,xy}+ \varphi_{1,yy}- \varphi_{1,xx})]$ and reactive transverse stresses with sum j = 1, 2. 3 are

$$[\varepsilon^*_x, \varepsilon^*_y]_1= [(\tilde{\varepsilon}_{x1} + \alpha\, \gamma_{xz0,x}), (\tilde{\varepsilon}_{y1} + \alpha\, \gamma_{xz0,y})] \quad (26a)$$

$$\gamma^*_{xy1} = [\tilde{\gamma}_{xy1} + \alpha\, (\gamma_{xz0,y} + \gamma_{yz0,x})] \quad (26b)$$

$$\tau^*_{xz2} = \alpha\, [Q_{1j}(\tilde{\varepsilon}_j - S_{j6}\, \sigma_{z1})_{,x} + Q_{3j}(\tilde{\varepsilon}_j - S_{j6}\, \sigma_{z1})_{,y}] \quad (27a)$$

$$\tau^*_{yz2} = \alpha\, [Q_{2j}(\tilde{\varepsilon}_j - S_{j6}\, \sigma_{z1})_{,y} + Q_{3j}(\tilde{\varepsilon}_j - S_{j6}\, \sigma_{z1})_{,x}] \quad (27b)$$

$$\sigma_{z3} = -\alpha\, (\tau^*_{xz2,x} + \tau^*_{yz2,y}) \quad (28)$$

Due to σ_{z3} from equations (23, 28), one gets the equation governing in-plane displacements $[u, v]_1$ satisfying both static and integrated equilibrium equations in the form

$$\alpha\, \beta_1\, (\tau^*_{xz2,x} + \tau^*_{yz2,y}) = \alpha\, [(Q_{44}\, u_1 + Q_{45}\, v_1)_{,x} + (Q_{54}\, u_1 + Q_{55}\, v_1)_{,y}] \quad (29)$$

With the condition zero ω_z (i.e., $v_{,x}= u_{,y}$) required to decouple bending and torsion, equation (29) consists of Laplace equation $\Delta\varphi_1 = 0$ and a fourth order equation in ψ_1 to be solved with the following three conditions at each of x (and y) constant edges

$$\text{(i) } (u^* \text{ or } \sigma^*)_1 = 0,\ \text{(ii) } (v^* \text{ or } \tau^*_{xy})_1 = 0,\ \text{(iii) } \psi_1 \text{ or } \tau^*_{xz2} = 0 \quad (30)$$

C. *Supplementary problem in the face ply*

Transverse stresses in the face ply are

$$[\tau_{xz}, \tau_{yz}]= [\tau_{xz0}, \tau_{yz0}] + f_2\, [\tau_{xz2}, \tau_{yz2}] \quad (31)$$
$$\sigma_z = z\, \sigma_{z1} + f_3\, \sigma_{z3} \quad (32)$$

Corrective in-plane displacements in the supplementary problem are assumed as

$$[u, v]_s = [u_1, v_1]_s \sin(\pi z/2) \quad (33)$$

In-plane distributions u_{1s} and v_{1s} are added as corrections to the known in-plane displacements $[u_1, v_1]$ so that [u, v] in the supplementary problem are

$$[u, v] = [(u_1+ u_{1s}), (v_1+ v_{1s})] \sin(\pi z/2) \quad (34)$$

With the corresponding stresses and strains, one gets

$$\beta_1\sigma_{z3} = (2/\pi)^2\, \alpha^2\, [Q_{1j}\, \varepsilon_{1sj,xx} + 2\, Q_{3j}\, \varepsilon_{1sj,xy} + Q_{2j}\, \varepsilon_{1sj,yy}] \quad (35)$$

By expressing $[u_{1s}, v_{1s}] = -\alpha\, [\psi_{1s,x}, (\psi_{1s,y}]$, above equation becomes a fourth order equation in ψ_{1s} to be solved with two in-plane conditions along each one of x (and y) constant edges

$$u_{1s} \text{ or } \sigma_{xs1} = 0,\ v_{1s} \text{ or } \tau_{xys1} = 0 \quad (36)$$

Continuity of displacements and transverse stresses across interfaces

In-plane displacements and transverse stresses in the face ply become

$$u = z\, u_1 + (u_1 + u_{1s}) \sin(\pi z/2) \quad (37a)$$
$$v = z\, v_1 + (y_1 + v_{1s}) \sin(\pi z/2) \quad (37b)$$
$$\tau_{xz} = \tau_{xz0} + f_2\, \tau_{xz2} + (\tau_{xz2} + \tau_{xz2s})\, (\pi/2) \cos(\pi z/2) \quad (38a)$$
$$\tau_{yz} = \tau_{yz0} + f_2\, \tau_{yz2} + (\tau_{yz2} + \tau_{yz2s})\, (\pi/2) \cos(\pi z/2) \quad (38b)$$
$$\sigma_z = z\, (q/2) + [f_3 - \sin(\pi z/2)]\, \beta_1\sigma_{z3} - \sigma_{z3s} \sin(\pi z/2) \quad (39)$$

Continuity of u (with similar expressions for v) across interfaces is simply assured through the following recurrence relations

$$[u_{1s}^{(k)} - u_{1s}^{(k+1)}]\sin \tfrac{\pi}{2}\alpha_k = \alpha_k\, [u_1^{(k+1)} - u_1^{(k)}] + [\alpha_k + \sin \tfrac{\pi}{2}\alpha_k]\, [u_1^{(k+1)} - u_1^{(k)}] \quad (40)$$

Since $[\tau_{xz0}, \tau_{yz0}]$ and $\sigma_{z1} = z\, q/2$ are same throughout the laminate, recurrence relations for τ_{xz2} (with similar expressions for τ_{yz2}) and σ_{z3} are

$$f_2^{(k+1)}(\alpha_k)\, \tau_{xz2}^{(k+1)} = \{[\tau_{xz2s}^{(k)} - \tau_{xz2s}^{(k+1)}] + [\tau_{xz2}^{(k)} - \tau_{xz2}^{(k+1)}]\}\tfrac{\pi}{2} \cos \tfrac{\pi}{2}\alpha_k \quad (41)$$

$$\{[\sigma_{z3s}^{(k)}-\sigma_{z3s}^{(k+1)}]+\beta_1[\sigma_{z3}^{(k)}-\sigma_{z3}^{(k+1)}]\}\sin\tfrac{\pi}{2}\alpha_k= \beta_1[f_3^{(k+1)}(\alpha_k)\, \sigma_{z3}^{(k+1)}- f_3^{(k)}(\alpha_k)\, \sigma_{z3}^{(k)}] \quad (42)$$

With ε_{z1} from constitutive relation, vertical deflection w(x, y, z) is given by

$$w = w_0 - f_2\, \varepsilon_{z1} + (\pi/2)\, w_{0s} \cos(\pi z/2) \quad (43)$$

Note that vertical deflections w_0 and w_{0s} are obtained from integration of shear strain-displacement relations (Note that ε_{z1} is obtained from ε_6 in the constitutive relations (5) in the interior of each ply). They are

$$\alpha\, w_0 = \int [(\varepsilon_{40} - u_1)\, dx + (\varepsilon_{50} - v_1)\, dy] \quad (44)$$
$$\alpha\, w_{0s} = \int [(\varepsilon_{40} - u_{1s})\, dx + (\varepsilon_{50} - v_{1s})\, dy] \quad (45)$$

Continuity across interfaces gives the recurrence relation

$$\alpha[w_{0s}^{(k)} - w_{0s}^{(k+1)}]\, \tfrac{\pi}{2} \cos \tfrac{\pi}{2}\alpha_k = \alpha\, [w_0^{(k+1)} - w_0^{(k)}] - [f_2(\alpha_k)\, \varepsilon_{z1}]^{(k+1)} \quad (46)$$

B. *Initial solution of primary extension problem*

With the inclusion of gradients of the known $\tilde{\psi}_1$ in the normal stresses $[\sigma_x, \sigma_y]_0$ so that

$$\sigma_{x0} = Q_{1j}\, (\varepsilon_{j0} - S_{j6}\, \sigma_{z0}) - \alpha\, \tilde{\psi}_{1,x}$$
$$\sigma_{y0} = Q_{2j}\, (\varepsilon_{j0} - S_{j6}\, \sigma_{z0}) - \alpha\, \tilde{\psi}_{1,y}]$$

In-plane equilibrium equations (1) become

$$\alpha\{[Q_{1j}(\varepsilon_{j0} - S_{j6}\, \sigma_{z0}) - \alpha\, \tilde{\psi}_{1,x}]_{,x} + Q_{3j}\, (\varepsilon_{j0} - S_{j6}\, \sigma_{z0})_{,y}\} = 0 \quad (47a)$$
$$\alpha\{[Q_{2j}\, (\varepsilon_{j0} - S_{j6}\, \sigma_{z0}) - \alpha\, \tilde{\psi}_{1,y}]_{,y} + Q_{3j}\, (\varepsilon_{j0} - S_{j6}\, \sigma_{z0})_{,x}\} = 0 \quad (47b)$$

Above static equilibrium equations (47) along with two edge conditions (10) have to be solved for u_0 and v_0. They remain same in the integrated equations.

Above solutions for $[u, v]_0$ with reference to 3-D problem are in error in transverse shear strain-displacement relations due to $w = z\, \varepsilon_{z0}$ ($\varepsilon_{z0} = S_{6j}\, \sigma_{j0} + S_{66}\, q_0/2$, $j = 1, 2, 3$, from constitutive relation). In order to rectify this error, it is initially necessary to consider $f_2(z)\, [u_2, v_2]$ which, in turn, induce $[\tau_{xz1}, \tau_{yz1}]$. Displacements from strain-displacement relations consistent with $[\tau_{xz1}, \tau_{yz1}]$ and σ_z from equilibrium equation of transverse stresses are

$$w = z\, \varepsilon_{z0}\, ,\; u = u_0 + f_2\, u_2,\; v = v_0 + f_2\, v_2,\quad \sigma_z = f_2\, \sigma_{z2} \quad (48)$$

(Note that σ_{z2} is not priory known unlike $\sigma_{z1} = q_1/2$ in bending problem)

Since $w = z\, \varepsilon_{z0}$ as face variable should not participate in static equilibrium equations (1), displacements $[u_2, v_2]$ are modified in the form

$$[u_2, v_2]^* = [(u_2 - \alpha\, \varepsilon_{z0,x}), (v_2 - \alpha\, \varepsilon_{z0,y})] \quad (49)$$
$$\tau_{xz1} = -\, (Q_{44}u_2 + Q_{45}\, v_2) \quad (50a)$$
$$\tau_{yz1} = -\, (Q_{55}v_2 + Q_{45}\, u_2\,) \quad (50b)$$

In order to keep $[\tau_{xz3}, \tau_{yz3}]$ as free variables in the integrated equilibrium equations, $f_3(z)$ is modified with $\beta_1 = 1/3$ as $f^*_3(z) = f_3(z) - \beta_1 z$ so that

$$\tau_{xz} = z\, (\tau_{xz1} - \beta_1\, \tau_{xz3}) + f_3\, \tau_{xz3} \quad (51a)$$
$$\tau_{yz} = z\, (\tau_{yz1} - \beta_1\, \tau_{yz3}) + f_3\, \tau_{yz3} \quad (51b)$$

One gets from equilibrium equation of transverse stresses with first term in equations (51)

$$\alpha\, (\tau_{xz1,x} + \tau_{yz1,y}) = \beta_1 \sigma_{z4} \quad (52)$$

One has from strain-displacement relations

$$[\varepsilon_x, \varepsilon_y, \gamma_{xy}]_2^* = [\varepsilon_x, \varepsilon_y, \gamma_{xy}]_2 - \alpha^2\, [\varepsilon_{z0,xx}, \varepsilon_{z0,yy}, 2\, \varepsilon_{z0,xy}] \quad (53)$$

From integration of equilibrium equations, reactive transverse stresses are

$$\tau_{xz3}{}^* = \alpha\, (\sigma_{x,x} + \tau_{xy,y})_2{}^* \quad (56a)$$
$$\tau_{yz3}{}^* = \alpha\, (\sigma_{y,y} + \tau_{xy,x})_2{}^* \quad (56b)$$
$$\sigma_{z4} = -\, \alpha\, (\tau_{xz,x} + \tau_{yz,y})_3{}^* \quad (57)$$

One equation governing in-plane displacements $(u, v)_2$ from equations (52) and (57) is

$$\alpha\, \beta_1(\tau_{xz,x} + \tau_{yz,y})_3{}^* = \alpha\, [(Q_{44}\, u + Q_{45}\, v)_{2,x} + (Q_{54}\, u + Q_{55}\, v)_{2,y}] \quad (58)$$

With the second equation $v_{2,x} = u_{2,y}$, the above equation becomes a fourth order equation in ψ_2 to be solved along with harmonic function φ_2 with three conditions $u_2{}^* = 0$ or $\sigma_{x2}{}^* = 0$, $v_2{}^* = 0$ or $\tau_{xy2}{}^* = 0$ and $\psi_2 = 0$ or $\tau_{xz3}{}^* = 0$ along x (and y) constant edges.

Supplementary problem in the face ply

Here, corrective in-plane displacements in the face ply are assumed in the form:

$$[u, v]_s = \tfrac{\pi}{2}\, [u, v]_s \cos\, (\pi z/2) \quad (59)$$
$$\sigma_{0si} = Q_{ij}\, \varepsilon_{0sj} \quad (i, j = 1, 2, 3) \quad (60)$$

If transverse shear stresses are expressed in terms of $f_3{}^*$, their continuity across interface is simply given by continuity of ψ_1. With the in-plane stresses σ_{0is} along with $[\tau_{xz}, \tau_{yz}] = [\tau_{xz1}, \tau_{yz1}]_s \sin\, (\pi\, z/2)$ and $\sigma_{z2} = \sigma_{z2s} \cos\, (\pi\, z/2)$, integration of equilibrium equations give

$$\tau_{xz1s} = -\, (2/\pi)\, \alpha\, (Q_{1j}\, \varepsilon_{0j,x} + Q_{3j}\, \varepsilon_{0j,y})_s \quad (61a)$$
$$\tau_{yz1s} = -\, (2/\pi)\, \alpha\, (Q_{2j}\, \varepsilon_{0j,y} + Q_{3j}\, \varepsilon_{0j,x})_s \quad (61b)$$
$$\sigma_{z2s} = (2/\pi)^2\, \alpha^2\, [Q_{1j}\, \varepsilon_{0j,xx} + 2Q_{3j}\, \varepsilon_{0j,xy} + Q_{2j}\, \varepsilon_{0j,yy}]_s \quad (62)$$

In-plane distributions of $[u_0, v_0]_s$ are added as corrections to the known in-plane displacements $[u_0, v_0]$ so that $[u, v]$ in the supplementary problem are

$$[u, v] = [(u_0 + u_{0s}), (v_0 + v_{0s})] \cos\, (\pi z/2) \quad (63)$$
$$\beta_0\, \sigma_{z2c} = (2/\pi)^2\, \alpha^2\, [Q_{1j}\, \varepsilon_{0j,xx} + 2Q_{3j}\, \varepsilon_{0j,xy} + Q_{2j}\, \varepsilon_{0j,yy}]_s \quad (64)$$

By expressing $[u_{0s}, v_{0s}] = -\, \alpha\, [\psi_{0s,x}, \psi_{0s,y}]$, equation (64) becomes a fourth order equation in ψ_{0s} to be solved with two conditions along x (and y) constant edges

u_{0s} or $\sigma_{xs0} = 0$, v_{0s} or $\tau_{xys0} = 0$ (65)

Continuity of displacements and transverse stresses across interfaces

In-plane displacements and transverse stresses in the face ply from above analysis are

$$[u, v] = [(u_0 + u_{0s}), (v_0 + v_{0s})] \cos(\pi z/2) \quad (66)$$
$$\tau_{xz} = f_1\, \tau_{xz1} + (\tau_{xz1} + \tau_{xz1s})(\pi/2) \sin(\pi z/2) \quad (67a)$$
$$\tau_{yz} = f_1\, \tau_{yz1} + (\tau_{yz1} + \tau_{yz1s})(\pi/2) \sin(\pi z/2) \quad (67b)$$
$$\sigma_z = (q/2) + [f_2 - \cos(\pi z/2)]\, \beta_0\, \sigma_{z2} - -\sigma_{z2s} \cos(\pi z/2) \quad (68)$$

In the interior plies, displacements $[u_0, v_0]$, thereby, in the neighboring plies are obtained from solution of sixth order system of equations (1, 2) governing $[u_0, v_0]$. They are dependent on material constants but independent of lamination. Displacements $[u_0, v_0]_s$ and transverse stresses $[\tau_{xz1}, \tau_{yz1}, \sigma_{z2}]_s$ are obtained from continuity conditions across interfaces.

Continuity of u (and v) across interfaces is simply assured through the recurrence relations (similar relations for v)

$$[u_{0s}^{(k)} - u_{0s}^{(k+1)}] \cos(\pi\alpha_k/2) = \alpha_k [u_0^{(k+1)} - u_0^{(k)}] + + [\alpha_k + \cos(\pi\alpha_k/2)]\, [u_0^{(k+1)} - u_0^{(k)}] \quad (69)$$

Above analysis gives displacements and transverse stresses as

$$w = f_1 w_1 + (\pi/2)\, w_{1s} \sin(\pi z/2) + f^*_3\, w_3 \quad (70)$$
$$u = u_0 + u_{2s} \cos(\pi z/2) + f_2\, u_2 \quad (71a)$$
$$v = v_0 + v_{2s} \cos(\pi z/2) + f_2\, v_2 \quad (71b)$$
$$[\tau_{xz}, \tau_{yz}] = f_1 [\tau_{xz}, \tau_{yz}]_1 + f^*_3\, [\tau_{xz}, \tau_{yz}]_3 \quad (72)$$

$$\sigma_z = q_0(x, y)/2 + f_2\, \sigma^*_{z2} + \sigma_{z2s} \cos(\pi z/2) + +f_2\, \sigma_{z2} \quad (73)$$

IV. ASSOCIATED TORSION PROBLEMS

Here, the analysis requires only simple modification in the bending problem by replacing ψ_1 and ψ_3 with w_0 and w_2, respectively. Note that transverse shear strains from strain-displacement relations are $[-\alpha\varphi_{1,y}, \alpha\varphi_{1,x}]$ which correspond to self-equating stresses. From zero face shear conditions, one gets an additional face deflection $\tilde{w}_{0c}(x, y)$ given by

$$\tilde{w}_{0c} = \int [\varphi_{1,y}\, dx - \varphi_{1,x}\, dy] \quad (74)$$

Above $\tilde{w}_{0c}$ is a face variable which becomes zero with prescribed zero w_0 at the edge of the plate by preventing vertical movement of intersection of the face with cylindrical surface of the side wall of the plate.

In the associated torsion problem in bending of the plate, normal strains $[\varepsilon_x, \varepsilon_y, \varepsilon_z]_1$ are zero along mid-plane. Similarly, they are zero along faces of the plate with reference to second order corrections in the extension problem and shear stresses correspond to non-torsion problem. In the associated torsion problem, the analysis requires only simple modification by replacing ψ_2 and ψ_4 with w_1 and w_3, respectively.

V. Sequence of solutions from uncoupled 2-D problems

Disadvantage in the application of EPT is in the development of software for generation of $f_k(z)$ functions and β_{2k+1}, necessary for thickness ratio varying up to unit value. Errors in the analysis are due to statically equivalent transverse stresses associated with $f_2(z)$ and $f_3(z)$ in bending problems, and $f_3(z)$ and $f_4(z)$ in extension problems. In order to rectify these errors, it is more convenient to consider successive z-integrations of $f_1 = z$ in the suitable Fourier series expansion. For this purpose, we consider Fourier series of $f_1(z)$ in the form with $\lambda_n = 2/[(2n-1)\pi]$

$$f_1(z) = \sum A_n \sin(z/\lambda_n) \;\; \text{(sum on n)} \quad (75)$$

in which

$$A_n = \int_0^1 \sin(z/\lambda_n)\, dz = -(-1)^n (\lambda_n)^2 \quad (76)$$

Relevant $[f_2, f_3, f_4]$ functions are expressed (with sum on n), for convenience, in the form

$$f_2(z) = \sum A_n \lambda_n \cos(z/\lambda_n) \quad (77a)$$
$$f_3(z) = \sum A_n \lambda_n^2 \sin(z/\lambda_n) \quad (77b)$$
$$f_4(z) = \sum A_n \lambda_n^3 \cos(z/\lambda_n) \quad (77c)$$

(Term by term differentiation in each of the above series is valid. At the ply analysis, replace z to z/α_k so that λ_n becomes $\lambda_n\alpha_k$*)*

In the bending problem, displacements [u, v] due to σ_{z3} in constitutive relations are expressed as

$$[u, v] = \sum A_n \lambda_n^2 [u, v]_{2n+1} \sin(z/\lambda_n) \quad (78)$$

Correspondingly, transverse shear stresses are expressed as

$$[\tau_{xz}, \tau_{yz}] = \sum A_n \lambda_n [\tau_{xz}, \tau_{yz}]_2 \cos(z/\lambda_n) \quad (79)$$

Equations governing $[u, v]_n$ from equilibrium equations (9) with $v_{n,x} = u_{n,y}$ are

$$\alpha\, [Q_{1j}(\varepsilon_j - S_{j6}\, \sigma_{z3})_{,x} + Q_{3j}(\varepsilon_j - S_{j6}\, \sigma_{z3})_{,y}] + \tau_{xz2} = 0 \quad (80a)$$

$$\alpha\, [Q_{2j}(\varepsilon_j - S_{j6}\, \sigma_{z3})_{,y} + Q_{3j}(\varepsilon_j - S_{j6}\, \sigma_{z3})_{,x}] + \tau_{yz2} = 0 \quad (80b)$$

Above Poisson equations have to be solved with relevant homogeneous edge conditions.

In the extension problem, in-plane displacements [u, v] due to σ_{z4} are expressed as

$$[u, v] = \sum A_n \lambda_n^3 \cos (z/\lambda_n) [u, v]_{2n} \quad (81)$$

Correspondingly, transverse shear stresses are expressed as

$$[\tau_{xz}, \tau_{yz}] = \sum A_n \lambda_n^2 \sin (z/\lambda_n) [\tau_{xz3}, \tau_{yz3}] \quad (82)$$

Equations governing $[u, v]_{2n}$ from equilibrium equations (9) with $v_{,x} = u_{,y}$ are

$$\alpha [Q_{1j}(\varepsilon_j – S_{j6} \sigma_{z4})_{,x} + Q_{3j} (\varepsilon_j – S_{j6} \sigma_{z4})_{,y}] + \tau_{xz3}= 0 \quad (83a)$$

$$\alpha [Q_{2j}(\varepsilon_j – S_{j6} \sigma_{z4})_{,y} + Q_{3j} (\varepsilon_j – S_{j6} \sigma_{z4})_{,x}] + \tau_{yz3} = 0 \quad (83b)$$

Above Poisson equations have to be solved with relevant homogeneous edge conditions.

VI. UNSYMMETRICAL LAMINATES

Here, suffix n in ($-h_n$) of the bottom face is replaced by m with number of layers ‘m’ in the bottom-half $z \le 0$ need not be equal to number of layers ‘n’ in the upper-half $z \ge 0$. Initial set of solutions in the upper-half of the laminate in all the problems presented above are unaltered up to the reference plane z = 0. One has to consider continuity of non-zero displacements and transverse stresses across reference plane. They will be different due to asymmetry from similar analysis in the bottom-half of the laminate. A novel procedure is proposed here to maintain necessary continuity across z = 0 plane in bending problems and a similar procedure in extension problems. Corresponding procedures in torsion problems which involve simple modifications are not presented.

A. Bending problem

From analysis of the upper-half the laminate, $\sigma_z = 0$ and transverse shear stresses including first order corrections due to σ_{z1} in the in-plane constitutive relations along the reference plane z = 0 are

$$(\tau_{xz})_{z=0} = \tau_{xz0} + f_2(0)\, \tau_{xz2} + (\pi/2) [\tau^*_{xz2} + \tau_{xz2s}] \quad (84a)$$

$$(\tau_{yz})_{z=0} = \tau_{yz0} + f_2(0)\, \tau_{yz2} + (\pi/2) [\tau^*_{yz2} + \tau_{yz2s}] \quad (84b)$$

in which

$$\tau^*_{xz2}= \tau_{xz0} + (Q_{44}\, u_1 + Q_{45}\, v_1)_c \quad (85a)$$
$$\tau^*_{yz2} = \tau_{yz0} + (Q_{54}\, u_1 + Q_{55}\, v_1)]_c \quad (85b)$$

Continuation of the same analysis with $\bar{z}$ (= − z) ≥ 0, one obtains along $\bar{z}$ = 0 plane that normal stress $\bar{\sigma}_z$ = 0 and

$$\bar{\tau}_{xz}=\bar{\tau}_{xz0}+\bar{f}_2(0)\bar{\tau}_{xz2}+ (\pi/2)[\bar{\tau}^*_{xz2} +\bar{\tau}_{xz2s}] \quad (86a)$$
$$\bar{\tau}_{yz}=\bar{\tau}_{yz0} +\bar{f}_2(0)\bar{\tau}_{yz2}+(\pi/2)[\bar{\tau}^*_{yz2} + \bar{\tau}_{yz2s}] \quad (86b)$$

in which

$$\bar{\tau}^*_{xz2} = \bar{\tau}^*_{xz0} + (\bar{Q}_{44}\bar{u}_1 + \bar{Q}_{45}\bar{v}_1)_c \quad (87a)$$
$$\bar{\tau}^*_{yz2} = \bar{\tau}^*_{yz0} + (\bar{Q}_{45}\bar{u}_1 + \bar{Q}_{55}\bar{v}_1)_c \quad (87b)$$

Associated extension problem in bending

In the initial set of solutions, transverse shear stresses obtained along z = 0 plane are sum of the stresses in equations (85, 87). For continuity of these stresses across z = 0 interface, one has to consider the adjacent plies above and below the interface subjected to shear stresses

$$\tau_{xz}' = \pm [\bar{\tau}_{xz} - \tau_{xz}]_{z=0};\ \tau_{yz}' = \pm [\bar{\tau}_{yz} - \tau_{yz}]_{z=0} \quad (88)$$

Continuity of these stresses is ensured by adding solutions of the laminate with free top and bottom faces along with above stresses in the adjacent plies of the interface z = 0 to the solutions of problems in the initial set. Continuity of these stresses ensures also continuity of vertical displacement across z = 0 plane.

It is convenient to introduce the coordinate z' = (1− z) for (z ≥ 0) so that the reference plane z = 0 corresponds to z' = 1. $h_k' = 1- h_k$, interfaces $\alpha_k'= (1– \alpha_k)$. Here, q = 0 along z' = 1 and the faces z'=0 are free of transverse stresses. It is inconvenient to use linear $z'\tau_{xz}'$ along edges satisfying above face conditions since the corresponding solutions for in-plane displacements $[u_0', v_0']$ from in-plane equilibrium equations are lamination independent thereby not satisfying continuity across interfaces. As such, edge conditions at x = constant edges (and analogous conditions along y = constant edges) are assumed in the form

$$\tau_{xz} = \tau_{xz}'(y) \sin (\pi z'/2) \quad (89)$$

Since $[\tau_{xz}', \tau_{yz}']$ are gradients $\alpha'[\psi_{1,x}, \psi_{1,y}]$ of a harmonic function ψ_1 and $\partial/\partial z = -\partial/\partial z'$, in-plane static equilibrium equations governing $[u_0', v_0'] \cos(\pi z'/2)$ are

$$\alpha'[Q_{1j}\, \varepsilon'_{j,x} + Q_{3j}\, \varepsilon'_{j,y}] = \alpha' (\pi/2)\, \psi_{1,x} \quad (90a)$$
$$\alpha'[Q_{2j}\, \varepsilon'_{j,y} + Q_{3j}\, \varepsilon'_{j,x}] = \alpha' (\pi/2)\, \psi_{1,y} \quad (90b)$$

Solutions of the above equations with zero bending and twisting stresses along edges give $[u_0', v_0']$ in each ply independent of lamination. Continuity of these displacements across interfaces is through recurrence relations

$$[u'^{(k)}_0 - u'^{(k+1)}_0] \cos (\pi\alpha'_k/2) = [u'^{(k+1)}_0 - u'^{(k)}_0] \quad (91)$$

Normal strain $\varepsilon'_z = \varepsilon'_{z0} \cos(\pi z'/2)$ from constitutive relation in which ε'_{z0} is given by

$$\varepsilon'_{z0} = S_{ij}\, \sigma'_j \quad (i, j = 1, 2, 3) \qquad (92)$$

Vertical deflection $w' = -\, \varepsilon'_{z0}\, (2/\pi) \sin(\pi z'/2)$ from integration of ε'_z in the interior of the ply. This deflection along the interface is obtained from shear stress-strain and strain displacement relations as

$$\alpha\, w' = (\pi/2) \cos(\pi \alpha'_k/2) \int [(\varepsilon'_{40} - u'_0)\, dx + +(\varepsilon'_{50} - v'_0) dy] \qquad (93)$$

Its continuity across interfaces is through recurrence relations

$$\alpha(\varepsilon'^{(k)} - \varepsilon'^{(k-1)})_{z0}\, (2/\pi) \sin(\pi\, \alpha'_k/2) = (\pi/2) \int [(\varepsilon'_{40} - u'_0)dx + (\varepsilon'_{50} - v'_0)dy]^{(k)} \cos(\pi \alpha'_k/2) \qquad (94)$$

Similar analysis is to be carried out with $\bar{z}$ (= − z) ≥ 0 and $\bar{z}' = (1 - \bar{z})$ so that $\bar{z} = 0$ plane is $\bar{z}' = 1$ (this part of the analysis is omitted). By adding the above vertical displacements to the corresponding displacements obtained in the upper-half and bottom-half of the relevant symmetric laminate ensure continuity across reference plane. One can choose, in principle, any one interface (excluding faces of the laminate) as reference plane but from consideration of limitations of small deformation theory, it is better to choose either mid-plane or its adjacent interface as reference plane.

B. Extension problem

Along the reference plane z = 0, w and transverse shear stresses are zero and in-plane displacements and σ_z are (sum n ≥ 1)

$$u = [u_0 + (u^*_2 + u_{2s}) + f_{2n}(0)\, u_{2n} \qquad (95a)$$
$$v = [v_0 + [v^*_2 + v_{2s}] + f_{2n}(0)\, v_{2n} \qquad (95b)$$
$$\sigma_z = q_0/2 + [\sigma^*_{z2} + \sigma_{z2s}] + f_{2n}(0)\sigma_{z2n} \qquad (96)$$

Continuation of the same analysis with $\bar{z}$ (= − z) ≥ 0, $\bar{u}$, $\bar{v}$ and $\bar{\sigma}_z$ along $\bar{z} = 0$ plane are

$$\bar{u} = \bar{u}_0 + [\bar{u}_2^* + \bar{u}_{2s}] + \bar{f}_{2n}(0)\bar{u}_{2n} \qquad (97a)$$
$$\bar{v} = \bar{v}_0 + [\bar{v}_2^* + \bar{v}_{2s}] + \bar{f}_{2n}(0)\bar{v}_{2n} \qquad (97b)$$
$$\bar{\sigma}_z = q_0/2 + [\bar{\sigma}^*_{z2} + \bar{\sigma}_{z2s}] + \bar{f}_{2n}(0)\, \bar{\sigma}_{z2n} \qquad (98)$$

Continuity of σ_z and in-plane displacements across z = 0 plane requires them to be same in the adjacent ply on each side of the z = 0 plane. For this purpose, one needs solutions of associated bending problems.

Associated bending problem in Extension

In the initial set of solutions, in-plane displacements obtained along z = 0 plane are sum of the displacements in equations (95, 97). For continuity of these displacements across z = 0 interface, one has to consider the adjacent plies above and below the interface z = 0 with

$$[u', v', \sigma_z'] = \pm [(\bar{u} - u), (\bar{v} - v), (\bar{\sigma}_z - \sigma_z) \qquad (99)$$

Continuity of u, v and σ_z is ensured by adding solutions of the laminate with free top and bottom faces along with above displacements and σ_z' in the adjacent plies of the interface z = 0 to the solutions of problems in the initial set.

It is convenient to introduce the coordinate z' = (1− z) for (z ≥ 0) so that z'=1 is reference plane z = 0, $h_k' = 1 - h_k$ and $\alpha_k' = (1 - \alpha_k)$ are interfaces. Here, $\sigma_z' = [\bar{\sigma}_z - \sigma_z]$ and $[u', v'] = [(\bar{u} - u), (\bar{v} - v)]$ along z' = 1 plane and the faces z'=0 are free of transverse stresses. It is convenient to assume $\sigma_z' = [\bar{\sigma}_z - \sigma_z] \sin(\pi z'/2)$. Then, equation governing ψ' is $\alpha^2\, \Delta\, \psi'_1 = \sigma_z'$ with ∂/∂z = − ∂/∂z' and

$$\tau_{xz}'(\pi/2) \cos(\pi z'/2) = \alpha' \psi'_{1,x}\, (\pi/2) \cos(\pi z'/2) \qquad (100a)$$
$$\tau_{yz}'\, (\pi/2) \cos(\pi z'/2) = \alpha' \psi'_{1,y}\, (\pi/2) \cos(\pi z'/2) \qquad (100b)$$

Above equation is to be solved with zero normal gradient $(\psi'_1)_n$ along the edge of the plate.

In-plane static equilibrium equations governing $[u_0', v_0'] \sin(\pi z'/2)$ are

$$\alpha'[Q_{1j}\, \varepsilon'_{j,x} + Q_{3j}\, \varepsilon'_{j,y}] + \alpha' (\pi/2)\, \psi'_{1,x} = 0 \qquad (101a)$$
$$\alpha'[Q_{2j}\, \varepsilon'_{j,y} + Q_{3j}\, \varepsilon'_{j,x}] + \alpha' (\pi/2)^2\, \psi'_{1,y} = 0 \qquad (101b)$$

Solutions of the above equations with zero bending and twisting stresses along edges give $[u_0', v_0']$ in each ply independent of lamination. Continuity of these displacements and σ_z' across interfaces is through recurrence relations

$$[u'^{(k)}_0 - u'^{(k+1)}_0] \sin(\pi \alpha'_k/2) = [u'^{(k+1)}_0 - u'^{(k)}_0] \qquad (102a)$$

$$[v'^{(k)}_0 - v'^{(k+1)}_0] \sin(\pi \alpha'_k/2) = [v'^{(k+1)}_0 - v'^{(k)}_0] \qquad (102b)$$

$$[\sigma_z'^{(k)} - \sigma_z'^{(k+1)}] \sin(\pi \alpha'_k/2) = [\sigma_z'^{(k+1)} - \sigma_z'^{(k)}] \qquad (103)$$

Similar analysis is to be carried out with ($\bar{z}$ = − z) ≥ 0 and $\bar{z}' = (1 - \bar{z})$ so that $\bar{z} = 0$ plane is $\bar{z}' = 1$ (this part of the analysis is omitted). By adding in-plane displacements and normal stress σ_z' with those in the analysis of the upper-half and bottom-half of relevant symmetric laminates ensure continuity across reference plane.

CONCLUDING REMARKS

Poisson's theory developed in the present study for the analysis of bending of anisotropic plates within small deformation theory forms the basis for generation of proper sequence of 2D problems.

Analysis for obtaining displacements, thereby, bending stresses along faces of the plate is different from solution of a supplementary problem in the interior of the plate. A sequence of higher order shear deformation theories lead to solution of associated torsion problem only. In the primary analysis of homogeneous and laminated plates, transverse stresses are independent of material constants [6, 9].

One significant observation is that sequence of 2D problems converging to 3D problems in the analysis of extension, bending, and torsion problems are mutually exclusive to one other.

HIGHLIGHTS

Estimation of transverse stresses in the preliminary solution is independent of material constants through Poisson's theory in the analysis of primary bending problem defined from Kirchhoff's theory. Hence, it is unaltered in the analysis of bending of homogeneous and laminated plates with orthotropic and anisotropic material.

An attempt is made here to present a proper sequence of sets of 2-D problems necessary for analysis of laminated plates within small deformation theory. Emphasis is on the usage of vertical displacement variable. If it is used as a domain variable, analysis corresponds to the solution of associated torsion problem in which normal strains are not zero unlike in the St. Venant's torsion problem. In bending and extension problems, it cannot be used as a domain variable. In the interior of the domain, it is from thickness-wise integration of normal strain ϵ_z. Zero vertical displacement along the edge of the plate is to be replaced by zero ϵ_z. Displacement w(x, y) arising out of integration of ϵ_z is to be obtained as a face variable from integration of transverse strain-displacement relations. Zero w(x, y) along the prescribed edge condition requires only the prevention of vertical movement of line segment of intersection of face and wall of the plate.

Set of polynomials generated in z is necessary in satisfying both static and integrated equilibrium equations. (It is, however, not simple to develop software for generation of $f_k(z)$ functions and β_{2k+1}, necessary for application of the theory with thickness ratio varying up to unit value. This problem is avoided recently in [10].

The present theory needs exploitation in investigations on optimum ply lay-up, its utility in the analysis of associated Eigen-value problems of free vibration and buckling of plates, and even in the area of fracture mechanics. However, polynomials in z are not adequate for proper solutions of 3-D problems. Solution of a supplementary problem based on appropriate trigonometric function in z representing each of displacement and stress components is required. Solution of additional similar problem is required in the analysis of unsymmetrical laminates.

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