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Extension of global-local mean square error criterion to nonlinear oscillators under narrow band excitation

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***Abstract*—This paper is to continue our study on the global-local mean square error criterion (GLOMSEC) in Gaussian equivalent linearization (GEL) [15-16]. Herein, an extension of GLOMSEC to nonlinear oscillators under narrow band random input is presented. Two illustrative examples comprise Duffing system and Duffing one with nonlinear damping to be analyzed. The results show that the accuracy of outputs by the proposed criterion is significantly improved in comparison with the one by the classical criterion.**

***Keywords*—Gaussian equivalent linearization (GEL); stochastic equivalent linearization method; narrow band random process.**

I. INTRODUCTION

One popular class of methods for approximate solutions of nonlinear systems under random excitations is Gaussian equivalent linearization (GEL) techniques. These techniques are popular among structural dynamics and in the engineering mechanics community. This is partially due to its simplicity and applicability to systems with multi-degree-of-freedom (MDOF), and systems under various types of random excitations. The underlying idea of GEL is to replace the nonlinear system by a linear one such that the behavior of the equivalent linear approximates that of the original nonlinear oscillator. The standard way of implementing this technique is that the coefficients of linearization are to be found from the classical mean square error criterion that minimizes the equation error. Although the method is very efficient, but its accuracy decreases as nonlinearity is increasing and in many cases it results in very larger errors due to the non Gaussian property of the response. For this reason, a good deal of research has been published in recent decades on improving GEL that were reported in [1-7]. In 2006, Crandall's work [8] described a number of interesting episodes in the history of the linearization technique that have arisen in the past half century. In 1995 based on the assumption that the global integration domain taken in the mean square error criterion should be reduced to a local one where the response would be concentrated N. D. Anh and Di Paola [9] proposed a local mean square error criterion (LOMSEC). Further investigations by N. D. Anh and L. X. Hung [10-11] have showed a good accuracy of this criterion, however, the local domain in question was unknown and it has resulted in the main disadvantage of LOMSEC. Recently a dual conception was proposed in the study of responses to nonlinear systems [12] and has been developed in [13-14]. One of significant advantages of the dual conception is its consideration of two different aspects of a problem in question allows the investigation to be more appropriate. Using the dual approach to LOMSEC, a new technique namely global-local mean square error criterion (GLOMSEC) was proposed [15-16] for nonlinear systems under white noise excitation, in which new values of linearization coefficients are obtained as global averaged values of all local linearization coefficients. The applications to nonlinear oscillators under white noise excitation, MDOF included [15-16] show a significant improvement on the accuracy of solutions. The white noise excitation that has been widely used thus far is a mathematical idealization rather than an adequate representation of many excitation processes in reality. It is acknowledged that such an idealization can be used in the analysis of response that give important insight and useful results in the design process of a particular system. Quite frequently in real nature, the excitations should conceptually be better described as narrow band random processes. An important class of engineering systems modeled as dynamical ones under narrow band excitation is ships and ocean structures subjected to water waves, e.g. ship rolling motion, structures subjected to wind and earthquake loadings, vehicle vibration by road roughness. While with wide ban excitation, at least simple nonlinear systems can be solved, the corresponding solutions with narrow band excitation are not available. Due to a Gaussian narrow band process is essentially a filtered white noise, dimension of the response vector shall be at least four, the Fokker-Planck (FP) equation for the probability density can be easily built but solving this equation is a formidable task. The stochastic averaging method can reduce the number of variables to three, but the obtained FP equation is still difficult to solve. Thus GEL is still an efficient tool to get approximate solutions. Applications of GEL to nonlinear oscillators under narrow band excitations were reported in [17-24]. As the above-mentioned, the improvement of GEL for more accurate solutions is needful. This paper presents an extension of GLOMSEC with the input to be narrow band excitation. Two illustrative examples comprise Duffing system and

Duffing one with nonlinear damping to be analyzed. The applications show that the accuracy of outputs by the proposed criterion is significantly improved in comparison with the one by the classical GEL.

II. GLOMSEC FOR NONLINEAR OSCILLATORS UNDER NARROW BAND EXCITATION

Consider a nonlinear oscillator under narrow band excitation governed by

$$\ddot{z}+g(z,\dot{z})=f \tag{1}$$

Where, $g(z,\dot{z})$ is a nonlinear function of $z,\dot{z}$, f is a narrow band excitation that can be obtained by filtering a stationary Gaussian white noise through a linear filter with center frequency ω_f and bandwidth α such that the equation of motion for the filter is

$$\ddot{f}+\alpha\dot{f}+\omega_f^2 f=\omega_f^2 w \tag{2}$$

Where, w is the zero mean Gaussian white noise whose spectral density is S. The form of filter (2) was presented in publications by Iyengar, R.N. [21] and Cho, W.S. To [24] when considering a cubic hardening oscillator under narrow band excitation. The spectral density function of f is

$$S_f(\omega)=\left|H(\omega)\right|^2\omega_f^4 S \tag{3}$$

Where, $\left|H(\omega)\right|^2=H(\omega)H(-\omega)$, $H(\omega)$ is the complex frequency response of the linear filter defined by

$$H(\omega)=\left(\omega_f^2-\omega^2+i\omega\alpha\right)^{-1} \tag{4}$$

Therefore, $$S_f(\omega)=\frac{\omega_f^4 S}{\left(\omega_f^2-\omega^2\right)^2+\alpha^2\omega^2} \tag{5}$$

The variance of the filter response f is

$$\sigma_f^2=\int_{-\infty}^{\infty}S_f(\omega)d\omega=\frac{\pi S\omega_f^2}{\alpha} \tag{6}$$

Suppose that a stationary solution to (1) exists. Following GEL, (1) is placed by the following equivalent linear equation

$$\ddot{x}+c\dot{x}+kx=f \tag{7}$$

Where, k,c are linearization coefficients that shall be determined by minimizing the error between the original nonlinear system in (1) and the equivalent linear system in (7) in some sense. In (7), $x(t)$ and $\dot{x}(t)$ assumed as zero mean Gaussian random processes.

Denote $L(\omega)$ the frequency response function of the linear system (7) that defined by

$$L(\omega)=\left(k-\omega^2+i\omega c\right)^{-1} \tag{8}$$

The spectral density function of process $x(t)$ can be obtained as follows.

$$S_x(\omega)=\left|L(\omega)\right|^2 S_f(\omega)=\frac{S_f(\omega)}{\left(k-\omega^2\right)^2+c^2\omega^2} \tag{9}$$

The variances of $x(t)$ and $\dot{x}(t)$ respectively are

$$\sigma_x^2=\int_{-\infty}^{\infty}S_x(\omega)d\omega;\quad \sigma_{\dot{x}}^2=\int_{-\infty}^{\infty}\omega^2 S_x(\omega)d\omega. \tag{10}$$

Combine (5), (6), (9), (10) then one can get the variances of $x(t),\dot{x}(t)$ as follows.

$$\sigma_x^2=\frac{\pi S\omega_f^2}{\alpha}\frac{ck+\alpha\left(\omega_f^2+c^2\right)+\alpha^2 c}{ck\left[\left(\omega_f^2-k\right)^2+(c+\alpha)\left(c\omega_f^2+\alpha k\right)\right]};$$
$$\sigma_{\dot{x}}^2=\frac{\pi S\omega_f^4}{\alpha}\frac{(c+\alpha)}{c\left[\left(\omega_f^2-k\right)^2+(c+\alpha)\left(c\omega_f^2+\alpha k\right)\right]}. \tag{11}$$

σ_x^2, $\sigma_{\dot{x}}^2$ are considered as the approximate solutions to original nonlinear oscillator in (1). However, in (11), the linearization coefficients k,c must be defined by minimizing the error between the original nonlinear oscillator in (1) and the equivalent linear oscillator in (7) in some sense. The error between (1) and (7) is

$$e(x,\dot{x})=g(x,\dot{x})-kx-c\dot{x}. \tag{12}$$

The classical GEL requires $\left\langle e^2(x,\dot{x})\right\rangle\to\min_{k,c}$, or in explicit form of

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^2(x,\dot{x})P(x,\dot{x})dxd\dot{x}\to\min_{k,c}. \tag{13}$$

Where, $P(x,\dot{x})$ is the normal joint probability density function (PDF) of the random variables $x(t)$ and $\dot{x}(t)$ that can be separated into two independent single PDFs $P(x,\dot{x})=P(x)P(\dot{x})$ with the assumption that $x(t)$ and $\dot{x}(t)$ are mutually independent. The necessary conditions for (13) to be true are

$$\left\langle\frac{\partial e^2(x,\dot{x})}{\partial k}\right\rangle=0,\left\langle\frac{\partial e^2(x,\dot{x})}{\partial c}\right\rangle=0. \tag{14}$$

Expanding (14) and noting that $\langle x\dot{x}\rangle=0$, one gets

$$k=\frac{\langle g(x,\dot{x})x\rangle}{\langle x^2\rangle},\quad c=\frac{\langle g(x,\dot{x})\dot{x}\rangle}{\langle \dot{x}^2\rangle}. \tag{15}$$

Equations (11) and (15) form a close system for determining the unknowns k, c, $x(t)$ and $\dot{x}(t)$. Since the integrations is taken over the entire coordinate space ($-\infty,+\infty$), criterion (13) may be called as global mean square error criterion. Basing on an assumption that the global integration domain taken in the classical criterion should be reduced to a local one where the response would be concentrated, N. D. Anh and Di Paola [9], N. D. Anh and L. X. Hung [10-11] proposed a local mean square error criterion (LOMSEC) as follows.

$$\int_{-x_0}^{x_0}\int_{-\dot{x}_0}^{\dot{x}_0}e^2(x,\dot{x})P(x,\dot{x})dxd\dot{x}\to\min_{k,c}. \tag{16}$$

Where, $x_0,\dot{x}_0$ are given positive values. The expected integrations in (16) can be transformed to nondimensional variables by $x_0=r\sigma_x$, $\dot{x}_0=r\sigma_{\dot{x}}$ with r is a given positive value, σ_x and $\sigma_{\dot{x}}$ are the normal deviations of the random variables $x(t)$ and $\dot{x}(t)$, respectively. Thus, criterion (16) leads to

$$\left[e^2(x,\dot{x})\right] = \int_{-r\sigma_x}^{r\sigma_x}\int_{-r\sigma_{\dot{x}}}^{r\sigma_{\dot{x}}} e^2(x,\dot{x})P(x,\dot{x})\,dxd\dot{x} \rightarrow \min_{k,c} \qquad (17)$$

Where, [.] denotes the local mean values of random variables which are taken as follows.

$$[.] = \int_{-r\sigma_x}^{r\sigma_x}\int_{-r\sigma_{\dot{x}}}^{r\sigma_{\dot{x}}} (.)P(x,\dot{x})\,dxd\dot{x} \;. \qquad (18)$$

Similar to the classical GEL, one gets

$$k(r) = \frac{[g(x,\dot{x})x]}{[x^2]},\quad c(r) = \frac{[g(x,\dot{x})\dot{x}]}{[\dot{x}^2]}. \qquad (19)$$

It is seen from (18) and (19) the linearization coefficients obtained by LOMSEC are functions depending on r ($k=k(r),\ c=c(r)$), and also depending on the local mean values of random variables that are not explicit expressed here. The appendix provides the calculation of local mean values of $x(t)$ and $\dot{x}(t)$ by LOMSEC. Formulas in (19) indicate that the linearization coefficients are functions depending on parameter *r* and when *r* is determined they become constant values. In this sense the linearization coefficients $k(r), c(r)$ can be called as local linearization coefficients. The unknowns $k(r), c(r)$, $x(t)$ and $\dot{x}(t)$ can be obtained when *r* is given. The main limitations of LOMSEC, is that the local domain of integration, namely in our case the value of *r*, is unknown and the open question is of how to find it. Using the dual conception to LOMSEC, the authors of [15-16] suggested that instead of finding a special value of *r* one may consider its varying in all the global domain of integration. Thus, the constant linearization coefficients k, c can be suggested as global mean values of all local linearization coefficients as follows.

$$k = \langle k(r)\rangle = \lim_{s\to\infty}\left(\frac{1}{s}\int_0^s k(r)dr\right),$$

$$c = \langle c(r)\rangle = \lim_{s\to\infty}\left(\frac{1}{s}\int_0^s c(r)dr\right). \qquad (20)$$

Where, <.> is used as the conventional notation for averaging operators of deterministic functions. Criterion (20) is called global-local mean square error criterion (GLOMSEC). Now (11), (19) and (20) allow finding the unknowns without specifying any value of *r*, in such way GLOMSEC is extended to nonlinear oscillators under narrow-band random excitation.

III. ILLUSTRATIVE EXAMPLES

A. *Duffing oscillator under narrow-band excitation*

The equation of oscillator is described by

$$\ddot{z} + \beta\dot{z} + \Omega^2(z + \varepsilon z^3) = f \qquad (21)$$

Where, $\beta, \varepsilon, \Omega^2$ are positive parameters, $g(z,\dot{z}) = \beta\dot{z} + \Omega^2(z+\varepsilon z^3)$ is a nonlinear function of $z, \dot{z}$, and f is a narrow-band excitation that its motion is defined by the above given (2). The original nonlinear equation (21) can be replaced by the equivalent linear equation as described by (7). The key procedure is that of defining the linearization coefficients of k, c. Apply (19) and the results from the appendix (a.5-a.7) to find $k(r), c(r)$ by LOMSEC.

$$k(r) = \frac{[g(x,\dot{x})x]}{[x^2]} = \frac{\beta[\dot{x}x] + \Omega^2[x^2] + \Omega^2\varepsilon[x^4]}{[x^2]} =$$

$$= \Omega^2\left(1 + \varepsilon\frac{2T_{2,r}\langle x^2\rangle^2 2T_{0,r}}{2T_{1,r}\langle x^2\rangle 2T_{0,r}}\right) = \Omega^2\left(1 + \varepsilon\langle x^2\rangle\frac{T_{2,r}}{T_{1,r}}\right), \qquad (22)$$

$$c(r) = \frac{[g(x,\dot{x})\dot{x}]}{[\dot{x}^2]} = \frac{\beta[\dot{x}^2] + \Omega^2[x\dot{x}] + \Omega^2\varepsilon[x^3\dot{x}]}{[\dot{x}^2]} = \beta.$$

The formula (22) indicates $c(r) = \beta$ (constant) while $k(r)$ depends on *r*, apply (20) to find this linearization coefficient by GLOMSEC and note that $\langle x^2\rangle = \sigma_x^2$.

$$k = \langle k(r)\rangle = \lim_{s\to\infty}\left(\frac{1}{s}\int_0^s k(r)dr\right) = \Omega^2 + \Omega^2\varepsilon\sigma_x^2 \lim_{s\to\infty}\left(\frac{1}{s}\int_0^s \frac{T_{2,r}}{T_{1,r}}dr\right). \qquad (23)$$

Where, $T_{n,r}\ (n=1,2)$ is defined by (a.5) in the appendix. The limitation value in (23) can be obtained by the computational approximate calculation as follows.

$$\lim_{s\to\infty}\left(\frac{1}{s}\int_0^s \frac{T_{2,r}}{T_{1,r}}dr\right) \approx 2.41189\,. \qquad (24)$$

The final result of the linearization coefficients obtained by GLOMSEC to be

$$k = \Omega^2 + 2.41189\Omega^2\varepsilon\sigma_x^2,\ c = \beta. \qquad (25)$$

Use (22) and let $r\to\infty$, one obtains the linearization coefficients as defined by the classical GEL

$$k = \Omega^2 + 3\Omega^2\varepsilon\sigma_x^2,\ c = \beta \qquad (26)$$

For the purpose of evaluating error of solutions obtained by GLOMSEC and the classical GEL while the exact solution of the considered original system does not exists, a rather accurate solution obtained by the energy balance method [4,7] is used. Following this method, the mean square differences of potential and dissipative energy between the governed nonlinear system and its equivalent linear one should be minimized. The potential function and dissipative function are defined by the following form

$$U(\theta) = \int_0^\theta g(\theta)d\theta\ . \qquad (27)$$

Where, $g(\theta)$ is the stiffness element causing the potential energy, or the damping element causing the dissipative energy. Apply (27) for the nonlinear system (21) and its equivalent linear system (7), one gets the respective potential functions and dissipative functions. The energy balance method requires

$$\left\langle\left(\int_0^x \Omega^2\left(x+\varepsilon x^3\right)dx-\int_0^x kxdx\right)^2\right\rangle=\left\langle\left(\Omega^2\left(\frac{x^2}{2}+\varepsilon\frac{x^4}{4}\right)-k\frac{x^2}{2}\right)^2\right\rangle \to \min_k, \left\langle\left(\int_0^{\dot{x}}\beta\dot{x}d\dot{x}-\int_0^{\dot{x}}c\dot{x}d\dot{x}\right)^2\right\rangle=\left\langle\left(\beta\frac{\dot{x}^2}{2}-c\frac{\dot{x}^2}{2}\right)^2\right\rangle\to\min_c. \quad (28)$$

The necessary conditions for (28) to be true are

$$\left\langle \partial\left(\Omega^2\left(\frac{x^2}{2}+\varepsilon\frac{x^4}{4}\right)-k\frac{x^2}{2}\right)^2 \Big/ \partial k\right\rangle=0, \quad \left\langle \partial\left(\beta\frac{\dot{x}^2}{2}-c\frac{\dot{x}^2}{2}\right)^2 \Big/ \partial c\right\rangle=0. \quad (29)$$

Expand (29) and note that $\langle x^2\rangle=\sigma_x^2$ the linearization coefficients by the energy balance method to be $k=\Omega^2+2.5\Omega^2\varepsilon\sigma_x^2,\ c=\beta.$ (30)
Combine in pairs each (25), (26), (30) with (11) resulting in the closed equation pairs that enable obtaining solutions by GLOMSEC ($\sigma_{x,G}^2$), the classical GEL ($\sigma_{x,C}^2$), the energy balance method ($\sigma_{x,E}^2$), respectively. It should be that such equation pairs will give multi-solution, in which just real and positive one is taken. Consider the system with parameters $\beta,\Omega^2,S,\alpha,\omega_f^2=1$ meanwhile ε varies. The results are presented in TABLE I. Denote $Err|\%|$ the relative errors of solutions compared to ones by the energy balance method.

TABLE I. THE MEAN SQUARE RESPONSES OF THE CONSIDERED SYSTEM WITH $\beta=\Omega^2=S=\alpha=\omega_f^2=1$ AND ε VARIES.

ε	$\sigma_{x,E}^2$	$\sigma_{x,C}^2$	$Err\|\%\|$	$\sigma_{x,G}^2$	$Err\|\%\|$
0.1	1.86038	1.75024	5.920	1.88195	1.159
1	0.66376	0.60015	9.583	0.67688	1.977
10	0.16687	0.14855	10.979	0.17072	2.307
100	0.03720	0.03296	11.398	0.03809	2.392

The relative errors indicate that in considered system, the solutions given by GLOMSEC are closer to the ones given by the energy balance method over the classical GEL, especially the nonlinearity is strong.

B. Duffing oscillator with nonlinear damping under narrow-band excitation

The equation of oscillator is described by
$\ddot{z}+\beta\dot{z}+\gamma\dot{z}^3+\varepsilon z^3=f$ (31)
Where, β,γ,ε are positive parameters, $g(z,\dot{z})=\beta\dot{z}+\gamma\dot{z}^3+\varepsilon z^3$ is a nonlinear function of $z,\dot{z}$, and f is a narrow-band excitation that has the same description as the previous example. The original nonlinear oscillator (31) is also replaced by the equivalent linear oscillator as described by (7). Apply (19) and the appendix (a.5-a.7) to find $k(r),c(r)$ by LOMSEC, the results as follows.

$$k(r)=\frac{[g(x,\dot{x})x]}{[x^2]}=\frac{\beta[\dot{x}x]+\gamma[\dot{x}^3x]+\varepsilon[x^4]}{[x^2]}=\varepsilon\frac{2T_{2,r}}{2T_{1,r}}\frac{\langle x^2\rangle^2 2T_{0,r}}{\langle x^2\rangle 2T_{0,r}}=\varepsilon\langle x^2\rangle\frac{T_{2,r}}{T_{1,r}},$$
$$c(r)=\frac{[g(x,\dot{x})\dot{x}]}{[\dot{x}^2]}=\frac{\beta[\dot{x}^2]+\gamma[\dot{x}^4]+\varepsilon[x^3\dot{x}]}{[\dot{x}^2]}=\beta+\gamma\frac{2T_{2,r}}{2T_{1,r}}\frac{\langle\dot{x}^2\rangle^2 2T_{0,r}}{\langle\dot{x}^2\rangle 2T_{0,r}}=\beta+\gamma\langle\dot{x}^2\rangle\frac{T_{2,r}}{T_{1,r}}. \quad (32)$$

Apply (20) to find the linearization coefficients by GLOMSEC and note that $\langle x^2\rangle=\sigma_x^2,\langle\dot{x}^2\rangle=\sigma_{\dot{x}}^2$

$$k=\langle k(r)\rangle=\lim_{s\to\infty}\left(\frac{1}{s}\int_0^s k(r)dr\right)=\varepsilon\sigma_x^2\lim_{s\to\infty}\left(\frac{1}{s}\int_0^s\frac{T_{2,r}}{T_{1,r}}dr\right),$$
$$c=\langle c(r)\rangle=\lim_{s\to\infty}\left(\frac{1}{s}\int_0^s c(r)dr\right)=\beta+\gamma\sigma_{\dot{x}}^2\lim_{s\to\infty}\left(\frac{1}{s}\int_0^s\frac{T_{2,r}}{T_{1,r}}dr\right). \quad (33)$$

Where, $T_{n,r}(n=1,2)$ and the limitation value are defined the same as the previous example. The final result of the linearization coefficients obtained by GLOMSEC:
$k=2.41189\varepsilon\sigma_x^2,\ c=\beta+2.41189\gamma\sigma_{\dot{x}}^2$ (34)

Use (32) and let $r\to\infty$, one obtains the linearization coefficients as defined by the classical GEL
$k=3\varepsilon\sigma_x^2, c=\beta+3\gamma\sigma_{\dot{x}}^2$ (35)

The energy balance method requires

$$\left\langle\left(\int_0^x\varepsilon x^3dx-\int_0^x kxdx\right)^2\right\rangle=\left\langle\left(\varepsilon\frac{x^4}{4}-k\frac{x^2}{2}\right)^2\right\rangle\to\min_k,$$
$$\left\langle\left(\int_0^{\dot{x}}\left(\beta\dot{x}+\gamma\dot{x}^3\right)d\dot{x}-\int_0^{\dot{x}}c\dot{x}d\dot{x}\right)^2\right\rangle=\left\langle\left(\beta\frac{\dot{x}^2}{2}+\gamma\frac{\dot{x}^4}{4}-c\frac{\dot{x}^2}{2}\right)^2\right\rangle\to\min_c. \quad (36)$$

The necessary conditions for (36) to be true are

$$\left\langle\partial\left(\varepsilon\frac{x^4}{4}-k\frac{x^2}{2}\right)^2\Big/\partial k\right\rangle=0,\left\langle\partial\left(\beta\frac{\dot{x}^2}{2}+\gamma\frac{\dot{x}^4}{4}-c\frac{\dot{x}^2}{2}\right)^2\Big/\partial c\right\rangle=0. \quad (37)$$

Expand (37), one gets the linearization coefficients by the energy balance method to be
$k=2.5\varepsilon\sigma_x^2,\ c=\beta+2.5\gamma\sigma_{\dot{x}}^2.$ (38)
Combine in pairs each (34), (35), (38) with (11) resulting in the closed equation pairs that enable obtaining solutions by GLOMSEC ($\sigma_{x,G}^2$), the classical GEL ($\sigma_{x,C}^2$), the energy balance method ($\sigma_{x,E}^2$), respectively. Just real and positive solution from each equation pair is taken. Consider the system with two cases of parameter set to be $\beta,\gamma,S,\alpha,\omega_f^2=1$, ε varies, and $\beta,\varepsilon,S,\alpha,\omega_f^2=1$, γ varies. The results are presented in TABLE II and TABLE III, respectively.

TABLE II. THE MEAN SQUARE RESPONSES OF THE CONSIDERED SYSTEM WITH $\beta=\gamma=S=\alpha=\omega_f^2=1$ AND ε VARIES.

ε	$\sigma_{x,E}^2$	$\sigma_{x,C}^2$	$Err\|\%\|$	$\sigma_{x,G}^2$	$Err\|\%\|$
0.1	2.40633	2.14097	11.028	2.46218	2.321
1	0.74346	0.65974	11.261	0.76111	2.374
10	0.18583	0.16413	11.677	0.19043	2.475
100	0.03968	0.03502	11.744	0.04067	2.495

TABLE III. THE MEAN SQUARE RESPONSES OF THE CONSIDERED SYSTEM WITH $\beta=\varepsilon=S=\alpha=\omega_f^2=1$ AND γ VARIES.

γ	$\sigma_{x,E}^2$	$\sigma_{x,C}^2$	$Err\|\%\|$	$\sigma_{x,G}^2$	$Err\|\%\|$
0.1	0.86998	0.77228	11.230	0.89053	2.362
1	0.74346	0.65974	11.261	0.76111	2.374
10	0.53843	0.47760	11.298	0.55128	2.387
100	0.36799	0.32617	11.364	0.37683	2.402

The relative errors indicate that for both the cases, the solutions given by GLOMSEC are closer to the ones given by the energy balance method over the classical GEL at any nonlinearity.

IV. CONCLUSION

Based on GLOMSEC was proposed for nonlinear oscillators under white noise random excitation [15], [16], the paper presents an extension of this criterion with the input to be narrow band excitation. The obtained results indicate two main outstanding achievements of GLOMSEC: The first, GLOMSEC allows obtaining significant improvement on the accuracy of solution over the classical GEL, especially in many cases when nonlinearity is strong. The second, GLOMSEC does not require given values of integration domain used in LOMSEC, consequently it becomes a more reliable and effective technique for analysis of nonlinear stochastic systems under not only white noise excitation but also narrow band excitation which widely exists in practice.

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APPENDIX

Assume that x and $\dot{x}$ are zero mean Gaussian independent random processes, denote [.] the local mean values of random variables which are taken as follows.

$$[.]=\int_{-x_0}^{+x_0}\int_{-\dot{x}_0}^{+\dot{x}_0}(.)P(x,\dot{x})dxd\dot{x} \quad (a.1)$$

Where, $x_0,\dot{x}_0$ are given positive values; $P(x,\dot{x})$ is their normal joint probability density function (PDF) that can be separated into two independent single PDFs: $P(x,\dot{x})=P(x)P(\dot{x})$,

$$P(x)=\frac{1}{\sqrt{2\pi}\sigma_x}e^{-x^2/2\sigma_x^2},\ P(\dot{x})=\frac{1}{\sqrt{2\pi}\sigma_{\dot{x}}}e^{-\dot{x}^2/2\sigma_{\dot{x}}^2}. \quad (a.2)$$

Where, σ_x and $\sigma_{\dot{x}}$ are the normal deviations of the random variables x and $\dot{x}$, respectively. The expected integrations in (a.1) can be transformed to non-dimensional variables by $x_0=r\sigma_x,\dot{x}_0=r\sigma_{\dot{x}}$ with r is a given positive value:

$$[.]=\int_{-r\sigma_x}^{+r\sigma_x}\int_{-r\sigma_{\dot{x}}}^{+r\sigma_{\dot{x}}}(.)P(x,\dot{x})dxd\dot{x} \quad (a.3)$$

As known that for a zero mean Gaussian random variable, all odd-order means are null, all higher even-order means can be expressed in terms of second order mean of the respective variable. By replacing variables $x=t\sigma_x$, $\dot{x}=t\sigma_{\dot{x}}$ and using formulas (a.2), (a.3), the higher even-order local means $\left[x^{2n}\right],\left[\dot{x}^{2n}\right]$ when using LOMSEC can be expressed in terms of second order global means $\langle x^2\rangle,\langle \dot{x}^2\rangle$, respectively. For the variable x:

$$\left[x^{2n}\right]=\int_{-r\sigma_x}^{+r\sigma_x}x^{2n}P(x)dx\int_{-r\sigma_{\dot{x}}}^{+r\sigma_{\dot{x}}}P(\dot{x})dx=$$
$$=\left(\int_{-r}^{r}t^{2n}\sigma_x^{2n}\frac{1}{\sqrt{2\pi}\sigma_x}e^{-t^2\sigma_x^2/2\sigma_x^2}\sigma_x dt\right)\left(\int_{-r}^{r}\frac{1}{\sqrt{2\pi}\sigma_{\dot{x}}}e^{-t^2\sigma_{\dot{x}}^2/2\sigma_{\dot{x}}^2}\sigma_{\dot{x}}dt\right)=$$
$$=\left(2\sigma_x^{2n}\int_0^r t^{2n}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt\right)\left(2\int_0^r\frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt\right) \quad (a.4)$$

Where, n is a natural number. Introduce $\sigma_x^{2n}=\langle x^2\rangle^n$ since x is a zero mean Gaussian random process, and the following replacements

$$\eta(t)=\frac{1}{\sqrt{2\pi}}e^{-t^2/2},\ T_{n,r}=\int_0^r t^{2n}\eta(t)dt,\ T_{0,r}=\int_0^r\eta(t)dt \quad (a.5)$$

Thus, formula (a.4) can be rewritten as follows

$$\left[x^{2n}\right]=\int_{-r\sigma_x}^{+r\sigma_x}x^{2n}P(x)dx\int_{-r\sigma_{\dot{x}}}^{+r\sigma_{\dot{x}}}P(\dot{x})d\dot{x}=2T_{n,r}\langle x^2\rangle^n 2T_{0,r} \quad (a.6)$$

Where, $\int_{-r\sigma_x}^{+r\sigma_x}x^{2n}P(x)dx=2T_{n,r}\langle x^2\rangle^n,\ \int_{-r\sigma_{\dot{x}}}^{+r\sigma_{\dot{x}}}P(\dot{x})d\dot{x}=2T_{0,r}$.

By the same way, we obtain the similar formula to (a.6) for the variable $\dot{x}$:

$$\left[\dot{x}^{2n}\right]=\int_{-r\sigma_{\dot{x}}}^{+r\sigma_{\dot{x}}}\dot{x}^{2n}P(\dot{x})d\dot{x}\int_{-r\sigma_x}^{+r\sigma_x}P(x)dx=2T_{n,r}\langle \dot{x}^2\rangle^n 2T_{0,r} \quad (a.7)$$

Where, $\int_{-r\sigma_{\dot{x}}}^{+r\sigma_{\dot{x}}}\dot{x}^{2n}P(\dot{x})d\dot{x}=2T_{n,r}\langle \dot{x}^2\rangle^n,\ \int_{-r\sigma_x}^{+r\sigma_x}P(x)dx=2T_{0,r}$.

When $r \to \infty$, the formulas (a.6) and (a.7) give the respective results for the classical criterion as follows.

$$\left\langle x^{2n} \right\rangle = \left[x^{2n} \right]_{-\infty}^{+\infty} = \int_{-\infty}^{+\infty} x^{2n} P(x) dx \int_{-\infty}^{+\infty} P(\dot{x}) d\dot{x} = 2T_{n,\infty} \left\langle x^2 \right\rangle^n \quad \text{(a.8)}$$
$$= (2n-1)!! \left\langle x^2 \right\rangle^n$$

Where, $\int_{-\infty}^{+\infty} x^{2n} P(x) dx = 2\sigma_x^{2n} \int_0^{\infty} t^{2n} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = 2T_{n,\infty} \left\langle x^2 \right\rangle^n$, $\int_{-\infty}^{+\infty} P(\dot{x}) d\dot{x} = 2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = 1.$

$$\left\langle \dot{x}^{2n} \right\rangle = \left[\dot{x}^{2n} \right]_{-\infty}^{+\infty} = \int_{-\infty}^{+\infty} \dot{x}^{2n} P(\dot{x}) d\dot{x} \int_{-\infty}^{+\infty} P(x) dx = 2T_{n,\infty} \left\langle \dot{x}^2 \right\rangle^n \quad \text{(a.9)}$$
$$= (2n-1)!! \left\langle \dot{x}^2 \right\rangle^n$$

Where, $\int_{-\infty}^{+\infty} \dot{x}^{2n} P(\dot{x}) d\dot{x} = 2\sigma_{\dot{x}}^{2n} \int_0^{\infty} t^{2n} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = 2T_{n,\infty} \left\langle \dot{x}^2 \right\rangle^n$, $\int_{-\infty}^{+\infty} P(x) dx = 2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = 1.$

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