# Poincare recurrence in open systems

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***Abstract*—In article in the context of the scientific field, physics of open systems, is considered the paradigm recurrence Poincare in nonlinear physical systems of fractional order. The problem is realized in terms of the synergetic approach.**

***Keywords*—physics dynamic systems, Poincare return time, semi-trajectories, generalized memory**

## I. Introduction

In most cases of new effects discovered in recent years include the research processes in nonlinear dissipative systems and environments. Problems outlined in the article will be considered in the context of modern scientific field – *physics of open systems* [1-4].

Noteworthy new features of so-called reversible mappings that allow for new look into the physical processes in the chaotic and stochastic systems.

It is known that the dynamic system with complex trajectories character can be described in terms of the geometry of limit sets in the phase space, as well as the evolution of the phase trajectories in time. Feature of the temporal dynamics of reversible is the so-called Poincare return, meaning that any trajectory, starting from a point $x_0$ of the phase space, eventually an infinite number of times will pass arbitrarily close to the initial conditions.

Depending of the system operation mode return will either (in the motion is stable periodic) or quasi-periodic (driving in $n$ - dimensional tory), or be a random sequence of time $\tau_k = t_{k+1} - t_k$ , when $t_k$ corresponds to time path to enter the $\varepsilon$ - neighborhood of $x_0$. So, for the first time of chaotic attractor's limited return: $\tau_k = t_{k+1} - t_k < z$ for all $k = 1,2,\ldots,$ which is a consequence of the presence of a minimal set in the system. Movement of the attraction satisfied specified properties, Poincare called Poisson stable.

It continues to be the actual problem analysis and synthesis of a large number of interacting heterogeneous information flow in complex structures. In the process of evolution, in an open system, increase information flows and objects leads to a complication of information components that causes a buildup of chaotic processes, which translate the system into a state of dynamic chaos.

Wherein the system produces a new random information, the rate of this process is the higher the greater the degree of randomness. In this important to keep track of Poincare return time as the main indicators and characteristics, show the dynamics of the system in time repeatability. Great interests are the processes of mixing multidimensional heterogeneous system. In this context, attention is drawn to the processes of mixing multidimensional heterogeneous systems. By mixing multidimensional systems may be coherent structures that require analysis and evolution on these structures.

The resonance caused works S.C.Shadden, Francois Lekien, Jerrold E. Marsden, G.Haller, I.M.Ottino, devoted to the study of coherent Lagrange structures, which are ranges of fields of finite-time Lyapunov exponent (FTLE). These ranges can be considered as a finite-time mixed formation. Concept of this work is applicable to flows with random time-dependent and in particular, flows which are defined in a finite-time interval. This problem is further updated in the analyses of nonlinear mixed physical systems, in which examples of Lagrangian coherent structures are stable and unstable manifolds of fixed points and periodic orbits. Along with the mixing process occurs problem of mixed transport flow. To their arises paradigm consideration of analysis and synthesis of structure as the "mixing-transport-control" of nonlinear physical processes.

It should be noted that the asymptotic distribution of Poincare recurrences is exponential for a wide class of mixing system, even if they uniformly hyperbolic.

Some preliminary investigations show that at least for the skew and for the mixing return times spectra also hold for the successive Poincare recurrences. Currently, there is intensification of research into the processes and phenomena, characterized by nonlocality, nonmarcov, hereditarily, fractality, nonhamiltonian. It is also paid great attention to studying the degree of non-locality and the power of long-term memory. Are mathematical methods of one of the modern areas of theoretical physics – *fractional dynamics*. This is especially true when it comes to fractal structure systems. It is important to note that the description of the properties of systems with fractal structure cannot be used Euclidean representation geometry. There is a need to analyze these processes in terms of fractional geometry.

Systems with fractal feature characterized by such effect as memory, complex spatial processes of mixing and self-organization. Thus formed a new scientific field-physic of open systems, in which the combined areas such as synergy, dissipative structures, deterministic chaos, fractional dynamics in the various branches of science. Methods of integral-differential fractional and fractional calculus, a history spanning more than three hundred years, back to the research of prominent mathematicians, such as Leibniz, Liouville, Riemann, Abel, Riesz, Veil.

New opportunities in mathematics and theoretical physics in the open systems when the order of $\alpha$ differential operator $D_x^\alpha$ becomes arbitrary parameter. Here fractional derivative index allows to consider features of open systems. The article uses fundamental research V.Afraimovich, B.Chirikov, G.Zaslavsky, V.Tarasov, A.Anishchenko and others.

In spite of the autonomy chapter, the consideration is in terms of Poincare return time. Important practical problem when working with nonlinear systems is their discrete in the time mapping, which allows to conclude that the nature of the continuous flow.

In addition, the article discusses issues of transient in multidimensional chaotic systems of fractional order and offered nontraditional chaotic and stochastic filters, the base of which is integrative component of the average Poincare return time.

Above actualizes the problem of the research of transients in multidimensional chaotic and stochastic systems of fractional-order. It should be noted there is a reassessment role of chaos in the process of evolution of nonlinear multidimensional systems. Noted that the chaos is necessary for the system output to one of the possible attractors; chaos is at the heart of combining mechanisms simpler structures in complex, and finally can act as a system of behavior change regimes.

It is important to note the features of a collective of multidimensional processes and phenomena in the fractional chaotic systems, in the context of the observed transients.

A big role in the description of the behavior of open systems plays a synergetic view of its evolution as a whole that is in terms of attractors, transition states, stability, bifurcations, of dynamical chaos and other. A basic element of this research is to follow the phase portrait and its changes when you change the synergetic model parameters.

*From the position of mathematics synergetic aspect of the process of evolution is a change in the topological structure of the phase space of an open system.*

Tracking change this structure, as the transition process requires the formation of a generalized criterion that recognizes entering the system in one or another state.

It is known that transient is a process whose parameters vary over time. Bat it should be noted that transition may occur both in domestic and external perturbation. The above describes the structure of the research multidimensional chaotic, stochastic and kinetic fractional-order systems (fig.1). Implementation of such a frame work should be based on some of the main provisions.

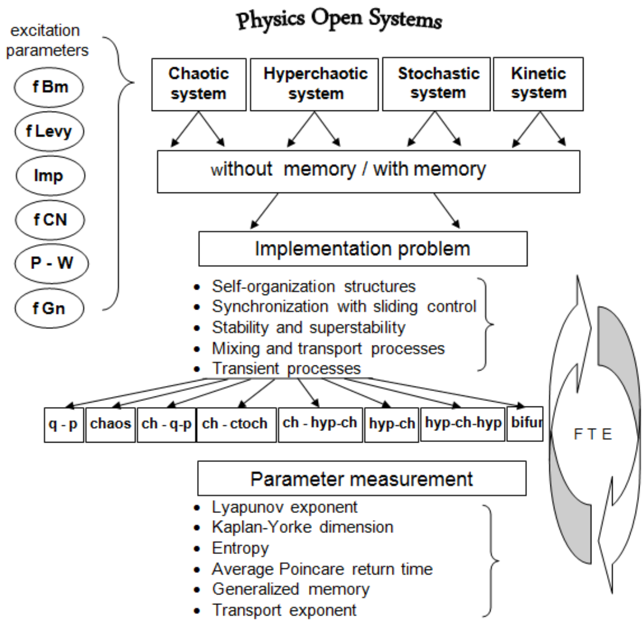


Fig.1. Research of fractional chaotic systems.

Legend:

**fBm** – fractional Brownian motion,
**fLevy** – fractional Levy motion,
**Imp** – impulsive function,
**fCN** – fractional Colored noise,
**P-W** – piecewise function,
**fGn** – fractional Gaussian noise,
**q – p** – quasi-periodic,
**ch – q-p** – chaos – quasi-periodic,
**ch-stoch** – chaos – stochastic,
**ch – hyp-ch** – chaos – hyper-chaos,
**hyp – ch – hyp** - hyper-chaos - chaos - hyper-chaos,
**bifur** – bifurcation.
**F T E** – fractional time evolution

## II. BASIC PROVISIONS

### A. *Recurrence Poincare*

In the 1880-s Henri Poincare had obtained a number of important results, which formed the basis of the modern theory of dynamical systems.

In particular, he noted the complexity of the behavior of the system in the vicinity of the so-called homoclinic trajectory (trajectory tends to a fixed point or a periodic trajectory with both $t \to \infty$, and when $t \to -\infty$) [5], it was published in 1890, as the "recurrence theorem" [5].

This theorem is the basis of the modern of measure preserving transformations, known as the ergodic theory.

Let $X$ open area in $n$- dimensional space with a homeomorphism $T$ for $x$ yourself, keep the volume. With repeated MSE $T$ of any point $x$ generates a sequence $x, Tx, T^2x, \ldots, T^i_{(x)}, \ldots$, called positive semiorbit $x$. When $x \in G$ for any infinity set of positive values $i$, we speak of a recurring point $x$ of an open set $G \subset X$.

On a content level Poincare theorem states for any open set $G \subset X$ points, returning relatively $G$, are all points $G$, except for some set of the first category on measure zero.

Formally takes place

*Theorem 1* [5]. Let $T$- be a measure-preserving transformation of a probability space $(X, \mu)$ and let $A \subset X$ be a measurable set. Then for any natural number $N \subset \boldsymbol{N}$

$$A\left(\left\{x \in A : \left\{T^n(x)\right\}_{n \geq N} \subset (X \setminus A)\right\}\right) = 0$$

where: $T$ – recurrence time; $X$ - arbitrary measurable set; $\mu(\cdot)$ - probability measure; $x$ - parameter of normalized number; $N$ - set of natural numbers.

Short proof of this theorem in [5].

*B. Topological order space*

*Definition 1*. The number is called as a metric order of a compact $A$

$$k = \lim\left(-\ln N_A(\varepsilon)/\ln \varepsilon\right),$$

where $\varepsilon$ - the sphere of radius $\varepsilon$; $N(\varepsilon)$ - number of spheres in a final sub covering of a set.

The lower bound of metric orders for all metrics of a compact $A$ (called by metric dimension) is equal his Lebesgue to dimension.

However it appeared that the metric order entered in [6], coincides with the lower side the fractal dimension of Hausdorff-Bezikovich defined in the terms “box-counting”.

Takes place

*Theorem 2* [6]. For any compact metric space $X$.

$$\dim X = \inf\left\{\underline{\lim}_{\varepsilon \to 0} \frac{\log N_{\varepsilon,d}(X)}{-\log \varepsilon} : d \ \text{ is a metric on } X\right\}$$

where

$$N_{\varepsilon,d}(X) = \min\left\{\begin{matrix}|U| : U \text{ is a finite open} \\ \text{covering of } X \text{ with mesh} \leq \varepsilon\end{matrix}\right\}.$$

From here $(X, d_f)$ - compact fractal metric space with dimension $d_f$.

Here it is important to note that at the description of properties of systems with fractional structure it is impossible to use representation of Euclidean geometry. There is a need of the analysis of these processes for terms of geometry of fractional dimension.

*Remark*. In [7] presented results of communication of a fractional integro-differentiation (in Riman-Liouville or Gryunvalda-Letnikov’s terms) with Koch’s curves.

It is noted that *biunique communication between fractals and fractional operators does not exist*: fractals can be generated and described without use of fractional operations, and defined the fractional operator not necessarily generates defined (unambiguously with it connected) fractal process or fractal variety.

However use of fractional operations allows generating other fractal process (variety) which fractal dimension is connected with an indicator of a fractional integro-differentiation a linear ratio on the basis of the set fractal process (variety).

In [7] fractional integrals of Riman-Liouville are understood as integrals on space of fractional dimension. Thus the indicator of integration is connected with dimension of space an unambiguous ratio.

In this regard consideration of dimension of chaotic systems of a fractional order causes interest. So, in [8] was noted that dimension of such systems can be defined by the sum of fractional exponents $\Sigma$, and $\Sigma < 3$ is the most effective.

Let the chaotic fractional system of Lorenz take place [8]:

$$\frac{d^\alpha}{dt^\alpha} x = \sigma(y - x) \quad \frac{d^\beta}{dt^\beta} y = \rho x - y - xz' \quad \frac{d^\gamma}{dt^\gamma} z = xy - bz$$

Here

$$\sigma = 10,\ \rho = 28,\ b = 8/3;\ 0 < \alpha, \beta, \gamma \leq 1,\ r \geq 1.$$

Then fractional dimension of system of the equations (6) will have an appearance [8]:

$$\alpha + \beta + \gamma = \Sigma.$$

So, for example, for Lorentz’s system with fractional exponents $\alpha = \beta = \gamma = 0.99$, effective dimension $\Sigma = 2.97$.

This, in the context of fractional dynamics let $\tilde{X}$ - any set of nonlinear physical systems, $A^\alpha$ - a subset of a set $\tilde{X}$ of systems of a fractional order with memory $A^\alpha \subset \tilde{X}$. Then a triad $(\tilde{X}, A^\alpha, \Sigma)$– compact fractional metric space with dimension $\Sigma$.

Let’s designate $W \in (X, d_f)$. On the basis [9] and remarks $(\tilde{X}, A^\alpha, \Sigma) \subset W$.

Let’s consider transformation $W$ at an angle of communications of average time of return of Poincare $\langle\tau\rangle$ with $d_f$ and “residual” memory $J(t)$.

Here

$$g : \langle\tau\rangle \Rightarrow d_f l : d_f \Rightarrow J(t) \chi : \langle\tau\rangle \Rightarrow (g, l).$$

From here $U \in (X, \langle\tau\rangle)$ - the generalized compact metric space of Poincare with dimension $\langle\tau\rangle$.

### C. Generalized memory

The analysis and synthesis of multidimensional chaotic system of fractional-order there was a problem with memory estimation.

***Axiomatic***

Let the trajectory of the generalized memory system is of the form [10-12]:

$$Q_{GM} = Q_{\geq 0} \bigcup Q_{\leq 0} .$$

*Definition 2.* $Q_{\geq 0} \subset Q_{GM}$ - it called semi-trajectory to the trajectory $Q_{GM}$, if each $t > 0$ it the inclusion

$$Q_{\geq 0} \subset O_{\varepsilon}^{mem} \bigcup_{j=0}^{j-1} \left( \left[ t_j , t_{j+1} \right], j \right).$$

Here $Q_{\geq 0}\left[t_j , t_{j+1}\right]$ - segment of the semi-trajectory memory of responsible values $t \subset \left[t_j , t_{j+1}\right]$ and $O_{\varepsilon}^{mem}$ - $\varepsilon$ - neighborhood of the corresponding set.

*Definition 3.* $Q_{\leq 0} \subset Q_{GM}$ it called semi-trajectory to the trajectory $Q_{GM}$, if each $t < 0$ it the inclusion

$$Q_{\leq 0} \subset O_{\varepsilon}^{lm} \bigcup_{k=1}^{k} \left( \left[ s_k , s_{k-1} \right], -k+1 \right).$$

Here $Q_{\leq 0}\left[s_k , s_{k-1}\right]$ - segment of the semi-trajectory "loss memory" of responsible values $t \in \left[s_k , s_{k-1}\right]$, and $O_{\varepsilon}^{lm}$ - $\varepsilon$ - neighborhood of the corresponding set.

*Definition 4.* Semi-trajectory $Q_{\geq 0} \subset Q$ recurrent if for every $t > 0$ inclusion

$$Q_{\geq 0} \subset \mathrm{O}_{\varepsilon} \bigcup_{j=0}^{j-1} \left( \left[ t_j , t_{j+1} \right], j \right).$$

Here $Q_{\geq 0}\left[t_j , t_{j+1}\right]$ - segment of the semi-trajectory recurrent of responsible values $t \subset \left[t_j , t_{j+1}\right]$.

*Definition 5.* Semi-trajectory $Q_{\leq 0} \subset Q$ not recurrent if for every $t < 0$ it the inclusion

$$Q_{\leq 0} \subset \mathrm{O}_{\varepsilon} \bigcup_{k=1}^{k} \left( \left[ s_k , s_{k-1} \right], \ k+1 \right).$$

*Remark*. It is known the average return time of Poincare is determined by the fractal dimensions the trajectories of generalized memory, GM. Hence the memory loss will be determined by the difference between global and local fractal dimensions, which means that the recurrence and no-recurrence semi-trajectories respectively.

### D. Formation of loss memory

It is a known that during the Poincare recurrence characterizes as a "residual", and the real memory of the fractional-order system. Hence the equivalence between the spectrums of the Poincare returns time and distribution of generalized memory.

Los memories are determined by the difference between the global and the local fractal dimensions, which means, respectively, reversible and irreversible processes. Loss of information numerically defines the entropy.

## III. Conclusion

In the article the complexes of problems at an angle of analysis of the Poincare return times. It noted the use of multi-dimensional fractional-order chaotic systems in the implementation of the task synchronization and control. Briefly presented axiomatic basic research positions and structure.

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