Generalized Rough Ideals In Semigroups

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***Abstract*—In this paper, we have introduced Generalized Rough Left [Right, two-sided, Bi-, Interior] Ideal in a semigroup, which is an extended notion of a Rough Left [Right, two-sided, Bi-, Interior] Ideal in a semigroup, and describe few of the properties of these type of ideals. We also explain the relations between the upper [lower] approximation and upper [lower] generalized rough ideals along with their homomorphism images.**

Keywords—Generalized Rough Sets; Generalized Rough Ideals; Generalized Rough Bi-ideal Ideals; Generalized Rough Interior Ideals; Homomorphism.

I. INTRODUCTION

Rough set theory was introduced by Pawalak [10] in 1991 to deal granularity and ambiguity in the information system. Biswas and Nand [1] introduce rough subgroups. In [8] Kuroki discussed new properties of the lower and upper approximations corrsponding to the normal subgroups and the fuzzy normal subgroups. In [7] he introduce rough ideals in semigroups. In [14] William Zhu defined generalized rough set based on binary relation.

In this paper, we have define generalized rough ideals in semigroups based on binary relation. The realistic requirements in classification and model construction with incomplete construction [32] motivated the researchers to introduce the idea of rough sets. The usefulness and versatility of the rough set models is very visibly applicable in a variety of problems [21,25]. An equivalence class can be expressed by the description that when two distinct objects are perceived as the same or being indistinguishable.

II. A SURVEY OF SEMIGROUPS

In this section we will give some basics of semigroup. A semigroup is a non-empty set S together with an associative binary operation " $\cdot$ " .Consider two subset X and Y of a semigroup S, then the product XY is defined as:

$$XY := \{xy \mid x \in A, y \in B\}.$$

For any subset X of S if $xy \in X$ for all $x, y \in X$ then X is said to be a subsemigroup of S. In addition if $XSX \subseteq X$, then the subsemigroup X of S is said to be a bi-ideal of S. A left (right) ideal of a semigroup S is a subset X of S such that $SX \subseteq X (XS \subseteq X)$. A two sided ideal is an ideal which is both a left and right a ideal of S. A nonempty subset X of S is an interior ideal of S if $SXS \subseteq X$.

III. GENERALIZED ROUGH SETS

A binary relation θ on a semigroup S is a subset of $S \times S$. Let θ be a binary relation on S. Define the lower $(\theta_-(X))$ and upper $(\theta_+(X))$ approximation operations of a subset X of S as follows:

θ_-, $\theta_+ : P(S) \to P(S)$ are such that

$$\theta_-(X) = \{x \in S : \forall y, x\theta y \Rightarrow y \in X\} = \{x \in S : \theta N(x) \subseteq X\}$$

$$\theta_+(X) = \{x \in S : \exists y \in X, \text{ Such that } x\theta y = \{x \in S : \theta N(x) \cap X \neq \phi\}.$$

where $\theta N(x) = \{y \in S : x\theta y\}$ and $P(S)$ is the collection of all subsets of S.

The collection $\theta(X) = \{\theta_-(X), \theta_+(X)\}$ is called a Generalized rough set w.r.t θ if $\theta_-(X) \neq \theta_+(X)$. A relation θ on S is said to be reflexive if $a\theta a$ for all $a \in S$. Recall that a binary relation θ in a semigroup S is called compatible if $a\theta b \Rightarrow as\theta bs$ and $sa\theta sb$ for all $s \in S$.

3.1 Theorem

Let θ and Φ be reflexive and compatible relations on a semigroup S and X, Y be non-empty subsets of S. Then the following conditions satisfied:

(i) $\theta_-(X) \subseteq X \subseteq \theta_+(X)$;

(ii) $\theta_+(X \cup Y) = \theta_+(X) \cup \theta_+(Y)$;

(iii) $\theta_-(X \cap Y) = \theta_-(X) \cap \theta_-(Y)$;

(iv) $X \subseteq Y$ implies $\theta_-(X) \subseteq \theta_-(Y)$;

(v) $X \subseteq Y$ implies $\theta_+(X) \subseteq \theta_+(Y)$;

(vi) $\theta_{-}(X \cup Y) \supseteq \theta_{-}(X) \cup \theta_{-}(Y)$;
(vii) $\theta_{+}(X \cap Y) \subseteq \theta_{+}(X) \cap \theta_{+}(Y)$;
(viii) $\theta \subseteq \Phi$ implies $\theta_{-}(X) \supseteq \Phi_{-}(X)$;
(ix) $\theta \subseteq \Phi$ implies $\theta_{+}(X) \subseteq \Phi_{+}(X)$.

Proof. (i) Let $x \in \theta_{-}(X)$, as θ is reflexive so $x\theta x$ implies $x \in X$, which implies $\theta_{-}(X) \subseteq X$. For any $x \in X$, $x\theta x$ which gives $x \in \theta_{+}(X)$. Thus $\theta_{-}(X) \subseteq X \subseteq \theta_{+}(X)$.

(ii) Let $x \in \theta_{+}(X \cup Y)$. then $\theta N(x) \cap (X \cup Y) \neq \phi$
$\Leftrightarrow (\theta N(x) \cap X) \cup (\theta N(x) \cap Y) \neq \phi$
$\Leftrightarrow \theta N(x) \cap X \neq \phi$ or $\theta N(x) \cap Y \neq \phi$
$\Leftrightarrow x \in \theta_{+}(X)$ or $x \in \theta_{+}(Y)$
$\Leftrightarrow x \in \theta_{+}(X) \cup \theta_{+}(Y)$.
Thus $\theta_{+}(X \cup Y) = \theta_{+}(X) \cup \theta_{+}(Y)$.

(iii) Let $x \in \theta_{-}(X \cap Y)$ then $\theta N(x) \subseteq X \cap Y$
$\Rightarrow \theta N(x) \subseteq X$ and $\theta N(x) \subseteq Y$
$\Leftrightarrow x \in \theta_{-}(X)$ and $x \in \theta_{-}(Y)$
$\Leftrightarrow x \in \theta_{-}(X) \cap \theta_{-}(Y)$.
Thus $\theta_{-}(X \cap Y) = \theta_{-}(X) \cap \theta_{-}(Y)$.

(iv) Since $X \subseteq Y$, so $X \cap Y = X$. It follows from(iii) that $\theta_{-}(X) = \theta_{-}(X \cap Y) = \theta_{-}(X) \cap \theta_{-}(Y)$.
This gives $\theta_{-}(X) \subseteq \theta_{-}(Y)$.

(v) Given $X \subseteq Y$, so $X \cup Y = Y$. It follows from (ii) that $\theta_{+}(Y) =$ $\theta_{+}(X \cup Y) = \theta_{+}(X) \cup \theta_{+}(Y)$.
so $\theta_{+}(X) \subseteq \theta_{+}(Y)$.

(vi) Given $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$, it follows from (vi) that
$\theta_{-}(X) \subseteq \theta_{-}(X \cup Y)$ and $\theta_{-}(Y) \subseteq \theta_{-}(X \cup Y)$
$\Rightarrow \theta_{-}(X) \cup \theta_{-}(Y) \subseteq \theta_{-}(X \cup Y)$.

(vii) Since $X \cap Y \subseteq X$ and $X \cap Y \subseteq y$, it follows from (v) that
$\theta_{+}(X \cap Y) \subseteq \theta_{+}(X)$ and $\theta_{+}(X \cap Y) \subseteq \theta_{+}(Y)$
$\Rightarrow \theta_{+}(X \cap Y) \subseteq \theta_{+}(X) \cap \theta_{+}(Y)$.
(viii) Since $\theta \subseteq \Phi$, so for all $x \in \Phi_{-}(X)$, we have
$\theta N(x) \subseteq \Phi N(x) \subseteq X$
$\Rightarrow \theta N(x) \subseteq X$
$\Rightarrow x \in \theta_{-}(X)$
so $\Phi_{-}(X) \subseteq \theta_{-}(X)$.

(ix) Let $x \in \theta_{+}(X)$, then $\theta N(x) \cap X \neq \phi$, so their exist $a \in \theta N(x) \cap X$. we have
$\theta N(x) \subseteq \Phi N(x)$ (since $\theta \subseteq \Phi$).
So $a \in \Phi N(a) \cap X$ implies $a \in \Phi_{+}(A)$. This implies
$\theta_{+}(X) \subseteq \Phi_{+}(X)$.

3.2 Theorem

Given θ a reflexive and compatible relation on a semigroup S . For any nonempty subsets X and Y of S . $\theta_{+}(X)\theta_{+}(Y) \subseteq \theta_{+}(XY)$.

Proof. Let $z \in \theta_{+}(X)\theta_{+}(Y)$ then $z = xy$ for some $x \in \theta_{+}(X)$ and $y \in \theta_{+}(Y)$. By definition there exists $a, b \in S$ such that $a \in X$ and $x\theta a$; $b \in Y$ and $y\theta b$.
Given θ a compatible relation on S, so $xy\theta ab$ .since $xy \in XY$, so $z = xy \in \theta_{+}(XY)$
$\Rightarrow \theta_{+}(X)\theta_{+}(Y) \subseteq \theta_{+}(XY)$.

3.3 Definition

Given a compatible relation θ on S then $\theta N(x)\theta N(y) \subseteq \theta N(xy)$ for all $x, y \in S$. If in addition $\theta N(x)\theta N(y) = \theta N(xy)$, then θ is said to be complete compatible relation.

3.4 Theorem

Given θ a reflexive and complete compatible relation on a semigroup S and X , Y are non-empty subsets of S . Then $\theta_{-}(X)\theta_{-}(Y) \subseteq \theta_{-}(XY)$.

Proof. Let z $z \in \theta_{-}(X)\theta_{-}(Y)$ then $z = xy$ for some $x \in \theta_{-}(X)$ and So
$\theta N(x) \subseteq X$ and $\theta N(y) \subseteq Y$.
$\Rightarrow \theta N(xy) = \theta N(x)\theta N(y) \subseteq XY$;
which implies that $xy \in \theta_{-}(XY)$.
Hence $\theta_{-}(X)\theta_{-}(Y) \subseteq \theta_{-}(XY)$.

3.5 Theorem

Let θ and Φ be reflexive and compatible relations on a semigroup S . If X is a non-empty subset of S then $(\theta \cap \Phi)_{+}(X) \subseteq \theta_{+}(X)\Phi_{+}(X)$.

Proof. Notice that $\theta \cap \Phi$ is also a reflexive and compatible relation on semigroup S . Let $z \in (\theta \cap \Phi)_{+}(X)$. Then $(\theta \cap \Phi)N(z) \cap X \neq \Phi$.
Let $x \in (\theta \cap \Phi)N(z) \cap X$, then $x \in (\theta \cap \Phi)N(z)$ and $x \in X$ .Now
$(z, x) \in (\theta \cap \Phi) \Rightarrow (z, x) \in \theta$ and $(z, x) \in \Phi$.
This implies that $x \in \theta N(z)$ and $x \in \Phi N(z)$. Now $x \in X$, so
$x \in \theta N(z) \cap X$ and $x \in \Phi N(z) \cap X$
$\Rightarrow z \in \theta_{+}(X)$ and $z \in \Phi_{+}(X)$,
and so $z \in \theta_{+}(X) \cap \Phi_{+}(X)$.
Hence $(\theta \cap \Phi)_{+}(X) \subseteq \theta_{+}(X)\Phi_{+}(X)$.

3.6 Theorem

Let θ and Φ be reflexive and compatible relations on a semigroup S. If X a non-empty subset of S, then $(\theta \cap \Phi)_-(X) = \theta_-(X) \cap \Phi_-(X)$.

Proof. Let $z \in (\theta \cap \Phi)_-(X)$ then $(\theta \cap \Phi)N(z) \subseteq X$

$\Leftrightarrow \theta N(z) \subseteq X$ and $\Phi N(z) \subseteq X$

$\Leftrightarrow z \in \theta_-(X)$ and $z \in \Phi_-(X)$

$\Leftrightarrow z \in \theta_-(X) \cap \Phi_-(X)$

Thus $(\theta \cap \Phi)_-(X) = \theta_-(X) \cap \Phi_-(X)$.

IV. Generalized Rough Ideals

4.1 Definition

Let θ be a binary relation on a semigroup S. If the upper approximation $\theta_+(X)$ is a subsemigroup of S for any nonempty subset X of S then X is said to be generalized upper rough subsemigroup of S. The set X is said to be generalized upper left (right, two-sided) ideal of S if $\theta_+(X)$ is a left (right, two-sided) ideal of S.

4.2 Theorem

Given θ a reflexive and compatible relation on a semigroup S. Then

(i) If X is a subsemigroup of S, then X is generalized upper rough subsemigroup of S.

(ii) If X is a left (right, two sided) ideal of S, then X is generalized upper rough left (right, two-sided) ideal of S.

Proof. (i) Given X a subsemigroup of S. It follows from Theorm 3.1(i),

$$\phi \neq X \subseteq \theta_+(X).$$

Now by Theorm 3.2

$$\theta_+(X)\theta_+(X) \subseteq \theta_+(XX) \subseteq \theta_+(X).$$

This gives $\theta_+(X)$ a subsemigroup of S and X a generalized upper rough subsemigroup of S

(ii) Given X as a left ideal of semigroup S. As we know that $\theta_+(S) = S$. It follows from Theorm 3.2 that

$$S\theta_+(X) = \theta_+(S)\theta_+(X) \subseteq \theta_+(SX) \subseteq \theta_+(X)$$

Hence $\theta_+(X)$ is a left ideal and so X is a generalized upper rough left ideal of S. The rest of the cases are follows in a similar way.

In the next example we will show that the converse of Theorem 4.2 does not hold in general.

Example 1 Given $S = \{a,b,c,d\}$ a semigroup with the multiplication table as follows:

*	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	c	c	c	c
d	d	c	b	a

Let θ be a compatible and reflexive relation on S such that $\theta N(a) = \{a\}$, $\theta N(b) = \{b,c\}$, $\theta N(c) = \{b,c\}$, $\theta N(d) = \{d\}$. Then $X = \{b\} \subseteq S$, $\theta_+(X) = \{b,c\}$ and $\{b,c\}S = S\{b,c\} = \{b,c\}$. This means that the set $\{b,c\}$ is a two sided ideal of S. It is clear that $X = \{b\}$ is not an ideal of S.

4.3 Definition

Let θ be a reflexive and compatible relation on a semigroup S. A non-empty subset X of S is said to be generalized lower rough subsemigroup of S if $\theta_-(X)$ is a subsemigroup of S. The set X is said to be generalized lower left (right, two-sided) ideal of S if the lower approximation of $X(\theta_-(X))$ is a left (right, two sided) ideal of S.

4.4 Theorem

Given θ a reflexive and complete compatible relation on S then

(i) $\theta_-(X)$, if it is non-empty, is a subsemigroup of S provided X is a subsemigroup of S.

(ii) $\theta_-(X)$, if it is non-empty, is a left (right, two-sided) ideal of S provided X is a left (right, two sided) ideal of S.

Proof. (i) Given X a subsemigroup of S. It follows from Theorm 3.4and Theorm 3.1(iv)

$$\theta_-(X)\theta_-(X) \subseteq \theta_-(XX) \subseteq \theta_-(X)$$

So $\theta_-(X)$ is a subsemigroup of S.

(ii) Given X be a left ideal of S. It follows from Theorm 3.4

$$S\theta_-(X) = \theta_-(S)\theta_-(X) \subseteq \theta_-(SX) \subseteq \theta_-(X).$$

Hence $\theta_-(X)$ is a left ideal of S. The remaining cases can be done in a similar way

4.5 Theorem

Let θ be reflexive and compatible relation on a semigroup S, then for any right ideal X and left ideal Y of S

$$\theta_+(XY) \subseteq \theta_+(X) \cap \theta_+(Y).$$

Proof. Given X a right ideal and Y a left ideal of S, so by definition $XY \subseteq XS \subseteq X$ and $XY \subseteq SY \subseteq Y$ which implies $XY \subseteq X \cap Y$. It follows from Theorm 3.1[(v)(vii)] that

$$\theta_+(XY) \subseteq \theta_+(X \cap Y) \subseteq \theta_+(X) \cap \theta_+(Y)$$

as required.

4.6 Theorem

Given θ a reflexive and compatible relation on a semigroup S and X is a right and Y is a left ideal of S, then $\theta_-(XY) \subseteq \theta_-(X) \cap \theta_-(Y)$.

Proof. As $XY \subseteq X \cap Y$. Then by Theorm 3.1(iv)

$$\theta_-(XY) \subseteq \theta_-(X) \cap \theta_-(Y).$$

as required.

V. Generalized Rough Interior Ideals

5.1 Definition

Let X be a non-empty subset of S and θ a binary relation on S. Then X is said to be generalized lower (upper) rough interior ideal of S if $\theta_-(X)(\theta_+(X))$ is an interior ideal of S.

5.2 Theorem

Given θ a reflexive and compatible relation on a semigroup S. If X is an interior ideal of S then X is a generalized upper rough interior ideal of S.

Proof. As X is an interior ideal of a semigroup S, so $SXS \subseteq X$. It follows from Theorm 3.2 that

$$S\theta_+(X)S = \theta_+(S)\theta_+(X)\theta_+(S) \subseteq \theta_+(SXS) \subseteq \theta_+(X),$$

so $\theta_+(X)$ is an interior ideal of S.

5.3 Theorem

Given θ be a reflexive and complete compatible relation on a semigroup S. If X is an interior ideal of S. Then $\theta_-(X)$ is, if it is a non-empty is an interior ideal of S.

Proof. Given X an interior ideal of S. Then by Theorm 3.1(iv) and Theorem 3.4

$$S\theta_-(X)S = \theta_-(S)\theta_-(X)\theta_-(S) \subseteq \theta_-(SXS) \subseteq \theta_-(X)$$

so $\theta_-(X)$ is an interior ideal of S.

5.4 Definition

The set X is said to be generalized rough interior ideal of S if it is a lower and upper generalized rough interior ideal of S.

5.5 Definition

Let θ and ϕ be binary relation on a semigroup S. Then the product $\theta \circ \phi$ of θ and ϕ defined as follows:

$\theta \circ \phi = \{(x,y) \in S \times S : (x,a) \in \theta \text{ and} (a,y) \in \phi \text{ for some } a \in S\}.$

5.6 Lemma

Let θ and ϕ be compatible relation on a semigroup S. Then $\theta \circ \phi$ is also a compatible relation on S.

Proof. Let $(x,y) \in \theta \circ \phi$ and $c \in S$. Then $(x,a) \in \theta$ and $(a,y) \in \phi$ for some $a \in S$. Now $(cx,ca) \in \theta$ and $(ca,cy) \in \phi \Rightarrow (cx,cy) \in \theta \circ \phi$.

Similarly $(xc,yc) \in \theta \circ \phi$. Thus $\theta \circ \phi$ is a compatible relation on S.

5.7 Theorem

Given θ and ϕ compatible relation on a semigroup S For a subsemigroup, X of S

$$\theta_+(X)\phi_+(X) \subseteq (\theta \circ \phi)_+(X).$$

Proof. Let z be any element of $\theta_+(X)\phi_+(X)$ then $z = xy$ where $x \in \theta_+(X)$ and $y \in \phi_+(X)$,

$$\Rightarrow a \in \theta N(x) \cap X \text{and} b \in \phi N(y) \cap X \text{forsome} a,b \in S$$

As $a,b \in X$ implies $ab \in X$, since X is a subsemigroup of S. Now $(a,x) \in \theta$ and $(b,y) \in \phi$

$\Rightarrow (ab,xb) \in \theta \text{and} (xb,xy) \in \phi$

(since θ and ϕ are compatible relations)

so

$$(xb,xy) \in \theta \circ \phi,$$

implies

$$ab \in (\theta \circ \phi)N(xy)$$

so

$$ab \in (\theta \circ \phi)N(xy) \cap X,$$

implies

$$z = xy \in (\theta \circ \phi)_+(X).$$

Hence

$$\theta_+(X)\phi_+(X) \subseteq (\theta \circ \phi)_+(X).$$

VI. Generalized Rough Bi-Ideals

6.1 Definition

Let X be a non-empty subset of S, with θ a compatible relation on a semigroup S. If $\theta_-(X)$ $(\theta_+(X))$ is a bi-ideal of S, then X is said to be generalized upper (lower) rough bi-ideal of S.

6.2 Theorem

Given θ a reflexive and compatible relation on a semigroup S. Then every bi-ideal X is a generalized upper rough bi-ideal of S.

Proof. Given X a bi-ideal of S. It follows from Theorem 4.2(i) that $\theta_+(X)$ is a subsemigroup of S. By Theorem 3.1(v) and Theorem 3.2

$$\theta_+(X)S\theta_+(X) = \theta_+(X)\theta_+(S)\theta_+(X) \subseteq \theta_+(XSX) \subseteq \theta_+(X)$$

so $\theta_+(X)$ is a bi-ideal of S.

6.3 Theorem

Given θ a reflexive and compatible relation on a semigroup S. Then every bi-ideal X is a generalized lower rough bi-ideal of S.

Proof. Given X a bi-ideal of S. It follows fom Theorem 4.4(i) $\theta_-(X)$ is a subsemigroup of S. By Theorem 3.1(iv) and Theorem 3.4

$$\theta_-(X)S\theta_-(X) = \theta_-(X)\theta_-(S)\theta_-(X) \subseteq \theta_-(XSX) \subseteq \theta_-(X),$$

so X is generalized lower rough bi-ideal of S.

VII. Problems of Homomorphisms

A mapping α from a semigroup S to a semigroup T is said to be homomorphism if $\alpha(ab)=\alpha(a)\alpha(b)$ for all $a,b\in S$.

7.1 Lemma

Let $\alpha: S\to T$ be a surject homomorphism and θ_2 be a compatible relation on T. Consider

$$\theta_1=\{(s_1,s_2)\in S\times S \mid \alpha(s_1),\alpha(s_2)\in\theta_2\}.$$

Then the following holds:

(i) θ_1 is a compatible relation on S.

(ii) θ_1 is complete provided that θ_2 is complete and α is single valued.

(iii) $\alpha(\theta_{1+}(X))=\theta_{2+}(\alpha(X))$ for $X\subseteq S$.

(iv) $\alpha(\theta_{1-}(X))\subseteq\theta_{2-}(\alpha(X))$, and if α is single valued, then

$$\alpha(\theta_{1-}(X))=\theta_{2-}(\alpha(X)).$$

Proof. (i) Let $(a,b)\in\theta_1$ .then $(\alpha(a),\alpha(b))\in\theta_2$. This implies

$$(t\alpha(a),t\alpha(b))\in\theta_2 \text{ forevery } t\in T (\text{since } \theta_2 \text{iscompatible}).$$

As α is surjective so for each $t\in T$ there is $s\in S$ such that $\alpha(s)=t$

$$(\alpha(s)\alpha(a),\alpha(s)\alpha(b))\in\theta_2$$

$$=(\alpha(sa),\alpha(sb))\in\theta_2,$$

which implies that $(sa,sb)\in\theta_1$. Similarly $(as,bs)\in\theta_1$. Thus θ_1 is compatible relation on semigroup S

(ii) Let $x\in\theta_1 N(ab)$. This implies

$$\alpha(x)\in\theta_2 N(\alpha(ab))=\theta_2 N(\alpha(a)\alpha(b))=\theta_2 N(\alpha(a))\theta_2 N(\alpha(b))$$

since α is surjective, there exists $x_1,x_2\in S$ such that

$$\alpha(x_1)\in\theta_2 N(\alpha(a)),\ \alpha(x_2)\in\theta_2 N(\alpha(b)),$$

and

$$\alpha(x)=\theta(x_1)\ \theta(x_2)=\alpha(x_1x_2)$$

since α is single valued, by definition of θ_1

$$x_1\in\theta_1 N(a), x_2\in\theta_1 N(b) \text{ and } x=x_1x_2.$$

Thus

$$x\in\theta_1 N(a)\theta_1 N(b).$$

This gives

$$\theta_1 N(ab)\subseteq\theta_1 N(a)\theta_1 N(b).$$

On the other hand,

$$\theta_1 N(a)\theta_1 N(b)\subseteq\theta_1 N(ab).$$

Thus θ_1 is complete.

(iii) Let $b\in\alpha(\theta_{1+}(X))$, then there exists $a\in\theta_{1+}(X)$ such that $\alpha(a)=b$, so

$$\theta_1 N(a)\cap X\neq\phi,$$

so there exists $x\in\theta_1 N(a)\cap X$. Now $\alpha(x)\in\alpha(X)$, and by definition of θ_1, $\alpha(x)\in\theta_2 N(\alpha(a))$, so

$$\theta_2 N(\alpha(a))\cap\alpha(X)\neq\phi,$$

this implies that $b=\alpha(a)\in\theta_{2+}(\alpha(X))$, so that we get

$$\alpha(\theta_{1+}(X))\subseteq\theta_{2+}(\alpha(X)).$$

Conversely, let $b\in\theta_{2+}(\alpha(X))$, then there exists $a\in S$ such that $\alpha(a)=b$. So,

$$\theta_2 N(b)\cap\theta(X)\neq\phi$$

$$\Rightarrow\theta_2 N(\alpha(a))\cap\alpha(X)\neq\phi$$

so that there exists $x\in X$ such that $\alpha(x)\in\alpha(X)$ and $\alpha(x)\in\theta_2 N(\alpha(a))$. Now by definition of θ_1, $x\in\theta_1 N(a)$. Thus

$$\theta_1 N(a)\cap X\neq\phi$$

$$\Rightarrow a\in\theta_{1+}(X)\Rightarrow b=\alpha(x)\in\alpha(\theta_{1+}(X)).$$

this implies that

$$\theta_{2+}(\alpha(X))\subseteq\alpha(\theta_{1+}(X))$$

Thus

$$\alpha(\theta_{1+}(X))=\theta_{2+}(\alpha(X))$$

(iv) Let $b\in\alpha(\theta_{1-}(X))$, then there exists $a\in\theta_{1-}(X)$ such that $\alpha(a)=b$, so we have

$$\theta_1 N(a)\subseteq X.$$

Let $b'\in\theta_2 N(b)$ then there exists an element $a'\in S$ such that

$\alpha(a')=b$ and $\alpha(a')\in\theta_2 N(\alpha(a))$.

Hence

$$a'\in\theta_1 N(a)\subseteq X$$

and so $b'=\alpha(a')\in\alpha(X)$. Thus

$$\theta_2 N(b)\subseteq\alpha(X)$$

which yields that $b\in\theta_{2+}(\alpha(X))$, so we have

$$\alpha(\theta_{1-}(X))\subseteq\theta_{2-}(\alpha(X)).$$

Suppose α is single valued and $b\in\theta_{2-}(\alpha(X))$, then there exists $a\in S$ such that $\alpha(a)=b$ and

$$\theta_2 N(\alpha(x))\subseteq\alpha(X).$$

Let $a'\in\theta_1 N(a)$, then

$$\alpha(a')\in\theta_2 N(\alpha(a))\subseteq\alpha(X)$$

and so $a'\in X$. Hence

$$\theta_1 N(a)\subseteq X,$$

which yields $a\in\theta_{1-}(X)$. and

$$b=\alpha(a)\in\alpha(\theta_{1-}(X))$$

so

$$\theta_{2-}(\alpha(X))\subseteq\alpha(\theta_{1-}(X))$$

Thus

$$\alpha(\theta_{1-}(X))=\theta_{2-}(\alpha(X)$$

This completes the proof.

7.2 Theorem

Let α be a surjective homomorphism of semigroup S to a semigroup T and θ_2 be reflexive and compatible relation on a T. Let $X \subseteq S$. Consider

$$\theta_1 = \{(a,b) \in S \times S : (\alpha(a), \alpha(b) \in \theta_2)$$

then $\theta_{1+}(X)$ is an ideal of S if and only if $\theta_{2+}(\alpha(X))$ is an ideal of T

Proof. Let $\theta_{1+}(X)$ is an ideal of S then $S\theta_{1+}(X) \subseteq \theta_{1+}(X)$. This implies

$$\alpha(S\theta_{1+}(X)) \subseteq \alpha(\theta_{1+}(X)).$$

Since α is surjective homomorphism, by above Lemma 5.1 we have

$$T\theta_{2+}(\alpha(X)) \subseteq \theta_{2+}(\alpha(X)).$$

Similarly

$$\theta_{2+}(\alpha(X))T \subseteq \theta_{2+}(\alpha(X))$$

so $\theta_{2+}(\alpha(X))$ is an ideal of T Conversely, suppose that $\theta_{2+}(\alpha(X))$ is an ideal of T it follows from Lemma 5.1 we have

$$\alpha(xx') \in \alpha(S\theta_{1+}(X)) = T(\alpha(\theta_{1+}(X))$$

$$= T\theta_{2+}(\alpha(X)) \subseteq \theta_{2+}(\alpha(X)) = \alpha(\theta_{1+}(X)$$

for any $x \in S$, $x' \in \theta_{1+}(X)$. Thus there exists an element $x_\circ \in \theta_{1+}(X)$, such that $\alpha(xx') = \alpha(x_\circ)$. So we have

$$\theta_1 N(x_\circ) \cap X \neq \phi$$

and $xx' \in \theta_1 N(x_\circ)$. This implies that

$$\theta_1 N(xx') \cap X \neq \phi$$

and so $xx' \in \theta_{1+}(X)$. Therefore

$$S\theta_{1+}(X) \subseteq \theta_{1+}(X).$$

Similarly,

$$\theta_{1+}(X)S \subseteq \theta_{1+}(X)$$

So $\theta_{1+}(X)$ is an ideal of S.

7.3 Theorem

Let $\alpha : S \to T$ be an isomorphism and θ_2 be complete compatible and reflexive relation on T and $X \subseteq S$. Consider

$$\theta_1 = \{(a,b) \in S \times S : (\alpha(a), \alpha(b) \in \theta_2)$$

Then $\theta_{1-}(X)$ is an ideal of S if and only if $\theta_{2-}(\alpha(X))$ is an ideal of T

Proof. It follows from Lemma 7.1 that $\alpha(\theta_{1-}(X)) = \theta_{2-}(\alpha(X))$. The similar argument as in Theorem 7.2 prove the result.

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