Journal of Multidisciplinary Engineering Science and Technology (JMEST)
ISSN: 2458-9403
Vol. 3 Issue 8, August - 2016

Mathematical Models For Population Projection In Albania

Alfred DACI
Department of Mathematics, Mathematical and Physical Engineering Faculty, Polytechnic University of Tirana
Tirana, Albania
alfreddaci@gmail.com

***Abstract* — This paper studies the logistic model, Malthus model, as well as its modified version suggested by Verhulst. First of all, it is presented the way the model has been built. Then it is solved analytically using mathematical techniques of differentiation and integration. Furthermore, there have done tests with practical data. There are given comparisons of predicted data and actual data of population growth for the models using charts, tables and figures. These models are applied to predict the growth population of Albania.**

***Keywords— population growth model, logistic growth model,* carrying capacity, Thomas R. Malthus's model**

I. Introduction

Mathematical applications and models are widely used in everyday life. Being able to forecast population in the future, and even being able to answer some interesting questions about population in the past, depends on developing accurate mathematical models of population growth. The aim of this paper is modeling in the population field. There will be discussed basic concepts and mathematical models of the dynamics of the population, including exponential and logistic growth. A *population* is "a group of plants, animals, or other organisms, all of the same species, that live together and reproduce." (Gotelli, 2) This paper will make it possible to see how the theoretical part of mathematics has several real-world applications. Using only two data points, an exponential growth population model is developed and used both to project future population and compare to past population data.

II. The Models Of Population (Logistic Equation)

How can the size of population of change? There are four factors which affect the population. The natality in a population increases the size of its, while the mortality decrease the size of population. Also, there are new entries in the population (migration) or population getaway (immigration). The above statements are presented in a simple equation [1]

$$\Delta P = B - D + I - E \qquad (1)$$

Where : ΔP = the change in the size of population in time t.

B = birth rate
D = death rate
I = migration rate
E = immigration rate

If the population is supposed to be 'closed' (habitat), then we do not have a mobility of the population, so that

$$\Delta P = B - D \qquad (2)$$

The simpliest model is when the growth of populationis considered to be densitity independent. Density independence means the birth and death rate are not affected by the size of population.
Thus, birth and death rate is in commensurate with population

$$P' = rP \qquad (3)$$

(this model was introduced by the English mathematician Thomas R. Malthus, in 1798, for the growth of populations). In this model, the velocity of the change per capita $r = P'/P$ is constant, positive when P increases, negative when P decreases.

This ia an equations with separable variables, let's separate the variables and then integrate

$$\frac{dP}{dt} = rP \Leftrightarrow \frac{dP}{P} = rdt$$

$$\int \frac{dP}{P} = \int rdt \Rightarrow \ln|P| = rt + c \Rightarrow P = e^{rt+c}$$

$$\Rightarrow P = e^{rt}C$$

at time $t = 0$ we have $C = P_0$, from which we have:

$$P(t) = P_0 e^{r\,t} \qquad (4)$$

where P_0 is the population at time $t=0$. If $r>0$, the exponential population growth can not continue forever. In an exponential plot, we can see that as $t\to\infty$, $P(t)\to\infty$.

In a real-life situation, it can not be possible for the population to grow in such dimension, because the population would exceed the weight of the earth.
What exactly limits the growth of population?
Food supply, territoriality, cannibalism, competition, predation, parasites, and diseases can all affect the growth.
In order to model the effects of overcrowding and limited sources of living, biologists who study populations and demographers often accept the following hypothesis:
The rate of growth per capita i.e. the rate P'/P decreases rapidly when P becomes bigger, as shown in Figure 1.

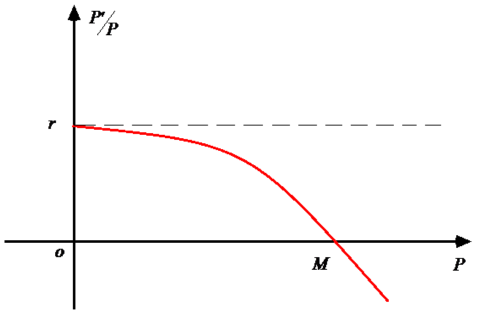

Fig. 1

For small P, P'/P nears the previous value r. However, for a population larger than the specific carrying capacity M, the growth rate becomes negative, which means the death rate is higher than the birth one.

A convincing mathematical manner to include these ideas is accepting that the rhythm of growth

P'/P decreases linearily with the growth of P, which means that the graph P'/P is a line. (Figure 2)
While writing the equation of the line which goes through the points $(0;r)$ and $(M;0)$

$$\frac{\frac{P'}{P}-r}{0-r}=\frac{P-0}{M-0}$$

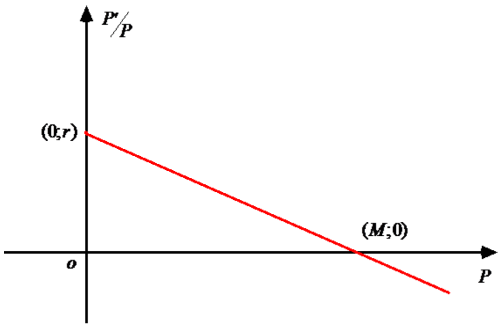

Fig. 2

obtain the equation (5)

$$P'=rP\left(1-\frac{P}{M}\right) \qquad (5)$$

which is called ***a logistic equation***.

This equation was suggested in 1838 by the demographe Verhulst to describe the growth of human population, which is a modification of Malthus model. [1], [5], [6].

Equation (5) has the form of $P'=f(P)$, where

$$f(P)=rP-\frac{r}{M}P^2 .$$

The graph of the function f is a parabola (Figure 3).
The dynamical system (5) has two fixed points, which are:

$$P_1^*=0,\quad P_2^*=M .$$

Let's solve this equation, separate the variables and integrate

$$\frac{dP}{dt}=rP\left(1-\frac{P}{M}\right)\Leftrightarrow\frac{M\,dP}{P(M-P)}=rdt$$

$$\int\frac{M\,dP}{P(M-P)}=r\int dt\Leftrightarrow\int\frac{dP}{P}+\int\frac{dP}{M-P}=r\int dt\Rightarrow$$

$$\ln|P|-\ln|M-P|=rt+c\Leftrightarrow\ln\left|\frac{P}{M-P}\right|=rt+c\Rightarrow$$

$$\frac{P}{M-P}=e^{rt+c}\Leftrightarrow\frac{P}{M-P}=e^{rt}C\Leftrightarrow$$

$$\frac{M-P}{P}=e^{-rt}C\Rightarrow P=\frac{M}{1+e^{-rt}C}$$

at time $t=0$ we have $C=\frac{M-P_0}{P_0}$, from which we have:

$$P(t)=\frac{MP_0}{P_0+(M-P_0)e^{-rt}} \qquad (6)$$

where P_0 is the population at time $t=0$.
If we go to the limit in the equation (6) when $t\to\infty$ we have the $\lim_{t\to\infty}P(t)=M$, so it approaches the carrying capacity.
In Figure 3, it is shown the phase portrait of the dynamical system for $P\geq 0$, because it does not make sense to talk about negative population. From this portrait we find out that the fixed point $P_1^*=0$ is an unstable fixed point, while $P_2^*=M$ is a stable fixed point.

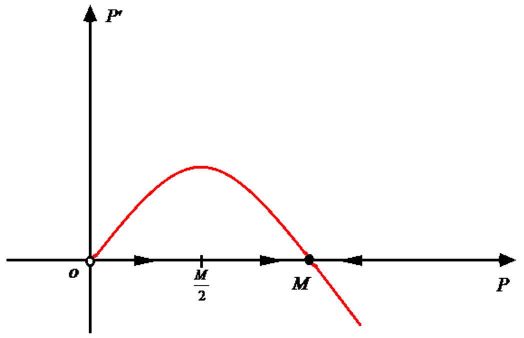

Fig. 3

In biological terms, the fackt that $P_1^* = 0$ shows an unstable fixed point, means that a little deviation of P from $P = 0$, that is even the presence of a small population, makes it grow rapidly, moving away from number $P = 0$ and nearing number M.

On the other hand, if P deviates even a little from M, then again $P(t) \to M$.

Figure 3, in fact, shows whatever be the initial population $P_0 > 0$, the population $P(t)$ goes to number M, so it heads towards the carrying capacity of the area.

Thus, population P always nears the carrying capacity M.

The phase portrait in figure 3 allows us to make a more qualitative analysis.

For example, if $P_0 < M/2$, then population P grows rapidly, because the derivative P' is a /an inceasing /growth function $(P''(t) > 0)$, and if $M/2 < P_0 < M$, population P grows again, but in a slow manner, because P' is a decreasing function $(P''(t) < 0)$.

In Figure 4 are plotted the graphs of the solutions (integral lines) of the equation (5) for any initial conditions along with the direction field using MAPLE. [7]

```
> with(DEtools):
>DEplot(diff(P(t),t)=0.03*P(t)*(1-0.001*P(t)),P(t),
t=0..300,{[0,2000], [0, 600],[0, 1000],[0, 200],
[0,0]},P=0..2000,arrows=slim,linecolour=blue);
```

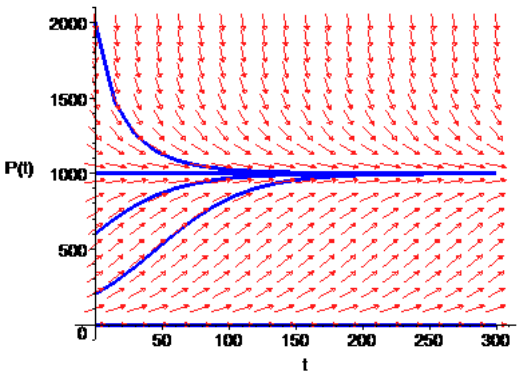

Fig. 4

III. Results And Discussion

A. *Exponential Growth Population Model*

Using the data with $t = 0$ corresponding to the year 1990, we have $P_0 = 3{,}188{,}380$. We can solve r using the fact that $P = 3{,}080{,}124$ when $t = 10$ which is the year 2000. By using (4) i.e. [3], [4], [5]

$$3{,}080{,}124 = 3{,}188{,}380\, e^{10r} \Rightarrow r = \frac{\ln\left(\frac{3{,}080{,}124}{3{,}188{,}380}\right)}{10}$$

$$= -0.00345$$

The general solution is given to us

$$P(t) = 3{,}188{,}380\, e^{-0.00345t}$$

We are now going to compute the population at later years and compare it to the actual data. The following chart represents it. [8], [9]

TABLE I.

Year	Actual	Predicted
2001	**3,060,173**	**3,069,648**
2002	**3,051,010**	**3,059,076**
2003	**3,039,616**	**3,048,540**
2004	**3,026,939**	**3,038,041**
2005	**3,011,487**	**3,027,577**
2006	**2,992,547**	**3,017,150**
2007	**2,970,017**	**3,006,759**
2008	**2,947,314**	**2,996,404**
2009	**2,927,519**	**2,986,084**
2010	**2,913,021**	**2,975,800**
2011	**2,904,780**	**2,965,551**

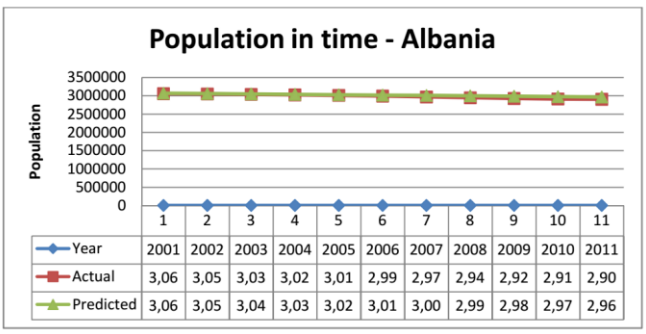

Fig. 5

This model can now be used to make future predictions about the population, as shown in Figure 5. We see an excellent agreement between actual data and predicted data between 2001 and 2011. It gives us some confidence about forecasting the future. Thus, using our model, we can plug in $t = 2030$ to estimate the population in that year in millions,

$$P(2030) = 3,188,380\, e^{-0.00345 \times 40} = 2,777,393$$

Also, the model can be used to determine the year in which a specific population target will be reached. Furthermore, we can use the exponential growth model to estimate the rate of growth as follows:

$$P(t) = P_0\, e^{rt}$$

gives us

$$P'(t) = rP_0\, e^{rt} = rP(t) = -0.00345\, P(t)$$

which of course is the differential equation that gives rise to the exponential model. Thus, the rate of population growth at any time, in thousands per year, is simply the population at that time multiplied by r , the relative growth rate, which we have already determined. For example, in 2001, $P(t) = 3,060,173$, giving us that $P'(t)$ is -105,575 thousand/year. This leads to a predicted population of $3,060,173 - 105,575 = 2,954,598$ thousand, which is quite close to the actual figure of 2010. [4]

IV. Conclusions

Malthus' model gives us an opportunity in predicting the population size. It is obviously clear that the predicted data is almost the same as the real one. The prediction about the Albanian population is an example of how fairly well this model works. There is an excellent fit between the predicted and actual data for the population of Albania in the years 2001-2011. This provides us an amazing tool to forecast the population growth in the future. Malthus' model assumes that the relative growth rate is constant. In fact, even if we ignore natural disasters, wars, and changes in social behaviour, the growth rate will change as the population increases due to crowding, disease, and lack of natural resources. The model predicts that the population would grow without bound. The logistic growth equation is a useful model for demonstrating the effects of density-dependent mechanisms. Under such model, it is possible for the population to overshoot its carrying capacity.

References

[1] M. Gaglione, Population Dynamics, A Mathematical Application of Biologocal Sciences.

[2] Boyce, William E. and DiPrima, Richard C. Elementary Differential Equations and Boundary Value Problems: Sixth Edition. John Wiley & Sons, Inc. 1997.

[3] S.R. Lubkin, Population Growth: the World.

[4] D. Hathout, Modeling population Growth: Exponantial Hyperbolic Modeling, *Applied Mathematics, 2013, 4, 299-304*

[5] M. Yi, W. Zizheng, Study of Mathematical Models fot population Projection.

[6] H. Luo, Population Modeling by Differential equations, Theses 2007.

[7] Lynch, S., Dynamical Systems with Applications using Maple TM, Second Edition, 2010.

[8] http://www.instat.gov.al/en/publications/books.aspx

[9] http://open.data.al/sq/lajme/lajm/id/669/titull/Popullsia-ne-Shtetin-Shqiptar-1870-2011