Uniqueness of weak solutions for a class of elliptic equations with weight

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Abstract—The uniqueness of weak solutions for a class of elliptic equations with weight is obtained.

Keywords—uniqueness; weak solution; elliptic equation.

I. INTRODUCTION

The elliptic equation has a strong background, and has many applications in physics and engineering. For the recent developments of weak solutions of elliptic equations, we refer the reader to [1-3]. The aim of this present paper is to obtain the uniqueness of weak solutions for a class of elliptic equations with weight. Our result is a generalization of the reference [4].

In this present paper, we consider the following A-harmonic equation

\[ \text{div} A(x, \nabla u) = 0 \]  

where \( A: \Omega \times \mathbb{R}^n \to \mathbb{R}^n \) satisfies the conditions

(H1) \( \{A(x, \xi), \xi\} \geq \alpha w(x)|\xi|^p \),

(H2) \( |A(x, \xi) - A(x, \xi_1)| \leq \beta w(x)|\xi_1 - \xi_2| \),

for almost every \( x \in \Omega \) and all \( \xi, \xi_1, \xi_2 \in \mathbb{R}^n \). Here \( \Omega \) is a bounded open set in \( \mathbb{R}^n \), \( \alpha, \beta > 0 \) are constants, \( 1 < p < n \).

The function \( w(x) \) in the conditions (H1-H2) satisfies \( w \in A_1 \) and \( w(x) > 0 \). Note that (H2) implies

\[ |A(x, \xi)| \leq \beta w(x)|\xi|^{p-1}. \]

Definition 1.1 A function \( u \in W_{\text{loc}}^{1,p}(\Omega, w) \) is called a weak solution of A-harmonic equation (1.1) if \( u \) satisfies

\[ \int_\Omega \langle A(x, \nabla u), \nabla \varphi \rangle \, dx = 0 \]  

for all \( \varphi \in C_0^\infty(\Omega) \).

The solutions belong to the local weighted Sobolev space \( u \in W_{\text{loc}}^{1,p}(\Omega, w) \) and we prove a uniqueness result for solutions provided that they belong to \( u \in W_{\text{loc}}^{1,p}(\Omega, w) \) for \( r < p \) and that they take the same boundary values in \( \mathcal{O} \setminus E \) where \( E \subset \mathcal{O} \) is a closed set and small in an appropriate capacity sense.

In order to formulate our theorem we need to consider local boundary values, see section 2.

For our main result we assume that the numbers \( p \in (1, n) \), \( q \in (1, \infty) \), \( s \in (1, \infty) \) and \( r \geq \max\{1, p-1\} \) satisfy

\[ t = \frac{sq}{sq - s - sq(p - 2)} > 1. \]  

(1.3)

Here \( (p - 2)^{r} = p - 2 \) if \( p \geq 2 \) and 0 otherwise.

Note. If \( t > n \), then \( \text{cap}_{\mathcal{E}} E = 0 \) implies \( E = \emptyset \). Hence only the values \( r \leq n \) are of interest. Let \( \mathcal{H}^s \) denote the \( s \)-dimensional Hausdorff measure. It is well known that \( \mathcal{H}^{n-1}(E) \subset \infty \) implies \( \text{cap}_{\mathcal{E}} E = 0 \).

Theorem 1.2 Suppose that \( \theta \in W_{\text{loc}}^{1,p}(\Omega, w) \) and that \( u_1, u_2 \) are weak solutions of A-harmonic equation (1.1) such that

(i) \( u_1 \) and \( u_2 \) have boundary values \( \theta \) in the \( W_{\text{loc}}^{1,p} \) sense at \( \mathcal{O} \setminus E \);

(ii) \( u_1 - u_2 \in L^s(\Omega, w) \);

(iii) \( \nabla u_1 - \nabla u_2 \in L^s(\Omega, w) \);

(iv) \( \nabla u_1, \nabla u_2 \in L^r(\Omega, w) \) if \( p > 2 \).

If \( \text{cap}_{\mathcal{E}} E = 0 \), then \( u_1 = u_2 \) in \( \Omega \).

Note. Let \( w = 1 \) conditions (H1)-(H2), then our main results is Theorem 1.1 in [4].

II. PRELIMINARY KNOWLEDGE

Before discussing we refer to some notations we shall use.

Throughout this paper, \( \Omega \) will denote bounded open set in \( \mathbb{R}^n \), and \( E \subset \mathcal{O} \) is a closed set and small in an appropriate capacity sense. In order to avoid some technical difficulties related to the imbedding theorem we shall illustrate our approach only for \( p \) smaller than the spatial dimension of \( \Omega \).
The function \( w(x) \) in the condition (H1-H2) is a locally integrable non-negative function in \( \mathbb{R}^n \). Assume that \( 0 < w < \infty \) almost everywhere. A Radon measure \( 0 < w < \infty \) is canonically associated with the weight \( w(x) \), \( \mu(\Omega) = \int_{\Omega} w(x)dx \). Thus \( d\mu = w(x)dx \), where \( dx \) is the \( n \)-dimensional Lebesgue measure. We say that \( w \) belongs to the Muckenhoupt class \( A_p \), 1 < \( p < \infty \), or that \( w \) is an \( A_p \) weight, if there is a constant \( A_p(w) \) such that

\[
\sup_B \left( \frac{1}{|B|} \int_B w(x)dx \right) \leq A_p \left( \frac{1}{|B|} \int_B w^{1/(1-p)}(x)dx \right)^{p-1} = A_p(w) < \infty \quad (2.1)
\]

for all balls \( B \) in \( \mathbb{R}^n \). We say that \( w \) belongs to \( A_1 \), or that \( w \) is an \( A_1 \) weight, if there is a constant \( A_1(w) \) such that

\[
\frac{1}{|B|} \int_B w(x)dx = A_1(w) \text{ess inf } w
\]

for all balls \( B \) in \( \mathbb{R}^n \). It is well known that \( A_1 \subset A_p \) whenever \( p > 1 \), see [2].

We say that a weight \( w \) is doubling if there is a constant \( C > 0 \) such that

\[
\mu(2B) \leq C \mu(B)
\]

whenever \( B \subset 2B \) are concentric balls in \( \mathbb{R}^n \), where \( 2B \) is the ball with the same center as \( B \) and with radius twice that of \( B \). Given a measurable subset \( E \) of \( \mathbb{R}^n \), we will denote by \( L^p(E, w) \), 1 < \( p < \infty \), the Banach space of all measurable functions \( f \) defined on \( E \) for which

\[
\|f\|_{L^p(E, w)} = \|f\|_{L^p(E, w)} = \left( \int_E |f(x)|^p w(x)dx \right)^{1/p} < \infty.
\]

The weighted Sobolev class \( W^{1,p}(E, w) \) consists of all functions \( f \), and its first general derivatives belong to \( L^p(E, w) \). The symbols \( L^p_{loc}(E, w) \) and \( W^{1,p}_{loc}(E, w) \) are self-explanatory.

We need to consider local boundary values. Let \( F \subset \partial\Omega \) and \( u \in W^{1,p}_{loc}(\Omega, w) \). We say that \( u \) has zero boundary values at \( F \) in the \( W^{1,p} \)-sense, abbreviated \( u \in W^{1,p}_{loc}(\Omega, w; F) \), if each \( x \in F \) has a neighborhood \( U \) and a function \( \eta \in C^{\infty}_{\text{loc}}(U) \) such that \( \eta = 1 \) in some neighborhood of \( x \) and \( \eta u \in W^{1,p}_{loc}(\Omega, w) \).

Suppose that \( \vartheta \in W^{1,p}_{loc}(\Omega, w) \). We say that \( u \) has the boundary values \( \vartheta \) at \( F \) in the \( W^{1,p} \)-sense if \( u - \vartheta \in W^{1,p}_{loc}(\Omega, w; F) \) and if for each \( x \in F \) there exists \( \eta \) as above with \( \eta \vartheta \in W^{1,p}(\Omega, w) \). Note that then also \( \eta u \) belongs to \( W^{1,p}(\Omega, w) \).

If \( u \) has the boundary values \( \vartheta \) at \( F \) in the \( W^{1,p} \)-sense, then \( u \) has the boundary values \( \vartheta \) at a neighborhood of \( F \). Hence we may always assume that \( F \) is open relative to \( \partial\Omega \).

III. PROOF OF THEOREM 1.2

Let \( u_1, u_2 \) be weak solutions of A-harmonic equation (1.1), \( \Omega \) be a bounded open set. Condition (i) in Theorem 1.2 implies that each \( \vartheta \in \partial\Omega \) has a neighborhood \( U = U(\vartheta) \) such that

\[
u_i \in W^{1,p}(U \cap \Omega, w), \quad i = 1, 2.
\]

Thus for each neighborhood \( V \) of \( W \) we have

\[
u_i \in W^{1,p}(\Omega \setminus \overline{V}, w), \quad i = 1, 2.
\]

Fix a ball \( B \subset \subset \Omega \). Since \( \text{cap}_p(E) = 0 \), we can choose an open set \( D \subset \mathbb{R}^n \) such that \( E \subset D \) \( B \subset \mathbb{R}^n \setminus D \) and \( \text{cap}_p(E; D) = 0 \). Here \( \text{cap}_p(E; D) = 0 \) refers to the usual variational \( \ell \)-capacity of the condenser \( (E; D) \) [2, Chapter 4]. Given \( \epsilon > 0 \) we can find an open set \( U_\epsilon \) and \( \xi \in C^\infty_{\text{loc}}(D) \) such that

\[
E \subset U_\epsilon \subset \subset \overline{D}, \quad 0 \leq \xi \leq 1, \quad \xi = 1 \text{ on } \overline{U_\epsilon}
\]

and

\[
\int_D |\nabla \xi|^p w(x)dx < \epsilon'.
\]

Hence

\[
\|\nabla \xi\|_{L^p(U_\epsilon)} < \epsilon.
\]

Write \( \eta = 1 - \xi \). Then \( \eta = 0 \) in \( \overline{U_\epsilon} \), \( \eta = 1 \) in \( \mathbb{R}^n \setminus D \), \( 0 \leq \eta \leq 1 \), \( \eta \in C^\infty(\mathbb{R}^n) \), and

\[
|\nabla \eta|_{L^p(U_\epsilon)} < \epsilon.
\]

The above inequality has important role in the following proof. Note that here we get \( \eta = 1 \) in \( B \).

Let \( W_\epsilon \) be a neighborhood of \( E \) with \( E \subset \subset W_\epsilon \). Let

\[
\varphi = \eta (u_1 - u_2).
\]

Since (3.2) holds and \( u_1 - u_2 \in W^{1,p}_{loc}(\Omega, w; \partial\Omega \setminus \{E\}) \) by Condition (i) in Theorem 1.2, then \( \varphi \in W^{1,p}_{loc}(\Omega, w) \), and the support of \( \varphi \) stays away from \( E \). Thus we can use \( \varphi \) as a test function in Definition 1.1, i.e.,

\[
\int_{\Omega \setminus W_\epsilon} \left\langle A(x, \nabla u_1), \nabla (\eta (u_1 - u_2)) \right\rangle dx = 0, \quad i = 1, 2.
\]

Hence

\[
\int_{\Omega} \left\langle A(x, \nabla u_1) - A(x, \nabla u_2), \nabla (\eta (u_1 - u_2)) \right\rangle dx = 0,
\]

here we have used \( \eta = 0 \) in \( \overline{W_\epsilon} \). Thus we obtain from the above formula
$$I = \int_\Omega \{ A(x, \nabla u_i) - A(x, \nabla u_j), \nabla (u_i - u_j) \} \, dx$$
$$= -\int_\Omega (u_i - u_j) \{ A(x, \nabla u_i) - A(x, \nabla u_j), \nabla \eta \} \, dx$$
$$\leq \int_\Omega A(x, \nabla u_i) - A(x, \nabla u_j) \| u_i - u_j \| \| \nabla \eta \| \, dx$$  \hspace{1cm} (3.9)

**Case 1:** \( p > 2 \). Using the condition (H2) and the Hölder inequality with

$$\frac{p-2}{r} + \frac{1}{s} + \frac{1}{q} = 1, \quad \text{and} \quad \frac{p-2}{r} + \frac{1}{s} + \frac{1}{q} = 1,$$

we have

$$I \leq \beta \int_\Omega \| u_i + u_j \|^{p-2} \| u_i - u_j \|^{p-1} \| \nabla u_i - \nabla u_j \| \| \nabla \eta \| \, dx$$
$$= \beta \int_\Omega \| u_i + u_j \|^{p-2} \| u_i - u_j \|^{p-1} \| \nabla u_i - \nabla u_j \| \| \nabla \eta \| \, dx$$
$$\leq \beta \left( \int_\Omega \| u_i + u_j \|^{p-2} \| u_i - u_j \|^{p-1} \| \nabla u_i - \nabla u_j \| \, dx \right)^{\frac{1}{p-2}} \left( \int_\Omega \| \nabla \eta \| \, dx \right)^{\frac{1}{p-2}}.$$

(3.11)

Thus by (3.5) and the conditions (ii)-(iv) in Theorem 1.2 we have

$$I \leq CE. \quad \text{(3.12)}$$

**Case 2:** \( p \leq 2 \). Let \( M = \{ x \in \Omega : |\nabla u_i| \leq 1 \text{ and } |\nabla u_j| \leq 1 \} \). Then by (3.9),

$$I \leq \int_\Omega A(x, \nabla u_i) - A(x, \nabla u_j) \| u_i - u_j \| \| \nabla \eta \| \, dx$$
$$+ \int_{\partial M} A(x, \nabla u_i) - A(x, \nabla u_j) \| u_i - u_j \| \| \nabla \eta \| \, dx.$$  \hspace{1cm} (3.13)

To obtain (3.12) in this case, we estimate the first term on the right-hand of the above inequality. Using the condition (H2) and the Hölder inequality, we have

$$I_i = \int_\Omega A(x, \nabla u_i) - A(x, \nabla u_j) \| u_i - u_j \| \| \nabla \eta \| \, dx$$
$$\leq \beta \int_\Omega \| u_i + u_j \|^{p-1} \| u_i - u_j \| \| \nabla \eta \| \, dx$$
$$\leq 2 \beta \int_\Omega \| u_i - u_j \| \| \nabla \eta \| \, dx$$
$$\leq 2 \beta \left( \int_\Omega \| u_i - u_j \|^q \, dx \right)^{\frac{1}{q}} \left( \int_\Omega \| \nabla \eta \|^q \, dx \right)^{\frac{1}{q}}.$$

(3.14)

where \( q^* = \frac{q}{q-1} \). Noticing that \( r = \frac{q}{q-1} - 1 \) for \( p \leq 2 \), then \( q^* \leq r \). Thus we obtain

$$I_i \leq 2 \beta \left( \int_\Omega \| u_i - u_j \|^q \, dx \right)^{\frac{1}{q}} \left( \int_\Omega \| \nabla \eta \|^q \, dx \right)^{\frac{1}{q}} \leq CE.$$  \hspace{1cm} (3.15)

Next we estimate the second term on the right-hand of the inequality (3.13). Using the condition (H2) and the Hölder inequality with

$$\frac{1}{s} + \frac{1}{q} + \frac{1}{r} = 1, \quad \text{and} \quad \frac{1}{s} + \frac{1}{q} + \frac{1}{r} = 1,$$

we have

$$I_2 = \int_{\partial M} \| A(x, \nabla u_i) - A(x, \nabla u_j) \| \| u_i - u_j \| \| \nabla \eta \| \, dx$$
$$\leq \beta \int_{\partial M} \| u_i + u_j \|^{p-2} \| u_i - u_j \|^{p-1} \| \nabla u_i - \nabla u_j \| \| \nabla \eta \| \, dx$$
$$\leq \beta \int_{\partial M} \| u_i - u_j \| \| \nabla \eta \| \, dx$$
$$\leq \beta \left( \int_{\partial M} \| u_i - u_j \|^q \, dx \right)^{\frac{1}{q}} \left( \int_{\partial M} \| \nabla \eta \|^q \, dx \right)^{\frac{1}{q}}.$$  \hspace{1cm} (3.16)

Thus by (3.5) and the conditions (ii)-(iv) in Theorem 1.2 we have

$$I \leq CE. \quad \text{(3.17)}$$

Hence in both cases we have the estimate

$$I \leq CE. \quad \text{(3.18)}$$

where \( C < \infty \) is independent of \( \epsilon \). This estimate together with the condition (H3) yields

$$0 \leq \int_{\partial \Omega} (A(x, \nabla u_i) - A(x, \nabla u_j), \nabla (u_i - u_j)) \, dx \leq CE. \quad \text{(3.19)}$$

Note that here we have used \( \eta = 1 \) in \( B \). Letting \( \epsilon \to 0 \) we obtain \( \nabla u_i = \nabla u_j \) a.e. in \( B \). Since \( B \subset \subset \Omega \)
was arbitrary, \( \nabla u_i = \nabla u_j \) in \( \Omega \) and hence \( u_i = u_j = C = \text{const.} \) in each component of \( \Omega \).

Now \( \text{cap}_p (E) = 0 \) and since the boundary of an arbitrary bounded domain cannot be of \( p \)-capacity zero, in each component \( \Omega \) the condition \( u_i - u_j \in W^{1,p}_0 (V, \omega \partial \Omega) \) implies \( C = 0 \). Thus \( u_i = u_j \) in \( \Omega \) and the theorem follows. This completes the proof of Theorem 1.2. \( \square \)

**REFERENCES**


