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Uniqueness of weak solutions for a class of elliptic equations with weight

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***Abstract*—The uniqueness of weak solutions for a class of elliptic equations with weight is obtained.**

Keywords—uniqueness; weak solution; elliptic equation.

I. INTRODUCTION

The elliptic equation has a strong background, and has many applications in physics and engineering. For the recent developments of weak solutions of elliptic equations, we refer the reader to [1-3]. The aim of this present paper is to obtain the uniqueness of weak solutions for a class of elliptic equations with weight. Our result is a generalization of the reference [4].

In this present paper, we consider the following A-harmonic equation

$$\mathrm{div}A(x,\nabla u)=0 \tag{1.1}$$

where $A:\Omega\times\mathbf{R}^n\to\mathbf{R}^n$ satisfies the conditions

(H1) $\langle A(x,\xi),\xi\rangle\ge\alpha w(x)|\xi|^p$,

(H2) $|A(x,\xi_1)-A(x,\xi_2)|\le\beta w(x)(|\xi_1|+|\xi_2|)^{p-2}|\xi_1-\xi_2|$,

(H3) $\langle A(x,\xi_1)-A(x,\xi_2),\xi_1-\xi_2\rangle>0$ whenever $\xi_1\ne\xi_2$.

for almost every $x\in\Omega$ and all $\xi,\xi_1,\xi_2\in\mathbf{R}^n$. Here Ω is a bounded open set in $\mathbf{R}^n$, $\alpha,\beta>0$ are constants, $1<p<n$.

The function $w(x)$ in the conditions (H1-H2) satisfies $w\in A_1$ and $w(x)>0$. Note that (H2) implies $|A(x,\xi)|\le\beta w(x)|\xi|^{p-1}$.

Definition 1.1 A function $u\in W^{1,p}_{loc}(\Omega,w)$ is called a weak solution of A-harmonic equation (1.1) if u satisfies

$$\int_\Omega\langle A(x,\nabla u),\nabla\varphi\rangle\,\mathrm{d}x=0 \tag{1.2}$$

for all $\varphi\in C_0^\infty(\Omega)$.

The solutions belong to the local weighted Sobolev space $u\in W^{1,p}_{loc}(\Omega,w)$ and we prove a uniqueness result for solutions provided that they belong to $u\in W^{1,r}_{loc}(\Omega,w)$ for $r<p$ and that they take the same boundary values in $\partial\Omega\setminus E$ where $E\subset\partial\Omega$ is a closed set and small in an appropriate capacity sense.

In order to formulate our theorem we need to consider local boundary values, see section 2.

For our main result we assume that the numbers $p\in(1,n)$, $q\in(1,\infty)$, $s\in(1,\infty)$ and $r\ge\max\{1,p-1\}$ satisfy

$$t=\frac{sq}{sq-s-q-\dfrac{sq(p-2)^+}{r}}>1. \tag{1.3}$$

Here $(p-2)^+=p-2$ if $p\ge2$ and 0 otherwise.

Note. If $t>n$, then $cap_tE=0$ implies $E=\varnothing$. Hence only the values $t\le n$ are of interest. Let $\mathcal{H}^s$ denote the s-dimensional Hausdorff measure. It is well known that $\mathcal{H}^{n-t}(E)<\infty$ implies $cap_tE=0$.

Theorem 1.2 Suppose that $\theta\in W^{1,p}_{loc}(\Omega,w)$ and that u_1,u_2 are weak solutions of A-harmonic equation (1.1) such that

(i) u_1 and u_2 have boundary values θ in the $W^{1,p}$-sense at $\partial\Omega\setminus E$;

(ii) $u_1-u_2\in L^q(\Omega,w)$;

(iii) $\nabla u_1-\nabla u_2\in L^s(\Omega,w)$;

(iv) $\nabla u_1,\nabla u_2\in L^r(\Omega,w)$ if $p>2$.

If $cap_tE=0$, then $u_1=u_2$ in Ω.

Note. Let $w=1$ conditions (H1)-(H2), then our main results is Theorem 1.1 in [4].

II. PRELIMINARY KNOWLEDGE

Before discussing we refer to some notations we shall use.

Throughout this paper, Ω will denote bounded open set in $\mathbf{R}^n$, and $E\subset\partial\Omega$ is a closed set and small in an appropriate capacity sense. In order to avoid some technical difficulties related to the imbedding theorem we shall illustrate our approach only for p smaller than the spatial dimension of Ω.

The function $w(x)$ in the condition (H1-H2) is a locally integrable non-negative function in $\mathbf{R}^n$. Assume that $0<w<\infty$ almost everywhere. A Radon measure $0<w<\infty$ is canonically associated with the weight $w(x)$, $\mu(\Omega)=\int_\Omega w(x)dx$. Thus $d\mu = w(x)dx$, where dx is the n-dimensional Lebesgue measure. We say that w belongs to the Muckenhoupt class A_p, $1<p<\infty$, or that w is an A_p weight, if there is a constant $A_p(w)$ such that

$$\sup_B \left(\frac{1}{|B|}\int_B w dx\right)\left(\frac{1}{|B|}\int_B w^{1/(1-p)}dx\right)^{p-1} = A_p(w)<\infty \quad (2.1)$$

for all balls B in $\mathbf{R}^n$. We say that w belongs to A_1, or that w is an A_p weight, if there is a constant $A_1(w)$ such that

$$\frac{1}{|B|}\int_B w dx = A_1(w)\operatorname{essinf}_B w$$

for all balls B in $\mathbf{R}^n$. It is well known that $A_1 \subset A_p$ whenever $p>1$, see [2].

We say that a weight w is doubling if there is a constant $C>0$ such that

$$\mu(2B)\le C\mu(B)$$

whenever $B \subset 2B$ are concentric balls in $\mathbf{R}^n$, where $2B$ is the ball with the same center as B and with radius twice that of B. Given a measurable subset E of $\mathbf{R}^n$, we will denote by $L^p(E,w)$, $1<p<\infty$, the Banach space of all measurable functions f defined on E for which

$$\|f\|_{E,w} = \|f\|_{L^p(E,w)} = \left(\int_E |f(x)|^p w(x)dx\right)^{1/p} < \infty.$$

The weighted Sobolev class $W^{1,p}(E,w)$ consists of all functions f, and its first generalized derivatives belong to $L^p(E,w)$. The symbols $L^p_{loc}(E,w)$ and $W^{1,p}_{loc}(E,w)$ are self-explanatory.

We need to consider local boundary values. Let $F\subset\partial\Omega$ and $u\in W^{1,p}_{loc}(\Omega,w)$. We say that u has zero boundary values at F in the $W^{1,p}$-sense, abbreviated $u\in W^{1,p}_{loc}(\Omega,w;F)$, if each $x\in F$ has a neighborhood U and a function $\eta\in C_0^\infty(U)$ such that $\eta=1$ in some neighborhood of x and $\eta u\in W_0^{1,p}(\Omega,w)$.

Suppose that $\theta\in W^{1,p}_{loc}(E,w)$. We say that u has the boundary values θ at F in the $W^{1,p}$-sense if $u-\theta\in W^{1,p}_{loc}(\Omega,w;F)$ and if for each $x\in F$ there exists η as above with $\eta\theta\in W^{1,p}(\Omega,w)$. Note that then also ηu belongs to $W^{1,p}(\Omega,w)$.

If u has the boundary values θ at F in the $W^{1,p}$-sense, then u has the boundary values θ at a neighborhood of F. Hence we may always assume that F is open relative to $\partial\Omega$.

III. Proof of Theorem 1.2

Let u_1,u_2 be weak solutions of A-harmonic equation (1.1), Ω be a bounded open set. Condition (i) in Theorem 1.2 implies that each $y\in\partial\Omega\setminus E$ has a neighborhood $U=U(y)$ such that

$$u_i\in W^{1,p}(U\cap\Omega,w),\quad i=1,2. \quad (3.1)$$

Thus for each neighborhood V of E we have

$$u_i\in W^{1,p}(\Omega\setminus\overline{V},w),\quad i=1,2. \quad (3.2)$$

Fix a ball $B\subset\subset\Omega$. Since $cap_t(E)=0$, we can choose an open set $D\subset\mathbf{R}^n$ such that $E\subset D$; $B\subset\mathbf{R}^n\setminus D$ and $cap_t(E;D)=0$. Here $cap_t(E;D)=0$ refers to the usual variational t-capacity of the condenser $(E;D)$ [2, Chapter 2]. Given $\varepsilon>0$ we can find an open set U_ε and $\xi\in C_0^\infty(D)$ such that $E\subset U_\varepsilon\subset\subset D$, $0\le\xi\le1$, $\xi=1$ on $\overline{U_\varepsilon}$ and

$$\int_D |\nabla\xi|^t w(x)dx<\varepsilon^t. \quad (3.3)$$

Hence

$$\|\nabla\xi\|_{t,w}<\varepsilon. \quad (3.4)$$

Write $\eta=1-\xi$. Then $\eta=0$ in $\overline{U_\varepsilon}$, $\eta=1$ in $\mathbf{R}^n\setminus D$, $0\le\eta\le1$, $\eta\in C^\infty(\mathbf{R}^n)$, and

$$\|\nabla\eta\|_{t,w}<\varepsilon. \quad (3.5)$$

The above inequality has important role in the following proof. Note that here we get $\eta=1$ in B.

Let W_ε be a neighborhood of E with $E\subset\subset W_\varepsilon$ $\subset\subset U_\varepsilon$. Let

$$\varphi=\eta(u_1-u_2). \quad (3.6)$$

Since (3.2) holds and $u_1-u_2\in W_0^{1,p}(\Omega,w;\partial\Omega\setminus E)$ by Condition (i) in Theorem 1.2, then $\varphi\in W_0^{1,p}(\Omega,w)$, and the support of φ stays away from E. Thus we can use φ as a test function in Definition 1.1, i.e.,

$$\int_{\Omega\setminus\overline{W_\varepsilon}}\left\langle A(x,\nabla u_i),\nabla\left(\eta(u_1-u_2)\right)\right\rangle \mathrm{d}x=0,\ i=1,2. \quad (3.7)$$

Hence

$$\int_\Omega\left\langle A(x,\nabla u_1)-A(x,\nabla u_2),\nabla\left(\eta(u_1-u_2)\right)\right\rangle \mathrm{d}x=0, \quad (3.8)$$

here we have used $\eta=0$ in $\overline{W_\varepsilon}$. Thus we obtain from the above formula

$$
\begin{aligned}
I &= \int_\Omega \eta \left\langle A(x,\nabla u_1) - A(x,\nabla u_2), \nabla(u_1-u_2)\right\rangle \mathrm{d}x \\
&= -\int_\Omega (u_1-u_2)\left\langle A(x,\nabla u_1) - A(x,\nabla u_2), \nabla \eta\right\rangle \mathrm{d}x \\
&\le \int_\Omega |A(x,\nabla u_1) - A(x,\nabla u_2)||u_1-u_2||\nabla\eta| \,\mathrm{d}x.
\end{aligned} \tag{3.9}
$$

Case 1: $p>2$. Using the condition (H2) and the Hölder inequality with

$$\frac{p-2}{r}+\frac{1}{s}+\frac{1}{q}+\frac{1}{t}=1, \tag{3.10}$$

we have

$$
\begin{aligned}
I &\le \beta\int_\Omega (|\nabla u_1|+|\nabla u_2|)^{p-2}|\nabla u_1-\nabla u_2||u_1-u_2||\nabla\eta|\,w\mathrm{d}x \\
&= \beta\int_\Omega (|\nabla u_1|+|\nabla u_2|)^{p-2} w^{\frac{p-2}{r}}|\nabla u_1-\nabla u_2| w^{\frac{1}{s}}|u_1-u_2| w^{\frac{1}{q}}|\nabla\eta| w^{\frac{1}{t}}\mathrm{d}x \\
&\le \beta\left(\int_\Omega (|\nabla u_1|+|\nabla u_2|)^{r} w\mathrm{d}x\right)^{\frac{p-2}{r}}\left(\int_\Omega |\nabla u_1-\nabla u_2|^{s} w\mathrm{d}x\right)^{\frac{1}{s}} \\
&\quad\cdot\left(\int_\Omega |u_1-u_2|^{q} w\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_\Omega |\nabla\eta|^{t} w\mathrm{d}x\right)^{\frac{1}{t}}.
\end{aligned} \tag{3.11}
$$

Thus by (3.5) and the conditions (ii)-(iv) in Theorem 1.2 we have

$$I\le C\varepsilon. \tag{3.12}$$

Case 2: $p\le 2$. Let $M=\{x\in\Omega : |\nabla u_1|\le 1$ and $|\nabla u_2|\le 1\}$. Then by (3.9),

$$
\begin{aligned}
I &\le \int_M |A(x,\nabla u_1) - A(x,\nabla u_2)||u_1-u_2||\nabla\eta| \,\mathrm{d}x \\
&\quad + \int_{\Omega\setminus M} |A(x,\nabla u_1) - A(x,\nabla u_2)||u_1-u_2||\nabla\eta| \,\mathrm{d}x.
\end{aligned} \tag{3.13}
$$

To obtain (3.12) in this case, we estimate the first term on the right-hand of the above inequality. Using the condition (H2) and the Hölder inequality, we have

$$
\begin{aligned}
I_1 &= \int_M |A(x,\nabla u_1) - A(x,\nabla u_2)||u_1-u_2||\nabla\eta| \,\mathrm{d}x \\
&\le \beta\int_M (|\nabla u_1|+|\nabla u_2|)^{p-1}|u_1-u_2||\nabla\eta|\,w\mathrm{d}x \\
&\le 2\beta\int_M |u_1-u_2||\nabla\eta|\,w\mathrm{d}x \\
&\le 2\beta\left(\int_M |u_1-u_2|^{q} w\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_M |\nabla\eta|^{q'} w\mathrm{d}x\right)^{\frac{1}{q'}}.
\end{aligned} \tag{3.14}
$$

where $q'=\frac{q}{q-1}$. Noticing that $t=\frac{q}{q-1-q/s}$ for $p\le 2$, then $q'\le t$. Thus we obtain

$$
\begin{aligned}
I_1 &\le 2\beta|\Omega|^{\frac{1}{q'}-\frac{1}{t}}\left(\int_\Omega |u_1-u_2|^{q} w\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_M |\nabla\eta|^{t} w\mathrm{d}x\right)^{\frac{1}{t}} \\
&\le C\varepsilon.
\end{aligned} \tag{3.15}
$$

Next we estimate the second term on the right-hand of the inequality (3.13). Using the condition (H2) and the Hölder inequality with

$$\frac{1}{s}+\frac{1}{q}+\frac{1}{t}=1, \tag{3.16}$$

we have

$$
\begin{aligned}
I_2 &= \int_{\Omega\setminus M} |A(x,\nabla u_1) - A(x,\nabla u_2)||u_1-u_2||\nabla\eta| \,\mathrm{d}x \\
&\le \beta\int_{\Omega\setminus M} (|\nabla u_1|+|\nabla u_2|)^{p-2}|\nabla u_1-\nabla u_2||u_1-u_2||\nabla\eta|\,w\mathrm{d}x \\
&\le \beta\int_\Omega |\nabla u_1-\nabla u_2||u_1-u_2||\nabla\eta|\,w\mathrm{d}x \\
&\le \beta\left(\int_\Omega |\nabla u_1-\nabla u_2|^{s} w\mathrm{d}x\right)^{\frac{1}{s}}\left(\int_\Omega |u_1-u_2|^{q} w\mathrm{d}x\right)^{\frac{1}{q}} \\
&\quad\cdot\left(\int_\Omega |\nabla\eta|^{t} w\mathrm{d}x\right)^{\frac{1}{t}} \\
&\le C\varepsilon.
\end{aligned} \tag{3.17}
$$

Hence in both cases we have the estimate

$$I\le C\varepsilon, \tag{3.18}$$

where $C<\infty$ is independent of ε. This estimate together with the condition (H3) yields

$$0\le \int_B \left\langle A(x,\nabla u_1) - A(x,\nabla u_2), \nabla(u_1-u_2)\right\rangle \mathrm{d}x \le C\varepsilon. \tag{3.19}$$

Note that here we have used $\eta=1$ in B. Letting $\varepsilon\to 0$ we obtain $\nabla u_1=\nabla u_2$ a.e. in B. Since $B\subset\subset\Omega$ was arbitrary, $\nabla u_1=\nabla u_2$ in Ω and hence $u_1=u_2=C$ $=const.$ in each component of Ω.

Now $cap_t(E)=0$ and since the boundary of an arbitrary bounded domain cannot be of t-capacity zero, in each component V of Ω the condition $u_1-u_2\in W_0^{1,p}(V,w;\partial V\setminus E)$ implies $C=0$. Thus $u_1=u_2$ in Ω and the theorem follows. This completes the proof of Theorem 1.2. □

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