Note on the invariance of the Fourier descriptor through different travelling directions

Wen-Yi Lin
Department of Mechanical Engineering
De Lin Institute of Technology
Taipei, Taiwan
wylin@dlit.edu.tw

***Abstract*—The understanding of invariance or variance of the Fourier descriptor (FD) through different travelling directions is important for the correct application of the FD, such as path synthesis problems of mechanisms. However, the understanding of invariance of the FD through different travelling directions is not clear discussed in the literature. A concise mathematical explanation is shown for the invariance of the FD for a scalar signature through different travelling directions. A practical example is shown that different travelling directions can influence the amplitudes of the FD for the signature of the complex coordinates.**

Keywords—Fourier descriptor; invariance; shape description

I. INTRODUCTION

Many techniques for shape description have been utilized in the literature due to a wide range of applications in pattern recognition, computer vision and image processing. It is not convenient to directly use the shape signature in the spatial domain to describe and match the shape of a closed curve due to the problem of invariance in the similarity transformation of a shape [1,2]. It is also troublesome to directly use a single shape signature as the objective function for shape optimization because the shape obtained based on the smallest value of the objective function is not guaranteed to be sufficiently similar [3]. One of the most widely used shape descriptors is the Fourier descriptor (FD), which depends upon the Fourier coefficient of the Fourier transform of the shape signatures of a closed curve. With this method, a shape may be represented by a shape feature vector, which consists of the normalized FD. The Fourier descriptor technique has been applied to solve difficult path synthesis problems of mechanisms during the past 20 years [3-7].

Discrete shape signature functions are periodic because of a closed curve, and they can be expressed in the frequency domain using the discrete Fourier transform. The so-called invariance in similarity transformation is that the shape descriptors or shape features are invariant through rotation, scale, translation and change of the travelling direction [8]. Moreover, shape descriptors or shape features should be independent of the choice of starting point on the shape boundary. The choice of the starting point and rotation of a shape only influence the phase of the FD [8]. The translation of a shape does not influence the Fourier descriptor, with the exception of the DC component (the fundamental harmonic) of the FD [8]. The DC component of the FD is not used for shape recognition because it only gives "shapeless" information such as scale or position [9]. Thus, the most popular technique is to utilize the magnitude of the FD and to ignore the phase information in order to achieve rotation invariance as well as make the descriptor independent of the starting point. Scale invariance can be achieved by dividing the magnitude of the FD by the maximum value of the magnitude of the FD. In the literature, the Fourier coefficients are often called the FD of the shape [9,10], or the magnitude of the Fourier coefficient after the scale normalization is termed the FD [2,8].

The mathematical proofs for the above-mentioned invariance in similarity transformation for the FD can be found in [11], except for the travelling direction. To the best of the author's knowledge, it is the first pointed out in [3] that the amplitudes of the FDs of the scalar signatures, e.g., the centroid distance (CD) and triangular centroid area (TCA) are independent of the direction of travel; however, the different direction of travel can influence the amplitudes of the FD of the complex coordinates (CCs). In other words, before the discussion in [3], the understanding of invariance of the FD through different travelling directions is not clear. However, the proof for the invariance of the FD through different travelling directions has not been shown in [3]. Therefore, in this work, a concise mathematical explanation is shown for the invariance of the FD for CD and TCA signatures through different travelling directions. In addition, different travelling directions can influence the amplitudes of the FD for CCs signature is shown using a practical example.

II. EXPLANATION OF THE INVARIANCE OR VARIANCE OF THE FD THROUGH DIFFERENT TRAVELLING DIRECTIONS

The CC signature [12] is formed by treating the positions of the boundary points (x_m , y_m) $(m = 0,1,2,\ldots,N-1)$ of the shape as the complex coordinate and is expressed as follows:

$$f_m = (x_m - x_c) + j(y_m - y_c) \tag{1}$$

where the centroid (x_c , y_c) is computed by

$$x_c = \frac{1}{N}\sum_{m=0}^{N-1} x_m, \qquad y_c = \frac{1}{N}\sum_{m=0}^{N-1} y_m \tag{2}$$

where N is the number of discrete (sampling) points.

The CD signature [12] represents the distance between the boundary points and the centroid and is expressed as follows:

$$f_m = \sqrt{(x_m - x_c)^2 + (y_m - y_c)^2} \quad (3)$$

The TCA signature [8] represents the area of the triangle formed by two adjacent boundary points and the centroid and is expressed as follows:

$$f_m = \frac{1}{2}\left|(x_m - x_c)(y_{m+1} - y_c) - (x_{m+1} - x_c)(y_m - y_c)\right| \quad (4)$$

with $(x_N, y_N) = (x_0, y_0)$.

The discrete Fourier transform of a shape signature f_m $(m = 0,1,2,\ldots,N-1)$ is defined as follows:

$$\boldsymbol{F}_n = \sum_{m=0}^{N-1} f_m e^{-j\frac{2\pi mn}{N}}, n = 0,1,2,\ldots,N-1 \quad (5)$$

If the travelling directions are different, the periodic shape signature functions should be different. Intuitively, different periodic functions will produce different Fourier descriptors. However, if the shape signature is a scalar (i.e., CD and TCA), the FD is independent of the travelling direction. A concise proof is given below.

Let the counterclockwise shape signature vector be $\{f_m\}=\{f_0, f_1, f_2, \ldots, f_{N-3}, f_{N-2}, f_{N-1}\}$ and the clockwise shape signature vector be $\{f_m^*\}=\{f_{N-1}, f_{N-2}, f_{N-3}, \ldots, f_2, f_1, f_0\}$.

The counterclockwise Fourier descriptor before being normalized can be expressed as follows:

$$\boldsymbol{F}_n = \sum_{m=0}^{N-1} f_m e^{-j\frac{2\pi mn}{N}} = f_0 e^{-j\frac{2\pi(0)n}{N}} + f_1 e^{-j\frac{2\pi(1)n}{N}} + f_2 e^{-j\frac{2\pi(2)n}{N}} + \ldots\ldots + f_{N-3} e^{-j\frac{2\pi(N-3)n}{N}} + f_{N-2} e^{-j\frac{2\pi(N-2)n}{N}} + f_{N-1} e^{-j\frac{2\pi(N-1)n}{N}} \quad (6)$$

The clockwise Fourier descriptor before being normalized can be expressed as follows:

$$\boldsymbol{F}_n^* = \sum_{m=0}^{N-1} f_m^* e^{-j\frac{2\pi mn}{N}} = f_{N-1} e^{-j\frac{2\pi(0)n}{N}} + f_{N-2} e^{-j\frac{2\pi(1)n}{N}} + f_{N-3} e^{-j\frac{2\pi(2)n}{N}} + \ldots\ldots + f_2 e^{-j\frac{2\pi(N-3)n}{N}} + f_1 e^{-j\frac{2\pi(N-2)n}{N}} + f_0 e^{-j\frac{2\pi(N-1)n}{N}} \quad (7)$$

Let

$$\tilde{\boldsymbol{f}}_{0,N-1} = f_0 e^{-j\frac{2\pi(0)n}{N}} + f_{N-1} e^{-j\frac{2\pi(N-1)n}{N}} \quad (8)$$

$$\tilde{\boldsymbol{f}}_{0,N-1}^* = f_{N-1} e^{-j\frac{2\pi(0)n}{N}} + f_0 e^{-j\frac{2\pi(N-1)n}{N}} \quad (9)$$

$$\tilde{\boldsymbol{f}}_{1,N-2} = f_1 e^{-j\frac{2\pi(1)n}{N}} + f_{N-2} e^{-j\frac{2\pi(N-2)n}{N}} \quad (10)$$

$$\tilde{\boldsymbol{f}}_{1,N-2}^* = f_{N-2} e^{-j\frac{2\pi(1)n}{N}} + f_1 e^{-j\frac{2\pi(N-2)n}{N}} \quad (11)$$

$$\tilde{\boldsymbol{f}}_{2,N-3} = f_2 e^{-j\frac{2\pi(2)n}{N}} + f_{N-3} e^{-j\frac{2\pi(N-3)n}{N}} \quad (12)$$

$$\tilde{\boldsymbol{f}}_{2,N-3}^* = f_{N-3} e^{-j\frac{2\pi(2)n}{N}} + f_2 e^{-j\frac{2\pi(N-3)n}{N}} \quad (13)$$

……

$$\tilde{\boldsymbol{f}}_{N/2-1,N/2} = f_{N/2-1} e^{-j\frac{2\pi(N/2-1)n}{N}} + f_{N/2} e^{-j\frac{2\pi(N/2)n}{N}} \quad (14)$$

$$\tilde{\boldsymbol{f}}_{N/2-1,N/2}^* = f_{N/2} e^{-j\frac{2\pi(N/2-1)n}{N}} + f_{N/2-1} e^{-j\frac{2\pi(N/2)n}{N}} \quad (15)$$

Therefore

,

$$\boldsymbol{F}_n = \tilde{\boldsymbol{f}}_{0,N-1} + \tilde{\boldsymbol{f}}_{1,N-2} + \tilde{\boldsymbol{f}}_{2,N-3} + \ldots\ldots + \tilde{\boldsymbol{f}}_{N/2-1,N/2} \quad (16)$$

$$\boldsymbol{F}_n^* = \tilde{\boldsymbol{f}}_{0,N-1}^* + \tilde{\boldsymbol{f}}_{1,N-2}^* + \tilde{\boldsymbol{f}}_{2,N-3}^* + \ldots\ldots + \tilde{\boldsymbol{f}}_{N/2-1,N/2}^* \quad (17)$$

It can be easily derived that $\left|\boldsymbol{f}_{0,N-1}\right| = \left|\boldsymbol{f}_{0,N-1}^*\right|$, $\left|\boldsymbol{f}_{1,N-2}\right| = \left|\boldsymbol{f}_{1,N-2}^*\right|$, $\left|\tilde{\boldsymbol{f}}_{2,N-3}\right| = \left|\tilde{\boldsymbol{f}}_{2,N-3}^*\right|$, ……, $\left|\tilde{\boldsymbol{f}}_{N/2-1,N/2}\right| = \left|\tilde{\boldsymbol{f}}_{N/2-1,N/2}^*\right|$.

Moreover, the following facts can be derived very straightforward using the inner product of two vectors: the angle between $\tilde{\boldsymbol{f}}_{0,N-1}$ and $\tilde{\boldsymbol{f}}_{1,N-2}$ and the angle between $\tilde{\boldsymbol{f}}_{0,N-1}^*$ and $\tilde{\boldsymbol{f}}_{1,N-2}^*$ are the same; the angle between $\tilde{\boldsymbol{f}}_{1,N-2}$ and $\tilde{\boldsymbol{f}}_{2,N-3}$ and the angle between $\tilde{\boldsymbol{f}}_{1,N-2}^*$ and $\tilde{\boldsymbol{f}}_{2,N-3}^*$ are the same;…; the angle between $\tilde{\boldsymbol{f}}_{N/2-2,N/2+1}$ and $\tilde{\boldsymbol{f}}_{N/2-1,N/2}$ and the angle between $\tilde{\boldsymbol{f}}_{N/2-2,N/2+1}^*$ and $\tilde{\boldsymbol{f}}_{N/2-1,N/2}^*$ are the same. Therefore, we obtain $\left|\boldsymbol{F}_n\right| = \left|\boldsymbol{F}_n^*\right|$.

Similarly, if the clockwise shape signature vector is $\{f_0, f_{N-1}, f_{N-2}, f_{N-3}, \ldots, f_2, f_1\}$ and/or N is an odd number, one can also derive $|F_n| = |F_n^*|$ without difficulty.

We use an example (not a symmetric triangular shape) to valiate different travelling directions can influence the amplitudes of the FD of CCs as follows:

A set of counterclockwise discrete points (N=8) is given: {0,0},{1,0},{2,0},{3,0},{2,2/3},{1,4/3},{0,2},{0,1}. A set of clockwise discrete points (N=8) is given: {0,0},{0,1},{0,2},{1,4/3},{2,2/3},{3,0},{2,0},{1,0}.

Using equations (1), (2) and (5), we obtain the spectrum of the FD of the CCs signature for the counterclockwise discrete points as follows: 2.22045E-16, 9.25362, 0.471405, 0.726113, 1.05409, 0.384984, 2.35702 and 3.25329, and the spectrum of the FD of the CCs signature for the clockwise discrete points as follows: 0, 3.25329, 2.35702, 0.384984, 1.05409, 0.726113, 0.471405 and 9.25362. Obviously, the corresponding amplitutes of the FD of the CCs signature are different for different travelling directions.

Using equations (3), (2) and (5), we obtain the same spectrums of the FD of the CD signature for the counterclockwise and clockwise discrete points as follows: 9.53375, 0.287225, 1.93259, 1.58187, 0.495886, 1.58187, 1.93259 and 0.287225. Using equations (4), (2) and (5), we obtain the same spectrums of the FD of the TCA signature for the counterclockwise and clockwise discrete points as follows: 3, 0.46194, 0.353553, 0.191342, 2.14635E-16, 0.191342, 0.353553 and 0.46194.

III. Conclusions

The understanding of invariance or variance of the FD through different travelling directions is important for the correct application of the FD, such as path synthesis problems of mechanisms. This work has demonstrated that the amplitudes of the FDs of the scalar signatures, i.e., CD and TCA are independent of the direction of travel; however, different travelling directions can influence the amplitudes of the FD of CCs.

References

[1] D. Zhang, G. Lu, A comparative study of curvature scale space and Fourier descriptors, Journal of Visual Communication and Image Representation 14 (2003) 39-57.

[2] A. El-ghazal, O. Basir, S. Belkasim, Farthest point distance: A new shape signature for Fourier descriptors, Signal Processing: Image Communication 24 (2009) 572-586.

[3] W.Y. Lin, Optimum synthesis of planar mechanisms for path generation based on a combined discrete Fourier descriptor. ASME Journal Mechanisms Robotics 7 (2015) 041023.

[4] J.R. McGarva, Rapid search and selection of path generating mechanisms from a library, Mechanism and Machine Theory, 29 (1994) 223-235.

[5] I. Ullah, S. Kota, Optimal synthesis of mechanisms for path generation using Fourier descriptor and global search methods, ASME Journal of Mechanical Design, 119 (1997) 504-510.

[6] J. Buśkiewicz, R. Starosta, T. Walczak, On the application of the curve curvature in path synthesis, Mechanism and Machine Theory 44 (2009). 1223-1239.

[7] J. Buśkiewicz, Use of shape invariants in optimal synthesis of geared five-bar linkage, Mechanism and Machine Theory, 45 (2010) 273-290.

[8] D. Zhang, G. Lu, Study and evaluation of different Fourier methods for image retrieval, Image and Vision Computing 23 (2005) 33-49.

[9] D. Hewitt, Two Dimensional Shape Recognition using Complex Fourier Analysis and Extension to Three Dimensional Shape Recognition, Master Thesis, The University of Texas at San Antonio, San Antonio, 2011.

[10] Y. Mei, Robust Affine Invariant Shape Descriptors, Ph.D. Thesis, Ryerson University, Toronto, 2010.

[11] J. Bigun, Vision with Direction, Springer-Verlag, Berlin Heidelberg, 2006.

[12] H. Kauppinen, T., Seppänen, M. Pietikäinen, An experimental comparison of autoregressive and Fourier-based descriptors in 2D shape classification, IEEE Transactions on Pattern Analysis and Machine Intelligence 17 (1995) 201-207.