Numerical solution for the generalized Burgers-Korteweg-de Vries equation

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***Abstract*—Exponential time differencing-Runge-Kutta schemes with spectral approximations are extended to deal with the generalized Burgers-Korteweg-de Vries equation. The problem is reduced to a stiff system of ordinary differential equations that is solved by combinations of exponential time differencing and Runge-Kutta schemes. The stability properties of the exponential time differencing methods and the exponential time differencing Runge-Kutta schemes up to fourth-order are discussed. Then it is shown that it is convenient to solve the generalized Burgers-Korteweg-de Vries equation by third- or fourth-order exponential time differencing Runge-Kutta schemes. The numerical results seem to be in good agreement with the exact solutions.**

Keywords—Exponential Time Differencing; Runge-Kutta Schemes; Nonlinear stiff ODEs; Burgers-Korteweg-de Vries Equation; Spectral Approximations

I. INTRODUCTION

Some physical problems can be described by Korteweg-de Vries (KdV) equation. It is well known that solitons and solitary waves are the class of special solutions of the KdV equation. Non-linear shallow-water waves and wave motion in plasmas can be described as in [3] by KdV. In order to study the problems of liquid flow containing gas bubbles [4], fluid flow in elastic tubes [5], the propagation of waves in an elastic tube filled with a viscous fluid [11], weakly nonlinear plasma waves with certain dissipative effects [12], [13] — to name a few, the basic corresponding governing equation can be reduced to the so-called Burgers-Korteweg-de-Vries (B-KdV) equation

$$\partial_t u(x,t)=-au(x,t)\partial_x u(x,t)+b\partial_{xx} u(x,t)-c\partial_{xxx} u(x,t), \quad a,b,c\neq 0. \tag{1}$$

The B-Kdv equation is a combination of the Burgers equation and KdV equation. It arises from many physical contexts and it is one of the simplest evolution equations that features nonlinearity uu_x, dissipation u_{xx} and dispersion u_{xxx}. Physical considerations require that the dissipative parameter b must always be positive, while the dispersive parameter c may be either positive or negative. The B-Kdv equation has been studied in many literatures (see e.x. [7,8,9]). Recently there has been a considerable interest in the numerical solution of the B-KdV equation. Some well known methods, such as finite difference and finite element schemes, Fourier spectral methods and exponential finite difference method are employed to solve the Burgers equation and KdV equation. However, numerical treatment for the B-KdV equation is rarely reported. A numerical investigation of the B-KdV equation was carried out by Canosa and Gazdag [6]. Darvishi and others have used in [10] a Numerical Solution of the B-KdV equation by Spectral Collocation Method and Darvishis Preconditionings.

This paper deals with the numerical solution of the B-KdV equation using exponential time differencing Runge-Kutta schemes with spectral approximations. In the next section the fourth order Runge-Kutta exponential time differencing scheme (ETDRK4) is introduced. The motivation for this selection and for the involvement of exponential time differencing Runge-Kutta schemes (ETDRK) in this study are described. As an unavoidable matter, the stability of the various exponential time differencing methods (ETD) and (ETDRK) is discussed in Section 3. Numerical results for the solution of B-KdV are reported in section 4. The efficiency and accuracy of the proposed numerical schemes are shown by considering a numerical example. Finally in section 5 some concluding remarks are done.

II. EXPONENTIAL TIME DIFFERENCING FOURTH-ORDER RUNGE-KUTTA SCHEME

Many physical problems are described by the general partial differential equation (PDE)

$$u_t=Lu+N(u,x,t), \tag{2}$$

where L is a linear eliptic operator and N is an nonlinear operator. In many practical cases one can ignore the influence of boundaries on (2) and therefore impose periodic boundary conditions. For problems with spatially periodic boundary conditions, the Fourier spectral methods can be used to discretize the spatial derivatives of (2), and therefore a stiff of coupled ODEs in time t would be obtained.

$$u_t=\bar{A}u+B(u,t). \tag{3}$$

Stiffness is a challenging property of differential equations that prevents conventional explicit numerical integrators from handling a problem efficiently. The

ETD integrator is one of the successful methods developed to solve stiff semi-linear problems. The linear term in (3) contains the stiffest part of the dynamics of the problem, and the nonlinear term varies more slowly than the linear one. The ETD methods solve the linear term exactly, and then explicitly approximate the remaining part by polynomial approximations.

Multiplying (3) by the integrating factor $e^{-\bar{A}t}$ and integrating over a single time step of length l, gives

$$u(t_{n+1})=e^{\bar{A}l}u(t_n)+e^{\bar{A}l}\int_0^l e^{-\bar{A}\tau}B(u(t_n+\tau),t_n+\tau)d\tau. \quad (4)$$

If u_n denotes the numerical approximation to $u(t_n)$, the first order exponential time differencing scheme (ETD1) would be given by

$$u_{n+1}=e^{\bar{A}l}u_n+\bar{A}^{-1}(e^{\bar{A}l}-I)B(u_n,t_n). \quad (5)$$

The ETD schemes obtained when $B(u(t_n+\tau),t_n+\tau)$ is approximated through interpolation polynomials. Cox and Matthews in [1] give ETD scheme using Runge-Kutta schemes, denoted ETDRK. Expecially the ETDRK4 scheme is of particular interest. A third-order and a second-order Runge-Kutta exponential time differencing scheme can be found in [1]. The ETDRK4 scheme applied to the prototype problem in equation (3) can be written in the form:

$$\tilde{a}_n= e^{\bar{A}l/2}u_n+\bar{A}^{-1}(e^{\bar{A}l/2}-I)B(u_n,t_n)$$
$$\hat{g}_n= e^{\bar{A}l/2}u_n+\bar{A}^{-1}(e^{\bar{A}l/2}-I)B\left(\tilde{a}_n,u_n,t_n+\frac{l}{2}\right)$$
$$\check{r}_n= e^{\bar{A}l/2}\alpha_n+\bar{A}^{-1}(e^{\bar{A}l/2}-I)[2B\left(\hat{g}_n,t_n+\frac{l}{2}\right)-B(u_n,t_n)]$$
$$u_{n+1}=$$
$$e^{\bar{A}l}u_n+l^{-2}\bar{A}^{-3}\{[-4l-l\bar{A}+e^{\bar{A}l}(4l-3l\bar{A}+(l\bar{A})^2)]B(u_n,t_n)]$$
$$+2[2l+l\bar{A}+e^{\bar{A}l}(-2l+l\bar{A})]\left(B\left(\tilde{a}_n,t_n+\frac{l}{2}\right)+Y\left(\hat{g}_n,t_n+\frac{l}{2}\right)\right)$$
$$+(-4l-3l\bar{A}-(l\bar{A})^2)+e^{\bar{A}l}(4l-l\bar{A})]B\left(\check{r}_n,t_n+\frac{l}{2}\right)\} \quad (6)$$

The experience in solving problems by ETDRK shows that equation (6) suffers from serious cancellation if it is implemented directly in this format. So, to efficiently implement the ETDRK4 scheme in general, Kassam and Trefethen in [2] explore the numerical instability, and propose a modification of the ETD schemes that solves these numerical problems. The key idea is to make use of complex analysis and evaluate certain coefficient matrices or scalars via contour integrals in the complex plane.

III. STABILITY REGION

The stability properties of the ETD and ETDRK methods up to fourth-order will be discussed in this section. Stability is related to the accuracy of the schemes and refers to errors not growing in subsequent steps. The stability region is the subset of the complex plane consisting of those $q\Delta t \in \mathbb{C}$ with time step Δt, for which the numerical approximation produces bounded solutions when applied to the scalar linear model problem $du/dt=qu$. The stability of the ETDRK4 can be analyzed by plotting its stability regions. Consider the nonlinear ODE

$$ut = pu + f(u), \quad (7)$$

with f(u) nonlinear part. It is supposed that there exists a fixed point u_0 such that $pu_0 + f(u_0) = 0$. Linearizing about this fixed point, if u is peturbation of u_0 and q is $f'(u_0)$, the following result is received

$$ut = pu + qu. \quad (8)$$

The fixed point is stable if Re(p+q)<0. The stability regions are discussed below for schemes ETDRK4 and RK4 against ETD1, ETD2, ETDRK2 and ETDRK3, in the complex planes of x=qt for different values of y=pt. In general, the parameters p and q may be both complex-valued so the resulting stability region is four-dimensional. By assuming that q is complex and that p is fixed, negative and real, we can plot the resulting stability regions in the complex plane. In Fig.1 they are plotted the stability region of the classical Runge-Kutta method (RK4) and those of ETDRK4 for some different negative values of y. These last are properly selected for good-looking performance. As it is shown in Fig.1a, the region of stability for the ETDRK4 schemes grows larger as y decreases. Meantime, the inner red curve at the Fig.1b — the RK4 region — corresponds to the case y=0 for ETDRK4. It can be generalized: the stability regions of the ETDRK schemes for y=0 coincide with the those of the corresponding order RK schemes. This is expected since in the limit as $y \to 0$, ETDRK schemes reduce to the corresponding order explicit RK scheme. Of course, the regions plotted in Fig.1 give only an indication of the stability of the methods.

If the eigenvalue of the linear operator is pure imaginary then the stability regions are quite different. In Fig. 2, the boundary stability regions of ETDRK2, ETDRK3 and ETDRK4, for y=—18i and y=18i are drawn. It can be observed from this figure that the stability regions include an interval of the imaginary axis, till exactly the value –imag(y). It is found that the above statement holds for each imaginary value of y.

The stability regions of ETDRK2, ETDRK3 and ETDRK4, for y=—4 and y=—9 are given in Fig. 3. From this figure it can be seen that for both the two values selected for y, the stability region increases as the order of the ETDRK schemes increases, i.e, the ETDRK2 scheme has the smallest stability region while the ETDRK4 scheme has the largest one. By other experiments it is observed that the statement holds for each real negative value of y.

The stability regions of ETDRK2, ETDRK3, ETDRK4, ETD1and ETD2 are compared in Fig. 4 for the values of y=—9 and y=—15. It can be observed from this figure that for both the two values of the stability region the ETDRK4 scheme contains those of the ETDRK3 and the ETDRK2 schemes, while it is contained by that of the ETD2 scheme. A small part of the stability region of the ETDRK4 scheme is not contained by that of the ETD1 scheme, however the first region is considerably smaller than the last one. Meantime the all stability regions for y=—9 are considerably smaller and contained by the corresponding ones for y=—15. From other experiments and observation done it is concluded that the above statements hold as $y \to 0$.

From the observation done in this section it can be concluded that against the evident fact of the stability region reduced as the order of a RK method is increased, the stability of an ETDRK scheme is improved when its order is increased. Instead of the balance between the stability of a method and its order to be achieved by a RK method, the only problem faced in implementing an ETDRK scheme is the computational effort increased progressively to its order. But the higher the order of the ETDRK used, the larger integration step (without risking the stability of the scheme) can be adopted. So, the compromise between the integration stepsize and the order of the ETDRK scheme used is the key factor to optimizing the computational effort.

In the next section some numerical experiments done with ETDRK2, ETDRK3 and ETDRK4 methods, in solving B-KdV equations, are motivated and supported by the above analysis.

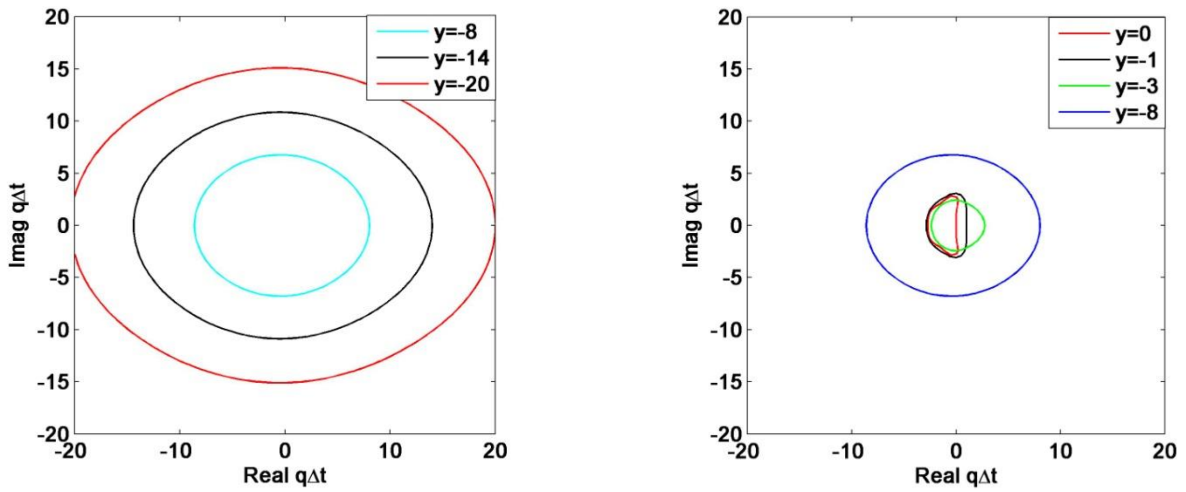


Fig. 1. *Stability regions for RK4 and ETDRK4 schemes in the complex plane of **x=qt** for several negative **y=pt***

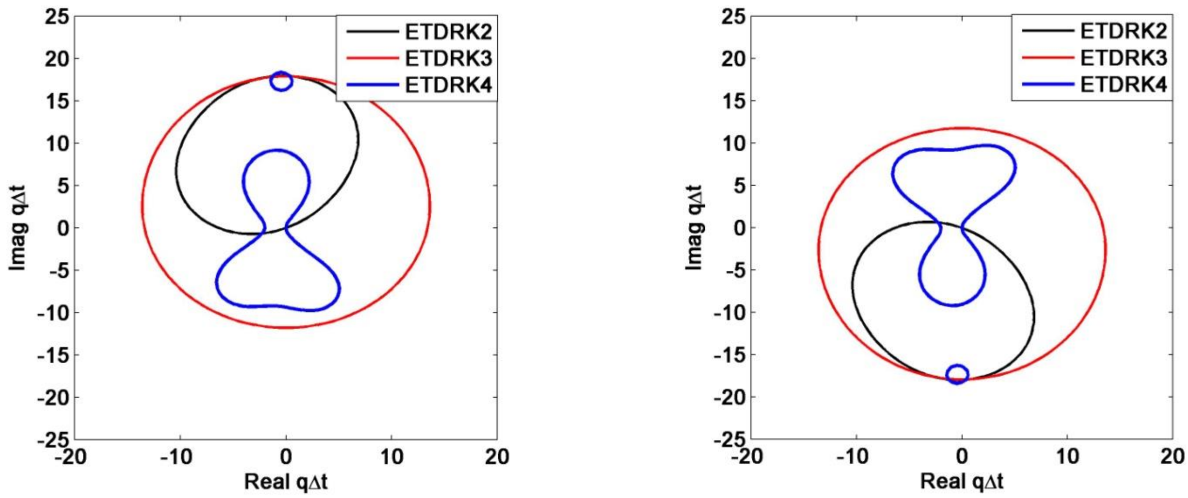


Fig. 2. *Boundary of stability regions :ETDRK2, ETDRK3, ETDRK4 (a) **y=—18i**, (b) **y=18i***

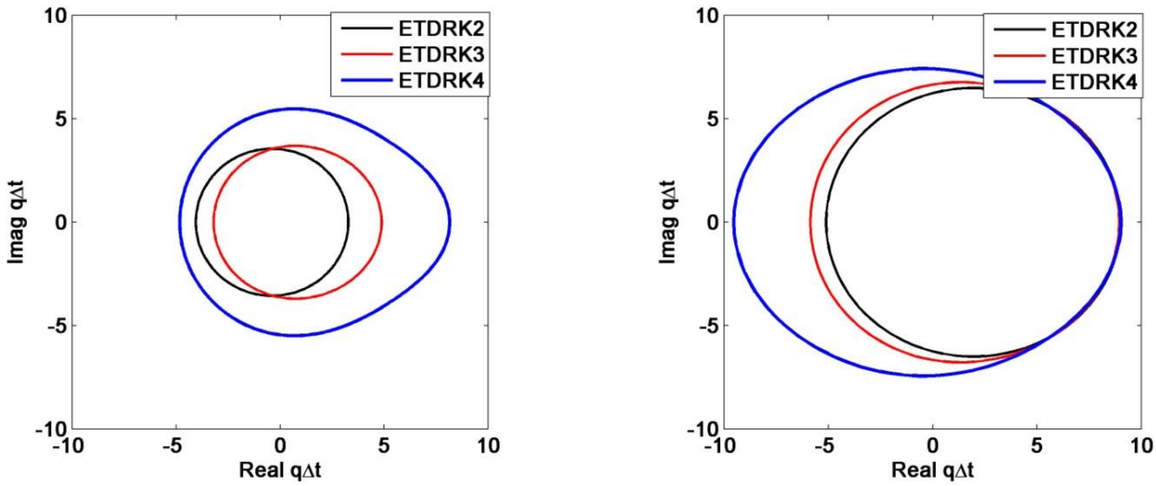


Fig. 3. *Stability regions: ETDRK2, ETDRK3, ETDRK4 (a)* ***y=—4*** *(b)* ***y=—9***

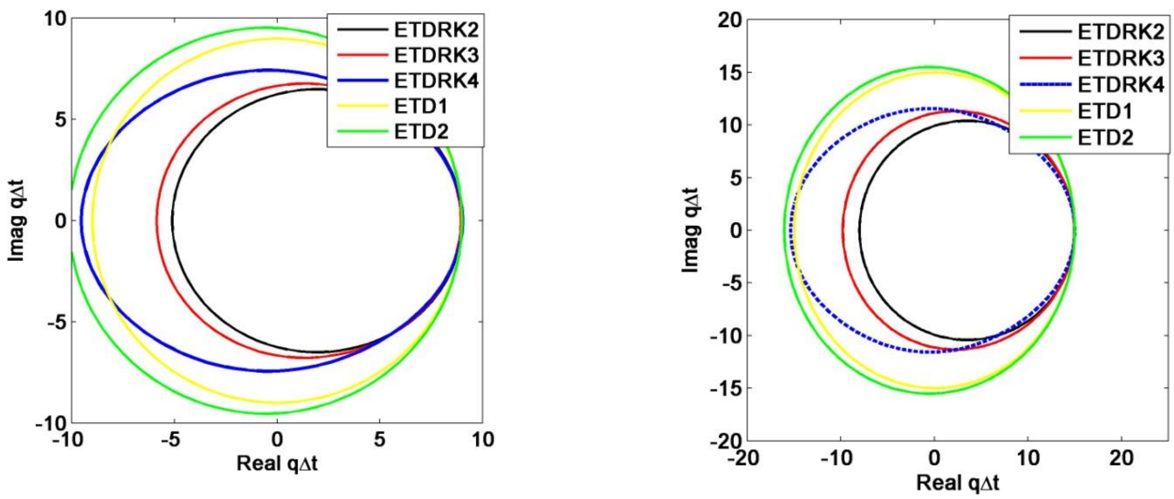


Fig. 4. *Stability regions: ETDRK2, ETDRK3, ETDRK4, ETD1, ETD2 (a)* ***y =—9*** *(b)* ***y=—15***

IV. NUMERICAL EXPERIMENTS

Consider now the B-KdV equation in [—π, π] for t>0, rewritten as

$$\partial_t u(x,t)=-au(x,t)\partial_x u(x,t)+b\partial_{xx} u(x,t)-c\partial_{xxx} u(x,t), \quad a, b, c\neq 0. \tag{9}$$

The Cauchy problem for the B-KdV (9) was investigated by Bona and Schonbek [15]. They proved the existence and uniqueness of bounded traveling wave solutions which tend to constant states at plus and minus infinity. Equation (9) can be written in integral form

$$\partial_t u(x,t)=-a\frac{1}{2}(\partial_x u(x,t))^2+b\partial_{xx} u(x,t)-c\partial_{xxx} u(x,t), \quad a, b, c\neq 0. \tag{10}$$

As it contains both second and third order derivatives (dissipation u_{xx}, and dispersion u_{xxx}), the B-KdV equation produces complex behavior. We can write the B-KdV equation (10) with 2L periodic boundary conditions in Fourier space as it follows

$$\partial_t \hat{u}(x,t)=-a\frac{ik}{2}(F((F^{1}(u))^2))-(bk^2-cik^3)\hat{u},$$

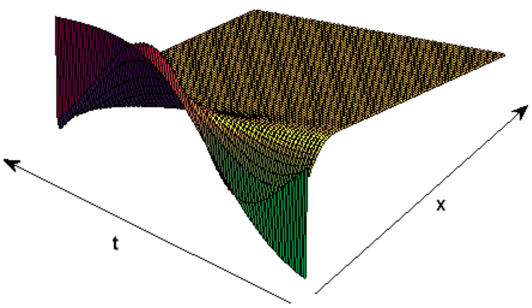
where F denotes the discrete Fourier transform. For the numerical experiments, in the example below, a reference solution is calculated on a grid with 2^7 x 2^7 points on the region [—π, π] with periodic boundary conditions. Matlab code ode15s, the best one in Matlab family for the accuracy provided for stiff problems, is used to generate some proper reference solutions. Let take the values of constants a=1, b=1.5, c=3 and take the following initial condition for the B-KdV (10)

$$u(x,0)=\frac{3b^2}{25ac}\ \mathrm{sech}^2\left(\frac{-b}{10c}x\right)-\frac{6b^2}{25ac}\tanh\left(\frac{-b}{10c}x\right)+\frac{6b^2}{25ac}. \tag{11}$$

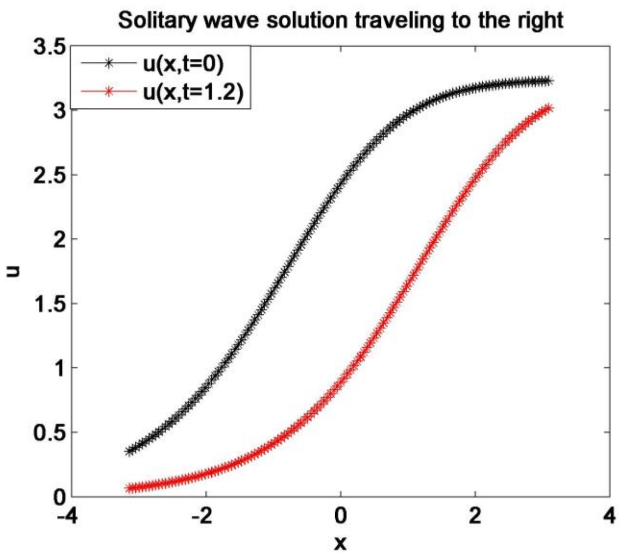
Following [16], the explicit solution for B-KdV (10)-(11) is given by the travelling wave

$$U(x,t)=\frac{3b^2}{25ac}\ \mathrm{sech}^2\left[\frac{1}{2}\left(\frac{-b}{5c}x+\frac{6b^3}{125c^2}t\right)\right]-$$
$$\frac{6b^2}{25ac}\tanh\left[\frac{1}{2}\left(\frac{-b}{5c}x+\frac{6b^3}{125c^2}t\right)\right]+\frac{6b^2}{25ac}.$$

A bounded traveling solitary wave solution to the B-KdV equation can be expressed as a composition of a bell-profile solitary wave and a kink-profile solitary wave. To get some insight, time evolution and solitary wave solution traveling to the right for B-KdV (10)-(11) are presented in Fig. 5.



(a)



(b)

Fig.5. *(a) Time evolution for B-KdV (10)-(11). The x axis runs from x=—4 to x=4 and the t-axis runs from t=0 to t=20. (b) Solitary wave solution traveling to the right.*

The errors and the local errors versus time step h in solving the B-KdV (10)-(11) with ETDRK2, ETDRK3 and ETDRK4 schemes are presented in Fig. 6. From both the two parts of this figure, it can be seen that for h ≤ 0.06, within the same level of accuracy, ETDRK4 can use a larger step size than ETDRK3, while ETDRK2 fails to solve the problem for the range of time step h described. The differences between ETDRK3 and ETDRK2 are reinforced as h is decreased.

Accuracy vs. time used for solving the B-KdV (10)-(11) with ETDRK3 and ETDRK4 schemes is presented in Fig. 7 below. From this figure, it can be seen that the error for ETDRK4 and the corresponding time used are linearly related in a stable manner. The same relationship for the ETDRK3 is less stable. However the graph describes clearly the differences in computational costs of the two methods.

In Fig. 8 the comparison between exact solution of B-KdV (10)-(11) and the simulated one by ETDRK4, is presented for $t=10^{-3}$.

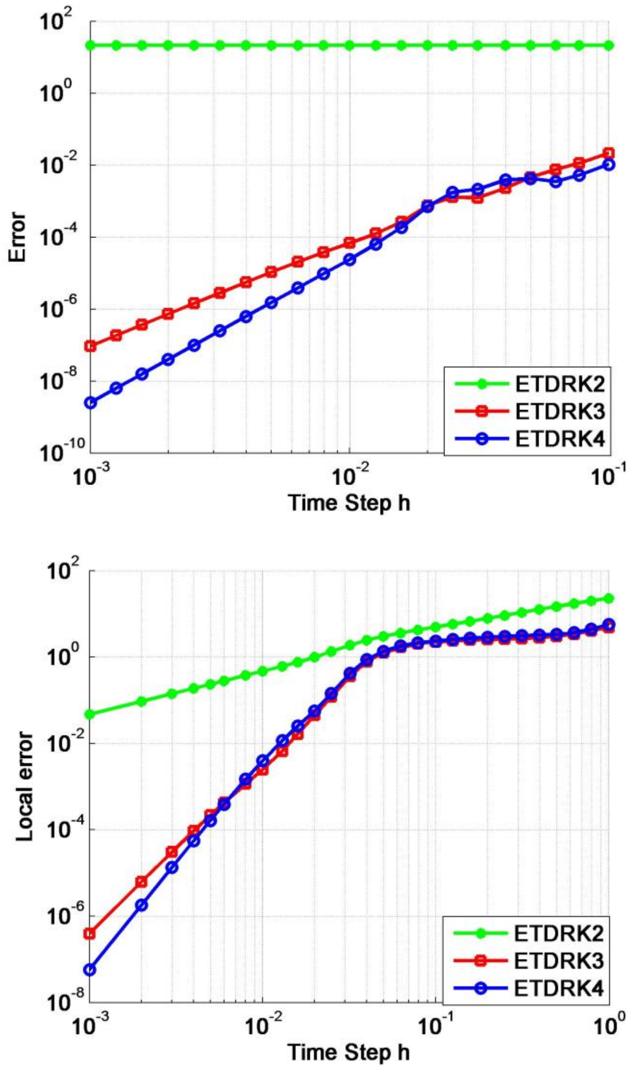


Fig. 6. *Accuracy vs. time step for solving the B-KdV (10)-(11) with ETDRK2, ETDRK3 and ETDRK4 schemes. Reference solver: ode15s*

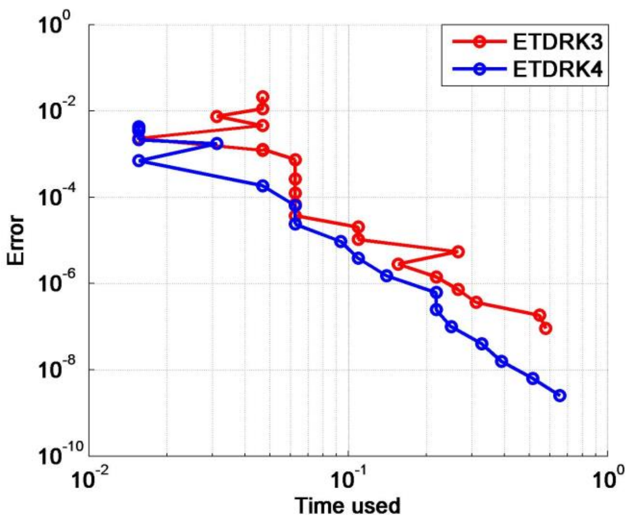


Fig. 7. *Accuracy vs. time used for solving the B-KdV (10)-(11) with ETDRK3 and ETDRK4. Reference solver: ode15s*

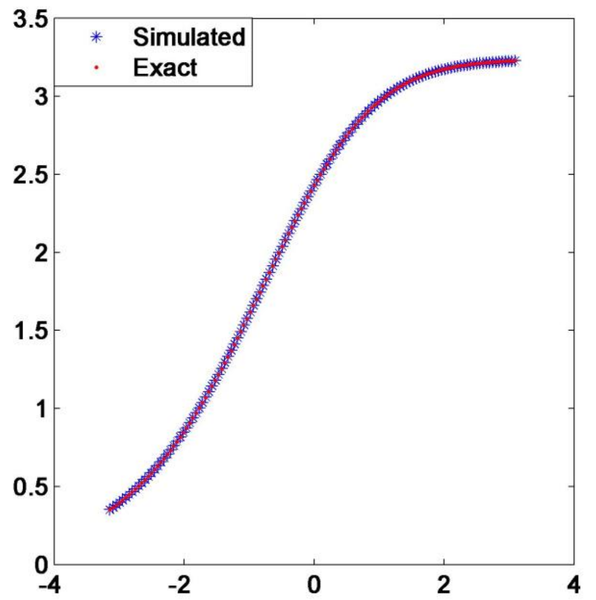


Fig. 8. *Comparison between the exact solution of B-KdV (10)-(11) and the simulated one by ETDRK4, $h=10^{-4}$, calculated at $t=10^{-3}$.*

Regarding the accuracy in the solving process, it can be concluded that the ETDRK4 scheme is clearly favored than the ETDRK3. It must be mentioned here that all the classical techniques used for stiff systems are implicit multistep or implicit RK methods of low orders. It can be concluded also that it is entirely practical to solve the difficult nonlinear partial differential equation (1) to high accuracy by ETDRK3 or ETDRK4 schemes.

V. Concluding Remarks

In this paper the Exponential time differencing-Runge-Kutta (ETDRK) schemes with spectral approximations are extended to obtain numerical solutions for the generalized Burgers-Korteweg-de Vries equation. The problem is reduced to a stiff system of ODEs in time t that is solved by exponential time differencing Runge-Kutta methods. A stability analysis is done for these schemes and numerical results for the generalized Burgers-Korteweg-de Vries (10)-(11) equation are obtained. In a series of numerical experiments Matlab codes are built and used. Matlab code ode15s is used to generate some proper reference solutions. The numerical solutions obtained are in good agreement with the known exact solutions. It is shown that it is entirely practical to solve this difficult nonlinear partial differential equation to high accuracy by third and more efficiently by fourth-order exponential time differencing Runge-Kutta scheme.

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