Journal of Multidisciplinary Engineering Science and Technology (JMEST)
ISSN: 3159-0040
Vol. 2 Issue 12, December - 2015

Modifications Of Extended Poisson Theory Of Plates

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***Abstract*—Author's earlier work is based on the presumption that the recently proposed extended Poisson theory (EPT) appears to be the best suitable theory to overcome lacuna in the classical theories of primary plate problems. Corrections to the solutions at each stage of the adapted iterative procedure are determined without disturbing solutions in the preceding stages of iterations. Disadvantage in the application of EPT is in the development of software for generation of $f_k(z)$ functions of thickness coordinate z necessary for analysis of plates with thickness ratio varying up to unit value. In the present work, new theories of plates are proposed to rectify errors in the initial set of solutions from EPT by incorporating them into uncoupled 2-D theories from Fourier series in terms of proper sinusoidal functions.**

Keywords—Elasticity; Plates; Bending; Torsion; Extension

I. INTRODUCTION

Analysis of plates within small deformation theory of elasticity is generally based on making suitable assumptions about thickness-wise distributions of displacements and/or stresses (or strains) to derive two-dimensional plate equations. In the energy methods based on stationary property of relevant total potential, equations governing 2D variables correspond to plate element equilibrium equations (PEEES). Prescribed or reactive static conditions along an edge of the plate are in terms of stress resultants which have no unique thickness-wise distributions. With assumed in-plane displacements, *transverse* stresses are obtained through post-processing of thickness-wise integration of equilibrium equations.

Displacements independent of z- coordinate are used as domain variables in most of the theories reported in the literature. Out of these three variables denoted by $[u, v, w]_0$, in-plane displacements $[u, v]_0$ are basic variables in extension problems. Solutions for these displacements satisfy both static and integrated equilibrium equations. Basic variable $w_0(x, y)$ in the bending problems is governed by a fourth order equation in Kirchhoff's theory [1] corresponding to plate element equilibrium equation. In higher order theories based on calculus of variations, it is implied that thickness-wise integrated equilibrium equations are satisfied with assumed displacements, in particular, if they are in terms of power series or various forms of polynomials.

PEEES are eliminated through the adapted iterative procedure in the recently proposed Extended Poisson Theory (EPT) of plates [2[. In expressing thickness-wise distribution of displacements, a complete set of $f_k(z)$ functions [3] is generated from recurrence relations with $f_0 = 1$, $f_{2k+1,z} = f_{2k}$, $f_{2k+2,z} = - f_{2k+1}$ such that $f_{2k+2}(\pm1) = 0$. The functions $f_{2k+1}(z)$ are replaced by $f^*_{2k+1} = (f_{2k+1} - \beta_{2k-1} f_{2k-1})$ so as to satisfy both static and integrated equations. Corrections to the solutions at each stage of the adapted iterative procedure are determined without disturbing solutions in the preceding stages of iterations. This facility is not available with solutions from PEEES. Only disadvantage in the application of the procedure is in the development of software for generation of $f_k(z)$ functions and evaluation of β_{2k+1} necessary for application of the theory with thickness ratio varying up to unit value. In EPT, however, one cannot avoid initial solutions of displacements with one and two term representation of displacements in bending (or associated torsion) and extension problems, respectfully. In view of prescribed or reactive asymmetric transverse shear stresses in primary extension problems, there is no need to apply EPT for finding $[u, v]_0$ but with w(x, y, z) as face variable, EPT requires two term representation of displacements. In view of inconvenience of generating and using $f_k(z)$ in higher order theories, we consider the sequence of uncoupled 2-D problems [4] governing displacement variables other than the basic variables in the initial set of EPT. The present work deals with development of new theories to rectify errors in the initial set of solutions in EPT by incorporating them into uncoupled 2-D theories.

II. PRELIMINARIES

For simplicity in presentation, a square plate bounded within $0 \le X, Y \le a$, $-h \le Z \le h$ with reference to Cartesian coordinate system (X, Y, Z) is considered. Material of the plate is homogeneous and isotropic with elastic constants E (Young's modulus), ν (Poisson's ratio) and G (Shear modulus) that are related to one other by E = 2(1+ν) G. For convenience, coordinates X, Y, Z and displacements (U, V, W) in non-dimensional form x= X/a, y=Y/a,

$z=Z/h$, $(u, v, w) = (U, V, W)/h$ and half- thickness ratio $\alpha = (h/a)$ are used.

With the above notation, equilibrium equations in stress components are:

$$\alpha\,(\sigma_{x,x}+ \tau_{xy,y}) + \tau_{xz,z} = 0 \quad (1a)$$
$$\alpha\,(\sigma_{y,y}+ \tau_{xy,x}) + \tau_{yz,z} = 0 \quad (1b)$$
$$\alpha\,(\tau_{xz,x}+ \tau_{yz,y}) + \sigma_{z,z} = 0 \quad (2)$$

in which suffix after ',' denotes partial derivative operator. Classical theory of extension problems deals with the two in-plane equilibrium equations of infinitesimal element where as Kirchhoff theory deals mainly with the equation of transverse stresses through PEEES.

In displacement based models, stress components are expressed in terms of displacements, via, six stress-strain constitutive relations and six strain-displacement relations. These relations within the classical small deformation theory of elasticity are:

Strain-stress and semi-inverted stress-strain relations:

$$E\,\varepsilon_x= \sigma_x - \nu\,(\sigma_y+ \sigma_z) \quad (3a)$$
$$E\,\varepsilon_y = \sigma_y - \nu\,(\sigma_x+ \sigma_z) \quad (3b)$$
$$E\varepsilon_z= \sigma_z - \nu\,(\sigma_x+ \sigma_y) \quad (3c)$$
$$G\,[\gamma_{xy}, \gamma_{xz}, \gamma_{yz}] = [\tau_{xy}, \tau_{xz}, \tau_{yz}] \quad (4)$$
$$\sigma_x = E'(\varepsilon_x + \nu\,\varepsilon_y) + \mu\,\sigma_z \quad (5a)$$
$$\sigma_y = E'(\varepsilon_y + \nu\,\varepsilon_x) + \mu\,\sigma_z \quad (5b)$$
$$\varepsilon_z = -\mu\,e + (1- 2\,\nu\,\mu)\,\sigma_z/E \quad (6)$$

in which $e = (\varepsilon_x + \varepsilon_y)$, $E' = E/(1- \nu^2)$ and $\mu = \nu/(1- \nu)$.

Strain-displacement relations:

$$[\varepsilon_x, \varepsilon_y, \varepsilon_z] = [\alpha u_{,x}, \alpha v_{,y}, w_{,z}] \quad (7a)$$
$$\gamma_{xy} = \alpha u_{,y}+ \alpha v_{,x} \quad (7b)$$
$$[\gamma_{xz}, \gamma_{yz}] = [u_{,z}+ \alpha w_{,x}, v_{,z}+ \alpha w_{,y}] \quad (8)$$

In-plane equilibrium equations in terms of displacements are

$$E'\,[\alpha^2\Delta u - ½(1+ \nu)\,\alpha^2(v_{,x} - u_{,y})_{,y}] + \mu\,\alpha\sigma_{z,x} + \tau_{xz,z} = 0 \quad (9a)$$
$$E'\,[\alpha^2\Delta v + ½(1+ \nu)\,\alpha^2(v_{,x} - u_{,y})_{,x}] + \mu\,\alpha\sigma_{z,y} + \tau_{yz,z} = 0 \quad (9b)$$

Prescribed upper and bottom face conditions along with edge conditions can be modified such that even functions $f_{2n}(z)$ and odd functions $f_{2n+1}(z)$ in the z-distribution of in-plane displacements are for analysis of extension and bending problems, respectively. Correspondingly, vertical displacement $w(x, y, z)$ is odd and even in the extension and bending problems, respectively, due to transverse shear strain-displacement relations. In displacement based models, classical theories of plates deal with determination of basic variables $[u, v, w]_0$. In the present work, role of linear thickness-wise distribution of each one of three displacements, six strains and six stress components is examined with reference to development of new theories. As such, we use suffixes $_0$ and $_1$ for the first and second terms, suffixes $_e$ and $_b$ for 2-D variables in extension and bending problems. In extension problems, displacements are $[u_0, v_0, w_1]$, in-plane strains are $[\varepsilon_x, \varepsilon_y, \gamma_{xy}]_0$ and transverse strains are $[\gamma_{xz1}, \gamma_{yz1}, \varepsilon_{z0}]$. Corresponding variables in bending problems are complementary to those in extension problems, i.e., they are $[u_1, v_1, w_0]$, $[\varepsilon_x, \varepsilon_y, \gamma_{xy}]_1$ and $[\gamma_{xz0}, \gamma_{yz0}, \varepsilon_{z1}]$.

III. INITIAL SETS OF SOLUTIONS IN PRIMARY PLATE PROBLEMS FROM EPT

In EPT of primary plate problems, in-plane displacements $[u, v]$ require two term representation in extension problems and one term representation in bending (or associated torsion) problems. Prescribed conditions at each of x = constant edge (with analogous conditions along y = constant edge) in the primary problems are

$$u = \tilde{u}_n(y) \text{ or } \sigma_{xn}(y) = T_{xn}(y) \quad (10a)$$
$$v = \tilde{v}_n(y) \text{ or } \tau_{xyn}(y) = T_{xyn}(y) \quad (10b)$$

in which n is 0 and 1 in extension and bending problems, respectfully. Similarly, prescribed transverse stresses along $z = \pm 1$ faces of the plate are $[T_{xz}(x, y), T_{yz}(x, y), T_z(x, y)]_n$. Due to odd and even z-distribution, however, $[T_{xz1}, T_{yz1}, T_{z0}]$ correspond to extension problems and vice versa in bending problems.

In auxiliary problems in EPT, transverse stresses in bending and extension problems are expressed as

$$[\tau_{xz}, \tau_{yz}]_{0b} = -\alpha\,[\psi_{0,x}, \psi_{0,y}] \quad (11a)$$
$$[\tau_{xz}, \tau_{yz}]_{1e} = \alpha\,[\psi_{1,x}, \psi_{1,y}] \quad (11b)$$

In bending problem with $\sigma_z = z\sigma_{z1}$, one gets from static equation (2) $\alpha^2\Delta\psi_0 = \sigma_{z1}$ ($\sigma_{z1} = q_1/2$) and the transverse shear stresses are independent of elastic deformations. In extension problem with $\sigma_z = f_2\,\sigma_{z2}$ with $f_2(z) = (1 - z^2)/2$, one gets $\alpha^2\Delta\psi_1 = \sigma_{z2}$ with σ_{z2} dependent on elastic deformations due to in-plane displacements $f_2[u, v]_2$.

In EPT, one should note that the error in the analysis with reference to the exact solution of 3-D problem is due to participation of $w_0(x, y)$ in the bending problem and $w_1(x, y)$ from $w = z\,w_1(x, y)$ in extension problem in the transverse strain-displacement relations (w_0 and w_1 are from z-integration of ε_z from constitutive relations). One should also note here that the determination of 2-D variables of $[u, v]$ are independent of w_0 and w_1. Moreover, σ_{z1} in bending problem is independent of elastic deformations where as σ_{z2} is dependent on elastic deformations in the extension problems. Analysis from EPT of primary extension problem in the presence of transverse stresses is presented later.

A. Initial solutions of primary bending problem

In EPT, w_0 is a face variable given by

$$w_{0f}(x, y) = \int[\tau_{xz0}\, dx + \tau_{yz0} dy] - \\ -\int [u_{,z}\, dx + v_{,z}\, dy]_{z=1} \quad (12)$$

Initial set of in-plane displacements and transverse stresses with $f_2(z) = (1 - z^2)/2$ and $f_3(z) = z(1-z^2/3)/2$ are

$$[u, v] = z\, [u, v]_1$$
$$[\tau_{xz}, \tau_{yz}] = f_2(z)\, [\tau_{xz2}, \tau_{yz2}] \quad (13a)$$
$$\sigma_z = f_3(z)\sigma_{z3} \quad (13b)$$

Note that f_2 and f_3 are due to mandatory requirement of z-integration of equilibrium equations for determination of $[u, v]_1$.

Since $f_3(1) \neq 0$, σ_{z3} becomes a free variable by replacing $f_3(z)$ with $f^*_3(z)$. Normal stress σ_z along with $(z\, q_1/2)$ in the extended Poisson's theory takes the form

$$\sigma_z = z\, [½\, q_1 - ⅓\, \sigma_{z3}] + f_3\, \sigma_{z3} \quad (14)$$

In order to determine $[u_1, v_1]$ satisfying static and integrated equilibrium equations due to elastic deformations, they are modified as

$$u_1^* = u_1 + \gamma_{xz0} - \alpha\, w_{0,x} \quad (15a)$$

$$v_1^* = v_1 + \gamma_{yz0} - \alpha\, w_{0,y} \quad (15b)$$

(Inclusion of w_0 term in the above equation is based on the fact that determination of w_0 is not from transverse shear strain-displacement relations as in Kirchhoff's theory and not correct from these relations in FSDT. Note, however, that $[\gamma_{xz0}, \gamma_{yz0}] = [(u_1 + \alpha\, w_{0,x}), (v_1 + \alpha\, w_{0,y})]$ after finding $[u_1, v_1]$). Transverse shear stresses along with those in the auxiliary problem are $[\tau_{xz}, \tau_{yz}] = [\tau_{xz0}, \tau_{yz0}] + f_2(z)\, [\tau_{xz2}, \tau_{yz2}]$ in which $[\tau_{xz2}, \tau_{yz2}] = G\, [u_1, v_1] + [\tau_{xz0}, \tau_{yz0}]$. Here, $[\tau_{xz0}, \tau_{yz0}]$ are included due to participation of σ_{z1} in the equilibrium equations (1). From equations (1, 14), one obtains

$$G\, \alpha\, (u_{1,x} + v_{1,y}) = \beta_1\, \sigma_{z3} \quad (16)$$

(Note that one cannot prescribe zero τ_{xz2} along x constant edges since τ_{xz0} is independent of u_1. Similarly, one cannot prescribe zero τ_{yz2} along y constant edges since τ_{yz0} is independent of v_1)

For the use of $[u_1^*, v_1^*]$ in the integration of equilibrium equations (1), it is convenient to express displacements $[u_1, v_1]$ in the form $[u_1, v_1] = -\alpha\, [\psi_{1,x} + \varphi_{1,y}, \psi_{1,y} - \varphi_{1,x}]$. Contributions of ψ_1 and w_0 in $[u_1, v_1]^*$ are one and the same in giving corrections to w(x, y, z) and transverse stresses (in fact, contribution of w_0 is through strain-displacement relations in static equilibrium equations and through constitutive relations in the z-integration of equilibrium equations. Since w_0 does not participate in the in-plane static equations, its contribution is through $[u_1, v_1]$ in the integrated in-plane equilibrium equations. Hence, w_0 in $[u_1, v_1]^*$ is replaced by ψ_1 (so as to be independent of w_0 used in strain-displacement relations) so that $[u_1, v_1, \varepsilon_{x1}, \varepsilon_{y1}, \gamma_{xy1}]^*$ are

$$u_1^* = -\alpha\, (2\psi_{1,x} + \varphi_{1,y}) + \gamma_{xz0} \quad (17a)$$
$$v_1^* = -\alpha\, (2\psi_{1,y} - \varphi_{1,x}) + \gamma_{yz0} \quad (17b)$$
$$\varepsilon_{x1}^* = \bar{\varepsilon}_{x1} + \alpha\, \gamma_{xz0,x} \quad (18a)$$
$$\varepsilon_{y1}^* = \bar{\varepsilon}_{y1} + \alpha \gamma_{xz0,y} \quad (18b)$$
$$\gamma_{xy1}^* = \bar{\gamma}_{xy1} + \alpha\, (\gamma_{xz0,y} + \gamma_{yz0,x}) \quad (18c)$$

In the above equations, $[\bar{\varepsilon}_{x1}, \bar{\varepsilon}_{y1}] = -\alpha^2\, [(2\psi_{1,xx} + \varphi_{1,xy}), (2\psi_{1,yy} - \varphi_{1,xy})]$, $\bar{\gamma}_{xy1} = -\alpha^2\, [(4\psi_{1,xy} + \varphi_{1,yy} - \varphi_{1,xx})]$.

From integration of equilibrium equations using the above strains along with $v_{1,x} = u_{1,y}$ in the equations (9), one gets reactive transverse stresses

$$\tau_{xz2}^* = \alpha\, (E'e_1^* + \mu\, \sigma_{z1})_{,x} \quad (19a)$$
$$\tau_{yz2}^* = \alpha\, (E'e_1^* + \mu\, \sigma_{z1})_{,y} \quad (19b)$$
$$\sigma_{z3} = \alpha\, (\tau_{xz2,x} + \tau_{yz2,y})^* \quad (19c)$$

One equation governing in-plane displacements (u_1, v_1), noting that σ_{z3} from eq. (16) is negative of the one from eq. (19c) due to $(f_{3,zz} + f_1) = 0$, is given by

$$\alpha\, \beta_1\, (\tau_{xz2,x} + \tau_{yz2,y})^* = G\, e_1 \quad (20)$$

From equations (18, 19c, 20), one gets

$$E'\, \beta_1\, \alpha^4\, \Delta\Delta\, (2\, \psi_1 + \psi_0/G) - \\ -G\alpha^2 \Delta\psi_1 - \mu\, \beta_1\, \alpha^2\, \Delta\sigma_{z1} = 0 \quad (21)$$

Above fourth order equation in ψ_1 along with $\Delta\varphi_1 = 0$ constitute a sixth order system to be solved with the following three conditions along x = constant edges (with analogue conditions along y = constant edges)

$$\psi_1 = 0 \text{ or } \tau^*_{xz2} = 0 \quad (22a)$$
$$u_1 = 0 \text{ or } \sigma_{x1} = T_{x1}(y) \quad (22b)$$
$$v_1 = 0 \text{ or } \tau_{xy1} = T_{xy1}(y) \quad (22c)$$

Note that 2-D variables $[\psi_1, \varphi_1]$, thereby, $[u_1, v_1]$ are determined from satisfying both static and integrated equilibrium equations of 3-D infinitesimal element.

With reference to solution of 3-D problem, above analysis in the determination of $[w_0, u_1, v_1]$ is in error in the transverse shear strain-displacement relations due to $[\tau_{xz}, \tau_{yz}] = f_2(z)\, [\tau_{xz2}, \tau_{yz2}]$, and in the constitutive relations due to $f_3(z)\sigma_{z3}$.

B. Initial solutions of primary extension problem

In a primary extension problem, the plate is subjected to symmetric normal stress $\sigma_{zo} = q_0(x, y)/2$, asymmetric shear stresses $[\tau_{xz1}, \tau_{yz1}] = \pm [T_{xz1}(x, y), T_{yz1}(x, y)]$ along top and bottom faces of the plate. Here, $\sigma_{zo} = q_0/2$ satisfying face condition does not participate in equilibrium equation of transverse stresses and the corresponding applied face shears $[T_{xz1}, T_{yz1}]$ are gradients of a given harmonic function $\tilde{\psi}_1$ so that $[T_{xz}, T_{yz}] = -\alpha [\tilde{\psi}_{1,x}, \tilde{\psi}_{1,y}]$. Transverse shear stresses and normal stress satisfying face conditions are $[\tau_{xz}, \tau_{yz}] = -\alpha z [\tilde{\psi}_{1,x}, \tilde{\psi}_{1,y}]$ and $\sigma_{z0} = q_0(x, y)/2$.

Since σ_{z0} does not participate in equilibrium equation of transverse stresses, $\tilde{\psi}_1$ remains as harmonic function in the integrated equilibrium equation of transverse stresses so that equilibrium equation of transverse stresses is ignored. With the inclusion of the above gradients of the known $\tilde{\psi}_1$ in the normal stresses, in-plane equilibrium equations (1) are

$$(E'/3)\,[\alpha^2 \Delta u_0 + \mu\, \alpha \sigma_{z0,x} - \tfrac{1}{2}(1+\nu)\, \alpha^2 (v_{0,x} - u_{0,y})_{,y}] = 0 \qquad (23a)$$
$$(E'/3)\,[\alpha^2 \Delta v_0 + \mu\, \alpha \sigma_{z0,y} + \tfrac{1}{2}(1+\nu)\, \alpha^2 (v_{0,x} - u_{0,y})_{,x}] = 0 \qquad (23b)$$

Above static equilibrium equations (23) along with two conditions (10) at x = constant edges (with analogue conditions along y = constant edges) have to be solved for u_0 and v_0. They remain same in the integrated equations.

The solutions of the above equations with reference to 3-D problem are in error in transverse shear strain-displacement relations due to $w = z\,\varepsilon_{z0}$ from constitutive relation. To rectify this error, one considers higher order in-plane displacement terms $f_2(z)[u_2, v_2]$ which, in turn, induce $[\tau_{xz1}, \tau_{yz1}]$ and zw_1. Displacements from strain-displacement relations consistent with the above $[\tau_{xz1}, \tau_{yz1}]$ and σ_z from equilibrium equation of transverse stresses take the form with $\varepsilon_{z0} = -\mu\, e_0 + (1 - 2\nu\mu)\, \sigma_{z0}/E$ in Eq. (6)

$$w = z\,(\varepsilon_{z0} + w_1)$$
$$u = u_0 + f_2\, u_2,\quad v = v_0 + f_2\, v_2,$$
$$\sigma_z = f_2\, \sigma_{z2} \qquad (24)$$

(Note that σ_{z2} is not priory known unlike $\sigma_{z1} = q/2$ in bending problem)

Since $w = z\,(\varepsilon_{z0} + w_1)$ as face variable should not participate in static equilibrium equations (1), displacements $[u_2, v_2]$ are modified in the form

$$u_2^* = u_2 - \alpha(\varepsilon_{z0} + w_1)_{,x} \qquad (25a)$$
$$v_2^* = v_2 - \alpha(\varepsilon_{z0} + w_1)_{,y} \qquad (25b)$$
$$[\tau_{xz1}, \tau_{yz1}] = -G[u_2, v_2] \qquad (26a)$$
$$\sigma_{z2} = G\,\alpha\,(u_{2,x} + v_{2,y}) \qquad (26b)$$

In order to keep $[\tau_{xz3}, \tau_{yz3}]$ as free variables in the integrated equilibrium equations, $f_3(z)$ is modified with $\beta_1 = 1/3$ as $f^*_3(z) = f_3(z) - \beta_1 z$ so that

$$\tau_{xz} = z\,(\tau_{xz1} - \beta_1\, \tau_{xz3}) + f_3\, \tau_{xz3} \qquad (27a)$$
$$\tau_{yz} = z\,(\tau_{yz1} - \beta_1\, \tau_{yz3}) + f_3\, \tau_{yz3} \qquad (27b)$$

From equilibrium equation of transverse stresses and $[\tau_{xz1}, \tau_{yz1}]$ from eq. (26) along with first term in equations (27), one gets

$$G\alpha\,(u_{2,x} + v_{2,y}) = \beta_1 \sigma_{z4} \qquad (28)$$

Strain-displacement relations give

$$\varepsilon_{x2}^* = \varepsilon_{x2} - \alpha^2\,(\varepsilon_{z0} + w_1)_{,xx} \qquad (29a)$$
$$\varepsilon_{y2}^* = \varepsilon_{y2} - \alpha^2\,(\varepsilon_{z0} + w_1)_{,yy} \qquad (29b)$$
$$\gamma^*_{xy2} = \gamma_{xy2} - 2\alpha^2(\varepsilon_{z0} + w_1)_{,xy} \qquad (29c)$$

For the use of $[u_2, v_2]^*$ in the integration of equilibrium equations (1), displacements $[u_2, v_2]$ are expressed in the form $[u_2, v_2] = -\alpha\,[\psi_{2,x} + \varphi_{2,y}, \psi_{2,y} - \varphi_{2,x}]$. Induced w_1 in $[u^*, v^*]_2$, like in EPT of bending problem, is replaced by ψ_2 so that $[u, v, \varepsilon_x, \varepsilon_y, \gamma_{xy}]^*_2$ are

$$u^*_2 = u_2 - \alpha\,(\psi_2 + \varepsilon_{z0})_{,x} \qquad (30a)$$
$$v^*_2 = v_2 - \alpha\,(\psi_2 + \varepsilon_{z0})_{,y} \qquad (30b)$$
$$\varepsilon^*_{x2} = \varepsilon_{x2} - \alpha^2\,(\psi_2 + \varepsilon_{z0})_{,xx} \qquad (31a)$$
$$\varepsilon^*_{y2} = \varepsilon_{y2} - \alpha^2\,(\psi_2 + \varepsilon_{z0})_{,yy} \qquad (31b)$$
$$\gamma^*_{xy2} = \gamma_{xy2} - 2\,\alpha^2(\psi_2 + \varepsilon_{z0})_{,xy} \qquad (31c)$$

Note that role of w_1 is in its contribution in the integrated equilibrium equations where as it is a virtual quantity in transverse strain-displacement relations since it should not alter ε_{zo} from constitutive relation.

From integration of equilibrium equations, reactive transverse stresses are

$$\tau^*_{xz3} = \alpha\,[\sigma^*_{x,x} + \tau^*_{xy,y}]_2 \qquad (32a)$$
$$\tau^*_{yz3} = \alpha\,[\sigma^*_{y,y} + \tau^*_{xy,x}]_2 \qquad (32b)$$
$$\sigma_{z4} = -\alpha\,(\tau_{xz,x} + \tau_{yz,y})^*_3 \qquad (33)$$

One equation governing in-plane displacements $(u, v)_2$, noting that σ_{z4} from eq. (28) is negative of the one from eq.(33) due to $(f_{3,zz} + f_1) = 0$, is given by

$$\alpha\, \beta_1(\tau_{xz,x} + \tau_{yz,y})^*_3 = G\alpha\,(u_{2,x} + v_{2,y}) \qquad (34)$$

With the second equation $v_{2,x} = u_{2,y}$, the above equation becomes a fourth order equation in ψ_2 to be solved along with harmonic function φ_2 with three conditions $u^*_2 = 0$ or $\sigma^*_{x2} = 0$, $v^*_2 = 0$ or $\tau^*_{xy2} = 0$ and $\psi_2 = 0$ or $\tau^*_{xz3} = 0$ along x = constant edges (with analogue conditions along y = constant edges)

With reference to solution of 3-D problem, above analysis in the determination of $[u_2, v_2, \varepsilon_{z2}]$ is in error in the transverse shear strain-displacement relations due to [τ_{xz}, τ_{yz}] = $f_3(z)$ [τ_{xz3}, τ_{yz3}], and in the constitutive relations due to $f_4(z)\sigma_{z4}$.

IV. RECTIFICATION OF ERRORS IN EPT

Disadvantage in the application of EPT is in the development of software for generation of $f_k(z)$ functions and β_{2k+1}, necessary for thickness ratio varying up to unit value. Errors in the analysis are due to statically equivalent transverse stresses associated with $f_2(z)$ and $f_3(z)$ in bending problems, and $f_3(z)$ and $f_4(z)$ in extension problems. In order to rectify these errors, it is more convenient to consider successive z-integrations of $f_1 = z$ in the suitable Fourier series expansion. For this purpose, we consider Fourier series of $f_1(z)$ in the form with $\lambda_n = 2/[(2n-1)\pi]$

$$f_1(z) = \sum A_n \sin (z/\lambda_n) \text{ (sum on n)} \quad (35)$$

in which

$$A_n = \int_0^1 z\sin (z/\lambda_n)\, dz = - (-1)^n \lambda_n^2 \quad (36)$$

Relevant [f_2, f_3, f_4] functions are expressed, for convenience, in the form

$$f_2(z) = \sum A_n \lambda_n \cos (z/\lambda_n) \quad (37a)$$
$$f_3(z) = \sum A_n \lambda_n^2 \sin (z/\lambda_n) \quad (37b)$$
$$f_4(z) = \sum A_n \lambda_n^3 \cos (z/\lambda_n) \quad (37c)$$

(Term by term differentiation in each of the above series is valid)

In the bending problem, in-plane displacements [u, v] due to σ_{z3} in constitutive relations are expressed as

$$[u, v] = \sum A_n \lambda_n^2 [u, v]_3 \sin (z/\lambda_n) \quad (38)$$

Correspondingly, transverse shear stresses are expressed as

$$\tau_{xz} = \sum A_n \lambda_n \cos(z/\lambda_n)\, \tau_{x2} \quad (39a)$$
$$\tau_{yz} = \sum A_n \lambda_n \cos(z/\lambda_n)\tau_{y2} \quad (39b)$$

Equations governing $[u, v]_3$ from equilibrium equations (9) with $v_{,x} = u_{,y}$ are

$$\lambda_n^2 [E' \alpha^2\Delta u_3 + \mu\, \alpha\sigma_{z3,x}] = \tau_{xz2} \quad (40a)$$
$$\lambda_n^2 [E' \alpha^2\Delta v_3 + \mu\, \alpha\sigma_{z3,y}] = \tau_{yz2} \quad (40b)$$

Above Poisson equations have to be solved with relevant homogeneous edge conditions.

In the extension problem, in-plane displacements [u, v] due to σ_{z4} in constitutive relations are expressed as

$$[u, v] = \sum A_n \lambda_n^3 \cos (z/\lambda_n) [u, v]_4 \quad (41)$$

Correspondingly, transverse shear stresses are expressed as

$$\tau_{xz} = \sum A_n \lambda_n^2 \sin (z/\lambda_n)\, \tau_{xz3} \quad (42a)$$
$$\tau_{yz} = \sum A_n \lambda_n^2 \sin (z/\lambda_n)\, \tau_{yz3} \quad (42b)$$

Equations governing $[u, v]_4$ from equilibrium equations (9) with $v_{,x} = u_{,y}$ are

$$\lambda_n^2 [E' \alpha^2\Delta u_4 + \mu\, \alpha\sigma_{z4,x}] + \tau_{xz3} = 0 \quad (43a)$$
$$\lambda_n^2 [E' \alpha^2\Delta v_4 + \mu\, \alpha\sigma_{z4,y}] + \tau_{yz3} = 0 \quad (43b)$$

Above Poisson equations have to be solved with relevant homogeneous edge conditions.

V. CONCLUDING REMARKS

Two term representation of displacements is mandatory for initial analysis of primary extension problems. The need for the use of higher order $f_k(z)$ polynomials for obtaining exact solutions of 3-D primary extension and bending problems is eliminated by the use of proper Fourier series expansion of displacements other than the basic variables in EPT of plates.

Acknowledgement

This is to express sincere thanks to M V AppaRao and Swarnalatha for their help in the publication of this paper.

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