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Membrane Solutions For Circular Toroidal Shells Under Internal Hydrostatic Pressure

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***Abstract*—Closed-form explicit results for membrane stresses and deformations in liquid-filled circular toroidal shells of revolution have been developed based on the linear theory of shells in this paper. For a toroidal mean radius of zero, the solutions coincide with those in literature for a spherical vessel under internal hydrostatic pressure. The analytical solution is adequate throughout the toroidal vessel, except in the zones surrounding the supports and the vicinity of the top and bottom circles of latitude, where additional bending will be required. The formulated results permit quick evaluation of membrane effects and the conduction of any desired parametric studies for shells of this type.**

Keywords—containment vessel; Circular toroids; internal hydrostatic pressure; membrane theory of shells; shell analysis

I. INTRODUCTION

Toroidal shells find applications in space, nuclear, under-water fields and containment vessels. Early investigations of these shell forms were based on pressure vessel application in mind [1-7]. It was noted that without additional bending, the membrane theory does not describe the state of stress and strain at the turning points of the toroidal shell where the curvature changes from positive to negative Gaussian curvature, even when loaded by a uniform pressure. Various approaches [4–6,8,9] were applied to obtain the exact solution valid for the entire toroid and for all opening ration A/a. This solution has proved to be very difficult. Depending on the kind of study, the formulation of toroidal shells has mainly been obtained through the Sanders shell theory [9] and Mushtari-Vlasov-Dennell shell theory [10,11]. Vibration analysis [12,13] and buckling investigation [14–16] of toroidal shells have been carried out using the Sanders shell theory. Various other studies [17–24] have been conducted on toroidal shells of revolution. Non-circular toroidal shells have also been studied [25–30]. With liquid-containment application in mind, this paper presents closed-form analytical expressions for the determination of membrane stresses and deformations in a relatively large liquid-filled circular toroidal shells of revolution, based on the linear theory of shells. For a toroidal limit $A=0$, the solutions coincide with those in literature for a spherical vessel under internal hydrostatic pressure. The results for the membrane stress resultants are also valid for non-uniform shell-thickness variations. A set of design recommendation is given. The formulated results can also be used for the conduction of any desired specialized studies for shells of this type.

II. GEOMETRY

A complete circular toroidal shell of revolution is generated by rotating a circular profile through 360^o about an axis lying outside the profile. This axis is referred to as the axis of revolution of the shell. Any given position of the circular profile as it is moved around the axis of revolution constitutes a *meridian* of the generated surface of revolution. Figure 1 shows the geometric parameters of a circular toroidal shell under internal hydrostatic pressure, where A is the toroid mean radius (the distance between the axis of rotation and the centre of the circular cross-section); a is the radius of the circular cross-section; t is the shell thickness; θ is the angular coordinate along the hoop circle of revolution for the toroid and ϕ is the angular coordinate along the meridian (measured from the vertical axis of the circular cross-section towards the outer surface the toroid). With this definition of ϕ, two points - one in the outer region and the other in the inner region of the toroid are located for one value of ϕ. To distinguish between these two points, when viewing the meridional profile to the right of the vertical axis of revolution Y-Y, we may consider all shells to the right side and left side of the local toroidal centreline y-y to be in the outer region of the toroid, while the shells to the left side of the local toroidal centreline y-y to be in the inter region of the toroid. This makes the segmented toroidal vessel to consist of four regions: the upper-outer, lower-outer, upper-inner, and lower-inner regions.

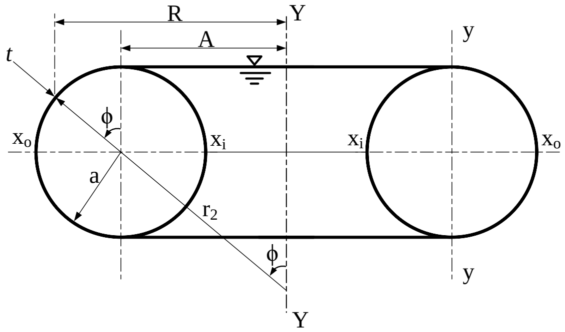

Fig. 1. *Geometric parameters of the liquid-filled circular toroidal shell*

The shape of the toroidal shell is characterized by r_1 and r_2 (the principal radii of curvature of the shell midsurface in the meridional plane and the second principal plane respectively):

$$r_1 = \pm a \tag{1}$$

$$r_2 = \frac{R}{\sin\phi} = \frac{A \pm a\sin\phi}{\sin\phi} \tag{2}$$

where for the sign ($\pm$), the plus sign applies to points in the outer region, while the minus sign applies to points in the inner regions of the toroid, and R is the horizontal coordinate measuring the distance between the vertical axis of revolution Y-Y and any point in the middle surface of the toroidal shell.

The vessel is assumed to be supported on axially stiff vertical columns at the extrados $(X_o - X_o)$ and intrados $(X_i - X_i)$ of the toroid. That is, the vertical supports are located at the outer and inner equatorial circles of latitude $(\phi = \pi/2)$ of the toroidal vessel, so that, the support reactions will be tangential to the shell middle surface. Therefore, the bending disturbances at the support regions will be minimal at these locations [1-5]. The support reactions on the shell are also assumed to be uniformly distributed around supports circumference. The latter assumption is justified, since the interposition of ring beams between the vertical columns (which are closely spaced) and the shell help to transfer the column reactions evenly to the shell, so that the support conditions of the vessel are essentially axisymmetric [1].

III. LOADING PRELIMINARIES

Assuming the vessel is completely filled with liquid of specific weight γ. The loading component p_r (per unit area of shell middle surface) due to the internal pressure loading from the contained liquid, acting normal to the shell middle surface is considered positive if pointing away from the axis of revolution of the shell (i.e outer region of the vessel), while negative if pointing towards the axis of revolution of the shell (i.e inner region of the vessel), may be expressed as:

$$p_r = \pm \gamma a(1 \mp \cos\phi) \tag{3}$$

where; for the double operations, the upper sign applies to the outer region, while the lower sign applies to the inner regions of the toroid.

For the axisymmetrically loaded vessel, the loading components p_ϕ and p_θ (per unit area of shell middle surface) in the meridional and hoop directions, respectively, are each equal to zero, since hydrostatic pressure acts purely perpendicular to the shell middle surface.

$$p_\phi = p_\theta = 0 \tag{4}$$

IV. MEMBRANE STRESS RESULTANTS

For shells of revolution subjected to distributed loadings that vary smoothly, continuously and 'not too rapidly' over the surface of the shell, the membrane hypothesis accurately predicts the state of stress in the interior of the shell, provided the shell geometry also exhibits the same smoothness proper-ties [6,34]. Both the loading and shell geometry of present considerations conform to these requirements. For toroidal shell geometry, the hydrostatic loading from the contained liquid is axisymmetric. The general expressions for the membrane stress resultants of axisymmetrically loaded shells of revolution have been presented [5,8]:

$$N_\phi = \frac{1}{r_2\sin^2\phi}\left[\int r_1 r_2 \left(p_r\cos\phi - p_\phi\sin\phi\right)\sin\phi d\phi\right] \tag{5}$$

$$N_\theta = r_2\left(p_r - \frac{N_\phi}{r_1}\right) \tag{6}$$

where; N_ϕ and N_θ are the membrane stress resultants in the meridional and hoop directions respectively. These are forces per unit length of the respective edge of a shell element, considered positive when tensile.

For the upper-outer region of the tank, using appropriate equations (1) - (4) to eliminate r_1, r_2, p_r and p_ϕ in expressing (5), and evaluating the integral, we obtain after some simplifications, the meridional stress resultant

$$\left(N_\phi\right)_1^o = \frac{\gamma a^2}{6\sin\phi(A + a\sin\phi)}\{a(-3 + 2\cos\phi)\cos^2\phi$$
$$-3A(\phi + (-2 + \cos\phi)\sin\phi) + C_1^o\} \tag{7}$$

where C_1^o is the constant of integration to be determined from a suitable boundary condition. At $\phi = 0$, $\left(N_\phi\right)_1^o$ must be zero. This condition gives $C_1^o = a$, so that

$$\left(N_\phi\right)_1^o = \frac{\gamma a^2}{6\sin\phi(A + a\sin\phi)}\{a(1 + (-3 + 2\cos\phi)\cos^2\phi)$$
$$-3A(\phi + (-2 + \cos\phi)\sin\phi)\} \tag{8}$$

With $\left(N_\phi\right)_1^o$ now known, the membrane stress resultant in the hoop direction follows from expression (6) which after eliminating r_1, r_2 and p_r for the upper-outer region of the vessel, may be written as

$$\left(N_\theta\right)_1^o = \frac{\gamma a}{6\sin^2\phi}\left\{4a(5 + 4\cos\phi)\sin^4\left(\frac{\phi}{2}\right) + 3A(\phi - \cos\phi\sin\phi)\right\} \tag{9}$$

For the lower-outer region of the vessel, N_ϕ must remain finite as $\phi \to \pi$. This condition gives $C_1^o = 5a + 3A\pi$, so that

$$(N_\phi)_2^o = \frac{\gamma a^2}{6\sin\phi(A + a\sin\phi)}\{a(5 + (-3 + 2\cos\phi)\cos^2\phi) + 3A(\pi - \phi - (-2 + \cos\phi)\sin\phi)\} \quad (10)$$

The membrane stress resultant in the hoop direction follows as:

$$(N_\theta)_2^o = -\frac{\gamma a}{12\sin^2\phi}\{a(1 + 6\cos\phi + 3\cos 2\phi - 2\cos 3\phi) + 6A(\pi - \phi + \cos\phi\sin\phi)\} \quad (11)$$

For the upper-inner region of the tank, using appropriate equations (1) - (4) to eliminate r_1, r_2, p_r and p_ϕ in expressing (5), and evaluating the integral, we obtain after some simplifications, the meridional stress resultant

$$(N_\phi)_1^i = \frac{\gamma a^2}{6\sin\phi(A - a\sin\phi)}\{a(3 + 2\cos\phi)\cos^2\phi + 3A(\phi + (2 + \cos\phi)\sin\phi) + C_1^i\} \quad (12)$$

where C_1^i is the constant of integration to be determined from a suitable boundary condition. At $\phi = \pi$, $(N_\phi)_1^i$ must be zero. This condition gives $C_1^i = -(a + 3A\pi)$, so that

$$(N_\phi)_1^i = \frac{\gamma a^2}{6\sin\phi(A - a\sin\phi)}\{a((1 + \cos\phi)^2(-1 + 2\cos\phi)) + 3A(-\pi + \phi + (2 + \cos\phi)\sin\phi)\} \quad (13)$$

With $(N_\phi)_1^i$ now known, the membrane stress resultant in the hoop direction follows from expression (6) which after eliminating r_1, r_2 and p_r for the upper-inner region of the vessel, may be written as

$$(N_\theta)_1^i = -\frac{\gamma a}{6\sin^2\phi}\{4a(-5 + 4\cos\phi)\cos^4\left(\frac{\phi}{2}\right) + 3A(\pi - \phi + \cos\phi\sin\phi)\} \quad (14)$$

For the lower-outer region of the vessel, N_ϕ must remain finite as $\phi \to 0$. This condition gives $C_1^i = -5a$, so that

$$(N_\phi)_2^i = \frac{\gamma a^2}{6\sin\phi(A - a\sin\phi)}\{a(-5 + (3 + 2\cos\phi)\cos^2\phi) + 3A(\phi + (2 + \cos\phi)\sin\phi)\} \quad (15)$$

The membrane stress resultant in the hoop direction follows as:

$$(N_\theta)_2^o = -\frac{\gamma a}{12\sin^2\phi}\{a(1 - 6\cos\phi + 3\cos 2\phi + 2\cos 3\phi) + 3A(-2\phi + \sin 2\phi)\} \quad (16)$$

As a check for $A = 0$, the vessel becomes a spherical tank, the above expressions for N_ϕ and N_θ at the upper and lower parts of the toroidal vessel respectively coincide with the well-known results for a spherical tank [5,8].

The difference in the values of N_ϕ (membrane stress resultants in the meridional direction) just below and above the supports ($\phi = \phi_s = \pi/2$) is

$$N_s = \frac{\gamma a^2(4a \pm 3A\pi)}{6\sin\phi_s(a\sin\phi_s \pm A)} = \frac{\gamma a^2(4a \pm 3A\pi)}{6(a \pm A)} \quad (17)$$

where for the sign ($\pm$), the plus sign applies to outer region, while the minus sign applies to the inner region of the toroid. N_s acts tangentially to the shell middle surface, in the direction of increasing ϕ for the outer region, while in the direction of decreasing ϕ for the inner region. N_s causes compressive or tensile actions in the vertical columns and ring beams depending on the region under consideration. The vertical columns and horizontal ring beams at the outer and inner regions of the toroid must be designed for their respective actions.

The actual membrane stresses σ_ϕ and σ_θ in the meridional and hoop directions respectively can be obtained from:

$$\sigma_\phi = \frac{N_\phi}{t}; \quad \sigma_\theta = \frac{N_\theta}{t} \quad (18)$$

V. MEMBRANE DEFORMATIONS

Displacements in shells are required for the assessment of deflections and distortions suffered by a shell under service condition; or relevant for evaluating edge or support effects by the flexibility method [5,9,10]. It is usually the horizontal displacement δ (considered positive when away from the axis of revolution) and the meridional rotation V at any point on the shell that need to be known. For axisymmetrically loaded shells of revolution, these deformations have been expressed in terms of membrane stress resultants N_ϕ and N_θ [1]:

$$\delta = \frac{1}{Et}(r_2\sin\phi)(N_\theta - \nu N_\phi) \quad (19)$$

$$V = \frac{1}{r_1}\left[\frac{\cot\phi}{Et}\{(r_1 + \nu r_2)N_\phi - (r_2 + \nu r_1)N_\theta\} - \frac{d}{d\phi}\left\{\frac{r_2}{Et}(N_\theta - \nu N_\phi)\right\}\right] \quad (20)$$

where E is the young's modulus of elasticity of the shell material, ν is its Poisson's ratio and other symbols are as previously defined.

A. Outer Region of the Vessel

When use is made of expressions (1), (3), (8), and (9) for the upper-outer region of the toroidal tank, the following closed-form explicit results for the deformations are obtained:

$$(\delta^m)_1^o = \frac{\gamma a}{6Et\sin\phi}\Bigg[\frac{(A+a\sin\phi)}{\sin\phi}\left\{4a(5+4\cos\phi)\sin^4\left(\frac{\phi}{2}\right)+3A(\phi-\cos\phi\sin\phi)\right\}$$

$$-a\nu\{a(1+(-3+2\cos\phi)\cos^2\phi)-3A(\phi+(-2+\cos\phi)\sin\phi)\}\Bigg] \quad (21)$$

and

$$(V^m)_1^o = -\frac{\gamma}{48Et\sin^4\phi(A+a\sin\phi)}\Bigg\{48a^3\sin^6\phi+96a^2A\sin^5\left(\frac{\phi}{2}\right)\Bigg(25\cos\left(\frac{\phi}{2}\right)$$

$$+11\cos\left(\frac{3\phi}{2}\right)+2\cos\left(\frac{5\phi}{2}\right)\Bigg)+aA^2(81-32\cos\phi-56\cos2\phi+7\cos4\phi$$

$$-36\phi\sin2\phi)+48A^3(-\phi\cos\phi+\sin\phi)\Bigg\} \quad (22)$$

Similarly, the deformations at the lower-outer region of the toroidal tank are:

$$(\delta^m)_1^o = \frac{\gamma a}{6Et\sin\phi}\Bigg[-\frac{(A+a\sin\phi)}{2\sin\phi}\{a(1+6\cos\phi+3\cos2\phi-2\cos3\phi)$$

$$+6A(\pi-\phi+cos\phi\sin\phi)\}-a\nu\{a(5+(-3+2\cos\phi)\cos^2\phi)$$

$$+3A(\pi-\phi-(-2+\cos\phi)\sin\phi)\}\Bigg] \quad (23)$$

and

$$(V^m)_1^o = -\frac{\gamma}{48Et\sin^4\phi(A+a\sin\phi)}\{48a^3\sin^6\phi+3a^2A(36\sin\phi+6\sin2\phi-10\sin3\phi$$

$$+\sin4\phi+2\sin5\phi)+aA^2(81+32cos\phi-56\cos2\phi+7\cos4\phi$$

$$+36(\pi-\phi)\sin2\phi+48A^3((\pi-\phi)\cos\phi+\sin\phi)\} \quad (24)$$

B. Inner Region of the Vessel

Using appropriate expressions (1), (2), (13), and (14) for the upper-inner region of the toroidal tank, the following closed-form explicit results for the deformations are obtained:

$$(\delta^m)_1^i = \frac{\gamma a}{6Et\sin\phi}\Bigg[-\frac{(A-a\sin\phi)}{\sin\phi}\left\{4a(-5+4\cos\phi)\cos^4\left(\frac{\phi}{2}\right)+3A(\pi+\cos\phi\sin\phi-\phi)\right\}$$

$$-a\nu\{a((1+\cos\phi)^2(-1+2\cos\phi))+3A(-\pi+\phi+(2+\cos\phi)\sin\phi)\}\Bigg] \quad (25)$$

and

$$(V^m)_1^i = \frac{\gamma}{48Et\sin^4\phi(A-a\sin\phi)}\Bigg\{-48a^3\sin^6\phi+96a^2A\cos^5\left(\frac{\phi}{2}\right)\Bigg(25\sin\left(\frac{\phi}{2}\right)$$

$$-11\sin\left(\frac{3\phi}{2}\right)+2\sin\left(\frac{5\phi}{2}\right)\Bigg)-aA^2(81+32\cos\phi-56\cos2\phi+7\cos4\phi$$

$$+36(\pi-\phi)\sin2\phi)+48A^3((\pi-\phi)\cos\phi+\sin\phi)\Bigg\} \quad (26)$$

Similarly, the deformations for the lower-inner region of the toroidal tank are:

$$(\delta^m)_1^o = \frac{\gamma a}{6Et\sin\phi}\Bigg[-\frac{(A-a\sin\phi)}{2\sin\phi}\{a(1-6\cos\phi+3\cos2\phi+2\cos3\phi)+3A(-2\phi$$

$$+\sin2\phi)\}-a\nu\{a(-5+(3+2\cos\phi)\cos^2\phi)$$

$$+3A(\phi+(2+\cos\phi)\sin\phi)\}\Bigg] \quad (27)$$

$$(V^m)_1^o = -\frac{\gamma}{48Et\sin^4\phi(A-a\sin\phi)}\{48a^3\sin^6\phi+3a^2A(-36\sin\phi+6\sin2\phi+10\sin3\phi$$

$$+\sin4\phi-2\sin5\phi)+aA^2(81-32cos\phi-56\cos2\phi+7\cos4\phi-36\phi\sin2\phi)$$

$$+48A^3(\phi\cos\phi-\sin\phi)\} \quad (28)$$

VI. NUMERICAL EXAMPLE AND DISCUSSION OF RESULTS

Let us consider a relatively large circular toroidal vessel made from steel plate of Young modulus $E = 200\times10^9\,N/m^2$, Poisson ratio $\nu = 0.3$ and constant thickness $t = 0.05m$ throughout. The vessel of toroidal mean radius $A = 30m$ and circular cross sectional radius $a = 15m$, is assumed to be completely filled with water of unit weight $\gamma = 10\times10^3\,N/m^3$. The variations of membrane stresses over the circular profile $(20^o \le \phi \le 160^o)$ of the toroidal vessel are shown in Figure 2. The values below $\phi = 20^o$ and beyond $\phi = 160^o$ are not presented, as it is well-known, membrane solution cannot be used to estimate the state of stress in the vicinity of the top and bottom circles of latitude without additional bending due to the incompatibility of deformations at the meeting points of the synclastic and anticlastic surfaces of the toroidal shell.

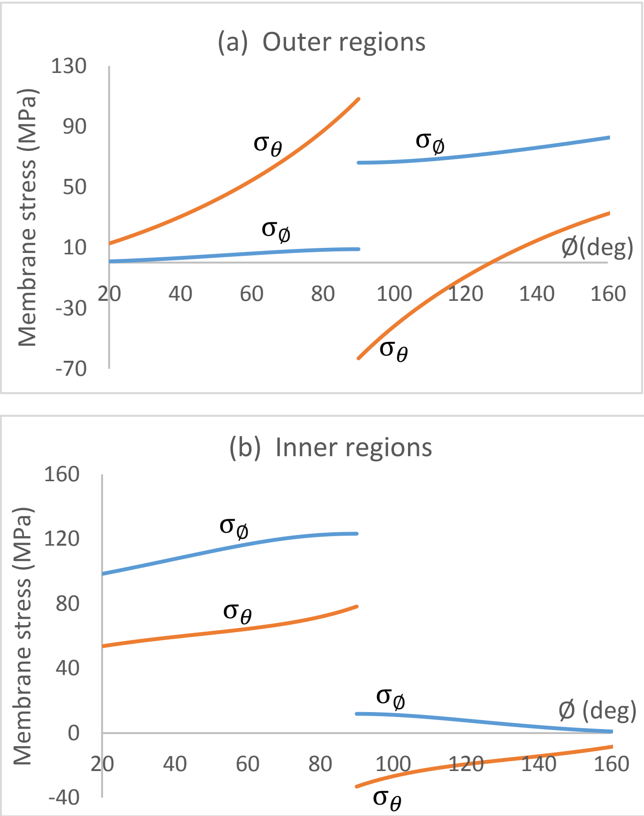

Fig. 2. *Variations of membrane stresses over the circular profile of the toroidal vessel: (a) outer regions, and (b) inner regions*

The σ_ϕ and σ_θ in the outer region of the vessel appreciably rise in tension as one moves from the apex towards the outer support $\phi = 90^o$, where these stresses attain values of 8.94 MPa and 108.19 MPa respectively. The corresponding membrane deformations δ and V are 23.74 mm and -2.03×10^3 respectively. As expected at the support junction between the upper-outer and lower-outer regions, there are discontinuities in the meridional stress and hoop stress in moving across the support junction, where the stress values become 66.06 MPa and -63.19 MPa respectively at the lower-outer region of the vessel, while the deformations δ and V are -18.68 mm and -2.03×10^3 (compressive) respectively. The σ_ϕ continues to rise, but gradually in tension as one moves from the support location towards the base of the vessel, while the σ_θ changes from being compressive to tensile at $\phi = 128^o$.

Similarly, the σ_ϕ and σ_θ in the inner region of the vessel gradually rise in tension as one moves from the bottom towards the inner support $\phi = 90^o$, where these stresses attain values of 123.19 MPa and 78.19 MPa respectively. The corresponding membrane deformations δ and V are -27.55 mm and 0.23×10^3 respectively. As also expected at the support junction between the upper- inner and lower- inner regions, there are discontinuities in the meridional stress and hoop stress in moving across the support junction, where the stress values become 11.81 MPa and -33.19 MPa (compressive) respectively at the upper-inner region of the vessel, while the deformations δ and V are 3.09 mm and 0.23×10^3 respectively. The σ_ϕ continues to decrease gradually in tension as one moves from the support location towards the apex of the vessel, while the σ_θ continued to be compressive.

It should be noted that V at the outer support level $\phi = 90^o$ is the same for the upper and lower regions of the vessel. This is also the case for the values of V at the inner support level of the vessel. But the δ values are different, just as the values of the membrane stresses σ_ϕ and σ_θ, implying a broken middle surface. Since the shell is physically continuous throughout, bending corrective actions are required in order to restore continuity so that the support circles are sources of bending disturbances. The magnitude of the bending disturbance is a function of support location, as earlier mentioned and the stiffness properties of the support ring beams [1, 5]. To cater for the localized bending stresses on either side of the support circles, stepwise local thickening of the shell should be adopted in a narrow band around the supports.

VII. CONCLUSION

Closed-form analytical expressions for membrane stresses and deformations in liquid-filled circular toroidal shells of revolution have been presented based on linear theory of shells. For a toroidal limit $A = 0$, the solutions coincide with those in literature for a spherical vessel under internal hydrostatic pressure. The results for the membrane stress resultants are also valid for non-uniform shell-thickness variations. A set of design recommendation has been given. The formulated results can also be used for the conduction of any desired specialized studies for shells of this type.

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