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Analytical and Numerical Solutions to a Rotating FGM Disk

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***Abstract*—Analytical and numerical solutions to a rotating uniform thickness functionally graded (FGM) disk are obtained. Solid and annular disk geometries are taken into consideration. The modulus of elasticity of the disk material is assumed to vary in the radial direction. A new one-parameter exponential model is used to express the variation of the modulus of elasticity. The results of the solutions are presented in tables and figures. Those presented in tables may form benchmark data for purely numerical calculations.**

Keywords—Rotating disk, Functionally graded material, von Mises criterion

I. Introduction

Research on the prediction of stress and deformation in rotating or stationary disks under different loading conditions and comprising different materials is unending because of the importance of these basic structures in various branches of engineering. Plane stress analytical solutions for rotating solid and annular disk problems in the elastic state of stress have been available for many years in standard textbooks [1-5]. Solutions involving thickness variability, partially plastic stress states, and material nonhomogeneity relevant to this investigation may be found in the most recent articles by Afsar and Go [6], Allam et al. [7], Apatay and Eraslan [8], Arani et al. [9], Argeso [10], Asghari and Ghafoori [11], Bagri and Eslami [12], Bayat et al. [13,14], Calderale et al. [15], Damircheli and Azadi [16], Durodola and Attia [17], Eraslan et al. [18], Eraslan and Akis [19], Eraslan [20], Eraslan et al. [21], Eraslan and Orcan [22-24], Eraslan [25], Eraslan and Argeso [26], Farshi et al. [27], Go et al. [28], Hassani et al. [29,30], Jafari et al. [31], Kordkheili and Naghdabadi [32], Nie and Batra [33], Peng and Li [34], Tutuncu and Temel [35,36], Vullo and Vivio [37], Vivio and Vullo [38], You et al. [39,40], You et al. [41], and Zenkour [42,43].

In this work, analytical and numerical solutions are obtained for a rotating uniform thickness FGM disk. The problem uses a one-parameter exponential variation [44] given by

$$\varphi(r) = \varphi_0 exp\left[\frac{\beta(r-a)}{b-a}\right]. \quad (1)$$

where β is the parameter, r the radial coordinate, a and b the inner and outer radii of the disk and φ_0 the value of φ at $r = a$. The variation of $\varphi(r)$ in a solid disk of unit radius for different values of parameter β is presented in Fig. 1. As seen in this figure, for $\beta > 0$, φ is an increasing, while it is a decreasing function of the radial coordinate for values of $\beta < 0$. In FGM disks studied here φ denotes the modulus of elasticity of the disk material. Similar variation has been used earlier in pressure chamber studies of Chen and Lin [44] but new in disk studies.

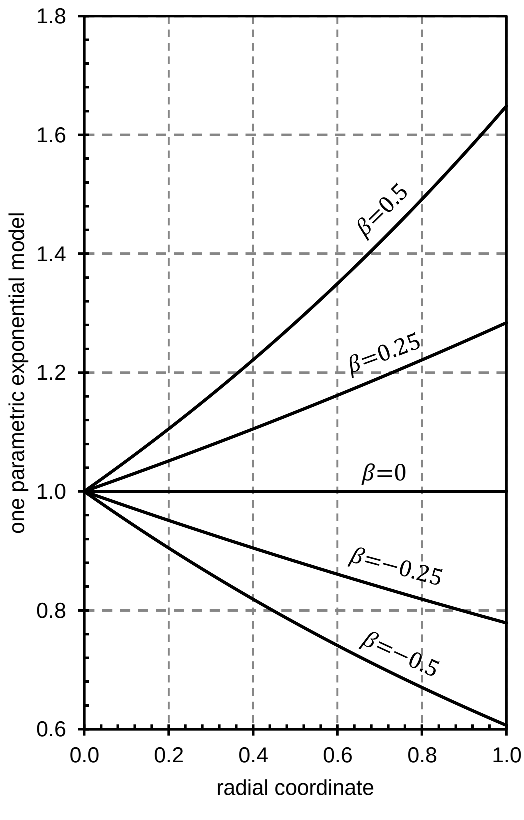

Fig. 1. *Variation of one-parameter exponential mode $\varphi(r)$ for different values of β.*

II. Mathematical Model

A. *Basic Equations*

Thin disk and hence a state of plane stress is assumed. The modulus of elasticity E of the disk material varies according to

$$E(r) = E_0 exp\left[\frac{\beta(\bar{r}-\bar{a})}{1-\bar{a}}\right]. \tag{2}$$

The strain displacement relations

$$\bar{\epsilon}_r = \frac{d\bar{u}}{d\bar{r}} \quad \text{and} \quad \bar{\epsilon}_\theta = \frac{\bar{u}}{\bar{r}} \tag{3}$$

the equation of equilibrium

$$\frac{d}{d\bar{r}}(\bar{r}\bar{\sigma}_r) - \bar{\sigma}_\theta + \Omega^2\bar{r}^2 = 0, \tag{4}$$

the equations of the generalized Hooke's Law

$$\bar{\epsilon}_r = \frac{1}{exp\left[\frac{\beta(\bar{r}-\bar{a})}{1-\bar{a}}\right]}[\bar{\sigma}_r - \nu\bar{\sigma}_\theta], \tag{5}$$

$$\bar{\epsilon}_\theta = \frac{1}{exp\left[\frac{\beta(\bar{r}-\bar{a})}{1-\bar{a}}\right]}[\bar{\sigma}_\theta - \nu\bar{\sigma}_r], \tag{6}$$

and the compatibility relation

$$\frac{d}{d\bar{r}}(\bar{r}\bar{\epsilon}_\theta) - \bar{\epsilon}_r = 0, \tag{7}$$

constitute the basic equations of the model in their dimensionless forms [2]. In these equations, $\bar{\epsilon}_r = \epsilon_r E_0/\sigma_0$ and $\bar{\epsilon}_\theta = \epsilon_\theta E_0/\sigma_0$ represent the normalized strains, $\bar{r} = r/b$ the dimensionless radial coordinate, $\bar{a} = a/b$ the dimensionless inner radius, $\bar{u} = uE_0/b\sigma_0$ the normalized and dimensionless radial displacement, $\bar{\sigma}_r = \sigma_r/\sigma_0$ and $\bar{\sigma}_\theta = \sigma_\theta/\sigma_0$ the dimensionless stresses, ν Poisson's ratio, $\Omega = \omega b\sqrt{\rho/\sigma_0}$ the dimensionless angular speed, ω the angular speed, ρ the mass density, and σ_0 the yield limit. Thereafter, overbars will not be used for simplicity. Introducing a stress function $F(r)$ of the form

$$F(r) = r\sigma_r, \tag{8}$$

we find from the equation of equilibrium, Eq. (4)

$$\sigma_\theta = \Omega^2r^2 + \frac{d}{dr}\left(r\frac{F}{r}\right) = \Omega^2r^2 + \frac{dF}{dr}, \tag{9}$$

so that the stress function F satisfies the equation of equilibrium. The equations of the generalized Hooke's Law now become

$$\epsilon_r = \frac{1}{exp\left[\frac{\beta(r-a)}{1-a}\right]}\left[\frac{F}{r} - \nu\left(\Omega^2r^2 + \frac{dF}{dr}\right)\right], \tag{10}$$

$$\epsilon_\theta = \frac{1}{exp\left[\frac{\beta(r-a)}{1-a}\right]}\left[\Omega^2r^2 + \frac{dF}{dr} - \nu\frac{F}{r}\right]. \tag{11}$$

Substituting the strains in terms of F into the compatibility relation, Eq. (7), leads to the governing differential equation

$$\frac{d^2F}{dr^2} + \left(\frac{1}{r} - \frac{\beta}{1-a}\right)\frac{dF}{dr} - \left(\frac{1}{r} - \frac{\beta\nu}{1-a}\right)\frac{F}{r} = Q_{FGM}, \tag{12}$$

where

$$Q_{FGM}(r) = -\left(3 - \frac{r\beta}{1-a} + \nu\right)r\Omega^2. \tag{13}$$

B. Analytical Solution

The governing equation, Eq. (12), is a second order, nonhomogeneous, linear ordinary differential equation with variable coefficients. The general solution is obtained by the power series method. The solution can be put into the form

$$F(r) = C_1F_1(r) + C_2F_2(r) + F_P(r), \tag{14}$$

where C_1 and C_2 are arbitrary integration constants and

$$F_1(r) = r + \frac{\beta(1-\nu)}{3(1-a)}r^2 + \frac{\beta^2(1-\nu)(2-\nu)}{24(1-a)^2}r^3 + \frac{\beta^3(1-\nu)(2-\nu)(3-\nu)}{360(1-a)^3}r^4 + \frac{\beta^4(1-\nu)(2-\nu)(3-\nu)(4-\nu)}{8640(1-a)^4}r^5 + \frac{\beta^5(1-\nu)(2-\nu)(3-\nu)(4-\nu)(5-\nu)}{302400(1-a)^5}r^6 + \cdots, \tag{15}$$

$$F_2(r) = lnr\left[\frac{\beta^2\nu(1+\nu)}{(1-a)^2}r + \frac{\beta^3\nu(1+\nu)(1-\nu)}{3(1-a)^3}r^2 + \frac{\beta^4\nu(1+\nu)(1-\nu)(2-\nu)}{24(1-a)^4}r^3 + \frac{\beta^5\nu(1+\nu)(1-\nu)(2-\nu)(3-\nu)}{360(1-a)^5}r^4 + \frac{\beta^6\nu(1+\nu)(1-\nu)(2-\nu)(3-\nu)(4-\nu)}{8640(1-a)^6}r^5 + \cdots\right] - \frac{2}{r} - \frac{2\beta(1+\nu)}{1-a} - \frac{\beta^2(1+2\nu)}{(1-a)^2}r - \frac{\beta^3(3+4\nu-9\nu^2-4\nu^3)}{9(1-a)^3}r^2 - \frac{\beta^4(24+26\nu-97\nu^2-2\nu^3+25\nu^4)}{288(1-a)^4}r^3 + \cdots, \tag{16}$$

$$F_P(r) = -\Omega^2r^3\left[\frac{1}{8}(3+\nu) + \frac{\beta(1-\nu^2)}{120(1-a)}r + \frac{\beta^2(4-\nu)(1-\nu^2)}{2880(1-a)^2}r^2 + \frac{\beta^3(5-\nu)(4-\nu)(1-\nu^2)}{100800(1-a)^3}r^3 + \cdots\right]. \tag{17}$$

Remark: For $\beta = 0$ the solution by Eq. (14) takes the form

$$F(r) = Ar - \frac{2B}{r} - \frac{1}{8}(3+\nu)\Omega^2r^3, \tag{18}$$

or

$$F(r) = C_1r + \frac{C_2}{r} - \frac{1}{8}(3+\nu)\Omega^2r^3. \tag{19}$$

Then from Eq. (11)

$$\epsilon_\theta = (1-\nu)C_1 - \frac{(1+\nu)C_2}{r^2} - \frac{1}{8}(1-\nu^2)\Omega^2r^2, \tag{20}$$

and from Eq. (3)

$$u = (1-\nu)C_1r - \frac{(1+\nu)C_2}{r} - \frac{1}{8}(1-\nu^2)\Omega^2r^3, \tag{21}$$

or

$$u = Ar + \frac{B}{r} - \frac{1}{8}(1-\nu^2)\Omega^2r^3, \tag{22}$$

which is the well known solution for a uniform thickness homogeneous disk [5].

The series in Eqs. (15)-(17) simplify notably if the Poisson's ratio ν is assigned to a numerical value. As an example, for $\nu = 3/10$ they take the forms:

$$F_1(r) = r + \frac{7\beta}{30(1-a)}r^2 + \frac{119\beta^2}{2400(1-a)^2}r^3 + \frac{357\beta^3}{40000(1-a)^3}r^4 + \frac{4403\beta^4}{3200000(1-a)^4}r^5 + \frac{29563\beta^5}{160000000(1-a)^5}r^6 + \cdots, \tag{23}$$

$$F_2(r) = lnr\left[\frac{39\beta^2}{100(1-a)^2}r + \frac{91\beta^3}{1000(1-a)^3}r^2 + \frac{1547\beta^4}{80000(1-a)^4}r^3 + \frac{13923\beta^5}{4000000(1-a)^5}r^4 + \frac{171717\beta^6}{320000000(1-a)^6}r^5 + \cdots\right] - \frac{2}{r} - \frac{13\beta}{5(1-a)} - \frac{8\beta^2}{5(1-a)^2}r - \frac{547\beta^3}{1500(1-a)^3}r^2 - \frac{15479\beta^4}{192000(1-a)^4}r^3 + \cdots, \quad (24)$$

and

$$F_P(r) = -\Omega^2 r^3\left[\frac{33}{80} + \frac{91\beta}{12000(1-a)}r + \frac{3367\beta^2}{2880000(1-a)^2}r^2 + \frac{22607\beta^3}{144000000(1-a)^3}r^3 + \cdots\right]. \quad (25)$$

The rapid convergent nature of these series is obvious as $\nu \approx 3/10$ always.

In the case of a rotating solid disk, the stresses must be finite at the center, hence $C_2 = 0$. The outer boundary is free of traction, and as a result, $\sigma_r(1) = 0$. Then, we find from Eq. (14)

$$C_1 = -\frac{F_P(1)}{F_1(1)} \quad (26)$$

It should be noted that, for a solid disk

$$\sigma_r(0) = \lim_{r\to 0}\left(\frac{F}{r}\right) = \frac{dF}{dr}. \quad (27)$$

The boundary conditions for a rotating annular disk are $\sigma_r(a) = \sigma_r(1) = 0$, which leads to

$$C_1 = \frac{F_2(a)F_P(1) - F_2(1)F_P(a)}{F_1(a)F_2(1) - F_1(1)F_2(a)};$$

$$C_2 = \frac{F_1(a)F_P(1) - F_1(1)F_P(a)}{F_1(a)F_2(1) - F_1(1)F_2(a)}. \quad (28)$$

C. Numerical Solution

The governing equation is written as

$$\frac{d^2F}{dr^2} = -\left(\frac{1}{r} - \frac{\beta}{1-a}\right)\frac{dF}{dr} + \left(\frac{1}{r} - \frac{\beta\nu}{1-a}\right)\frac{F}{r} + Q_{FGM}(r). \quad (29)$$

If we let $\phi_1 = F$, and $\phi_2 = dF/dr$, then by differentiating

$$\frac{d\phi_1}{dr} = \frac{dF}{dr} = \phi_2,$$

$$\frac{d\phi_2}{dr} = \frac{d^2F}{dr^2} = -\left(\frac{1}{r} - \frac{\beta}{1-a}\right)\frac{dF}{dr} + \left(\frac{1}{r} - \frac{\beta\nu}{1-a}\right)\frac{F}{r} + Q_{FGM}(r), \quad (30)$$

or

$$\frac{d\phi_1}{dr} = \phi_2,$$

$$\frac{d\phi_2}{dr} = -\left(\frac{1}{r} - \frac{\beta}{1-a}\right)\phi_2 + \left(\frac{1}{r} - \frac{\beta\nu}{1-a}\right)\frac{\phi_1}{r} + Q_{FGM}(r). \quad (31)$$

In this way, the governing equation is transformed into an initial value problem (IVP) consisting of two dependent variables. This IVP can accurately be integrated by using a state of the art ODE solver, starting with the initial conditions: ϕ_1^0, and ϕ_2^0. Since $\phi_1 = F = r\sigma_r$, for both solid and annular disks $\phi_1^0 = 0$ and $\phi_1(1) = 0$, but ϕ_2^0 is not known. This unknown initial value can be determined by shooting iterations. The condition that should be satisfied is

$$\phi_1(1) = 0.$$

Hence, iterations begin with an initial estimate ϕ_2^0 and continue until $\phi_1(1) = 0$ is satisfied.

At the $k-th$ iteration, the IVP described by Eq. (31) is solved 3 times: starting with ϕ_2^{k-1} to obtain $\Phi_1 = \phi_1(1)$, with $\phi_2^{k-1} + \Delta\phi$ to obtain $\Phi_2 = \phi_1(1)$, and finally with $\phi_2^{k-1} - \Delta\phi$ to obtain $\Phi_3 = \phi_1(1)$, where $\Delta\phi$ is a small increment of the order of ~10⁻³. A better approximation to ϕ_2^0 is then acquired from Newton's formula

$$\phi_2^0 = \phi_2^k = \phi_2^{k-1} - \frac{2\Delta\phi\Phi_1}{\Phi_2 - \Phi_3}. \quad (32)$$

When the iterations converge, the IVP system in Eq. (31) is solved once more with the converged ϕ_2^0 value in order to determine the stress and deformations. The Runge-Kutta-Fehlberg fourth-fifth order integration method is used with tight tolerances for the integration of the IVP.

The advantages of this method are the accuracy, stability, and rapid rate of convergence. With a reasonable initial estimate ϕ_2^0 only a few iterations are performed to reach convergence.

III. PRESENTATION OF RESULTS

In the following calculations $\nu = 0.3$. The von Mises yield criterion given by

$$\sigma_Y = \sqrt{\sigma_r^2 - \sigma_r\sigma_\theta + \sigma_\theta^2},$$

is used to determine the elastic limit of the disk [20]. As σ_r and σ_θ are dimensionless, the elastic limit corresponds to $\sigma_Y = 1$.

A. Solid Disk

Calculations are performed in order to determine the elastic limit of a rotating uniform thickness homogeneous solid disk. This is achieved by taking $\beta = 0$. The elastic limit angular speed is determined as $\Omega = 1.55700$, which is in perfect agreement with that reported by Eraslan [20]. The corresponding integration constants are $C_1 = 1.00000$ and $C_2 = 0.0$. In a related investigation by Eraslan and Akis [19], lower stresses are observed when the modulus of elasticity is an increasing function of radial coordinate. In this regard, calculations are performed for solid disks taking $\beta = 0.25$ and $\beta = 0.5$ at the speed $\Omega = 1.55700$. For $\beta = 0.25$, the nonzero integration constant is obtained as $C_1 = 0.946499$ and it is $C_1 = 0.893551$ for $\beta = 0.5$. Numerical solutions are also realized at the speed $\Omega = 1.55700$ for $\beta = 0.0$, $\beta = 0.25$ and $\beta = 0.5$. On average, 3 iterations are performed to reach converged ϕ_2^0. The results of these calculations are presented in Figs. 2 - 4 and in Table 1.

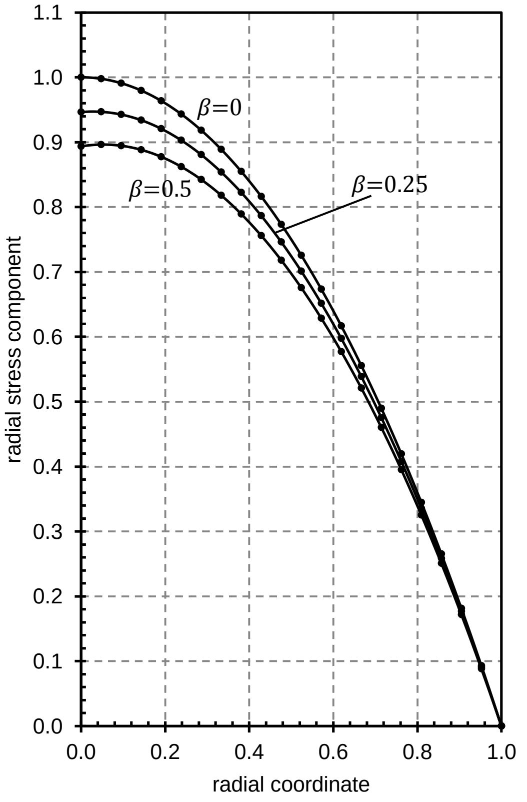

Fig. 2. *Variation of radial stress in a rotating FGM solid disk for different values of β. Solid lines belong to analytical, dots to numerical solutions.*

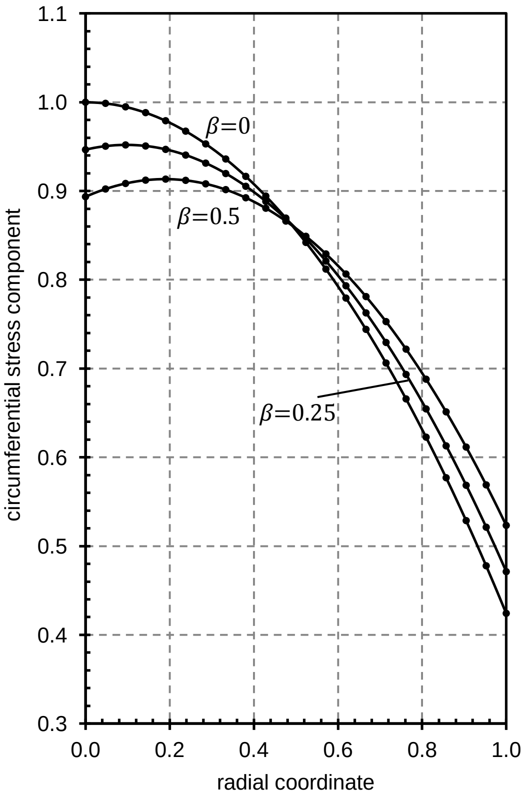

Fig. 3. *Variation of circumferential stress in a rotating FGM solid disk for different values of β. Solid lines belong to analytical, dots to numerical solutions.*

In all figures solid lines belong to analytical, while dots numerical solutions. Fig. 2 shows the variation of the radial stress, Fig. 3 circumferential stress and Fig. 4 radial displacement corresponding to different values of grading parameter β. The agreement between analytical and numerical solutions is obvious. Some selected points of the analytical solutions are also provided in Table 1.

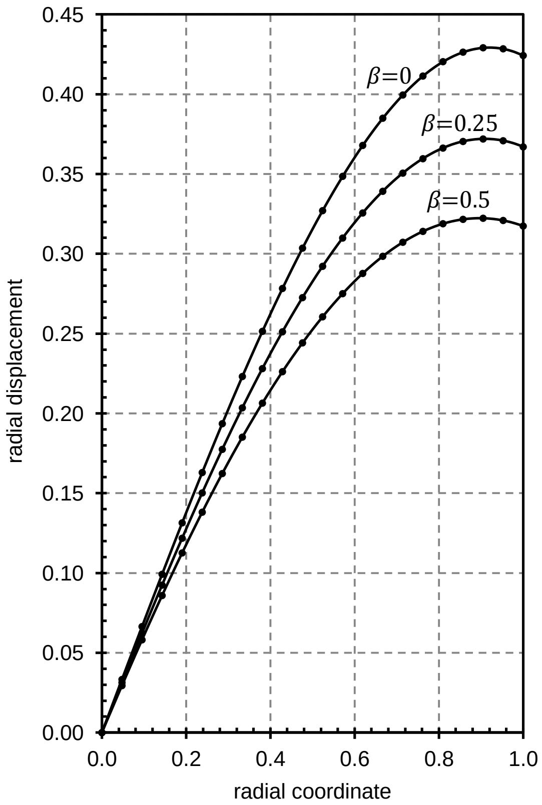

Fig. 4. *Variation of radial displacement in a rotating FGM solid disk for different values of β. Solid lines belong to analytical, dots to numerical solutions.*

B. Annular Disk

An annular disk with dimensionless inner radius $a = 0.2$ is considered. Assigning $\beta = 0$, calculations are performed to determine the elastic limit angular speed of this disk. The result turns out at $\Omega = 1.09632$ corresponding to the constants $C_1 = 0.515625$ and $C_2 = 0.00992$. Analytical calculations are also performed for $\beta = 0.25$ and $\beta = 0.5$ at the same speed. The integration constants corresponding to $\beta = 0.25$ and $\beta = 0.5$ are calculated as $C_1 = 0.486827$; $C_2 = 0.00874$ and $C_1 = 0.458594$; $C_2 = 0.00767$, respectively. The results of these calculations are compared with numerical ones in Figs. 5-7. Selected points of the analytical solutions are tabulated in Table 2.

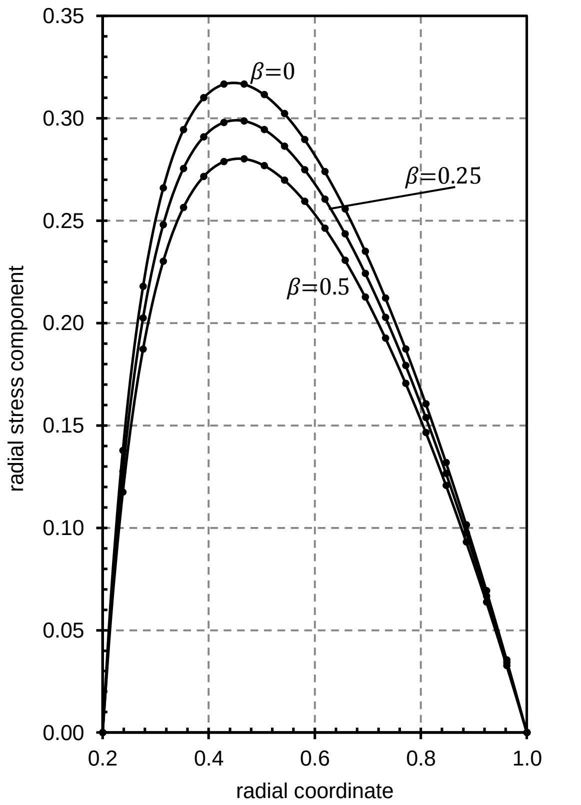

Fig. 5. *Variation of radial stress in a rotating FGM annular disk of $a = 0.2$ for different values of β. Solid lines belong to analytical, dots to numerical solutions.*

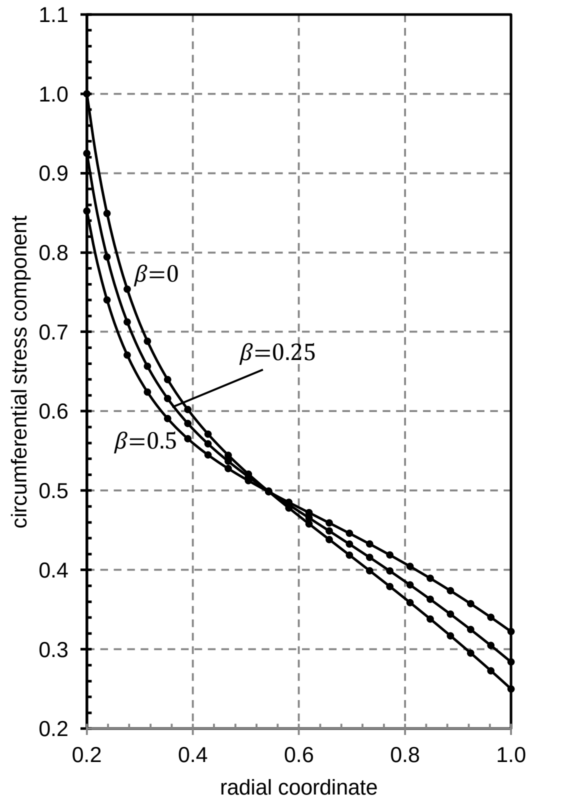

Fig. 6. *Variation of circumferential stress in a rotating FGM annular disk of $a = 0.2$ for different values of β. Solid lines belong to analytical, dots to numerical solutions.*

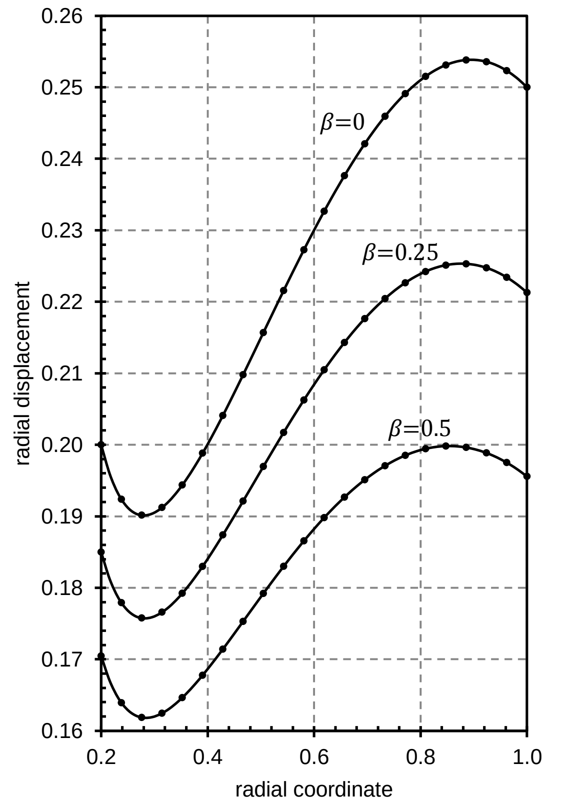

Fig. 7. *Variation of radial displacement in a rotating FGM annular disk of $a = 0.2$ for different values of β. Solid lines belong to analytical, dots to numerical solutions.*

TABLE I. ANALYTICAL SOLUTIONS TO ROTATING FGM SOLID DISKS.

r	σ_r			σ_θ			u		
	$\beta=0$	$\beta=0.25$	$\beta=0.5$	$\beta=0$	$\beta=0.25$	$\beta=0.5$	$\beta=0$	$\beta=0.25$	$\beta=0.5$
0.00	1.00000	0.94650	0.89355	1.00000	0.94650	0.89355	0.00000	0.00000	0.00000
0.05	0.99750	0.94677	0.89629	0.99856	0.95060	0.90262	0.03497	0.03291	0.03090
0.10	0.99000	0.94205	0.89408	0.99425	0.95185	0.90894	0.06972	0.06527	0.06095
0.15	0.97750	0.93233	0.88691	0.98705	0.95025	0.91251	0.10407	0.09688	0.08996
0.20	0.96000	0.91762	0.87478	0.97697	0.94576	0.91328	0.13779	0.12756	0.11778
0.25	0.93750	0.89791	0.85767	0.96402	0.93839	0.91124	0.17069	0.15712	0.14427
0.30	0.91000	0.87320	0.83560	0.94818	0.92811	0.90636	0.20256	0.18540	0.16931
0.35	0.87750	0.84349	0.80853	0.92947	0.91492	0.89861	0.23318	0.21225	0.19275
0.40	0.84000	0.80876	0.77648	0.90788	0.89879	0.88796	0.26235	0.23749	0.21451
0.45	0.79750	0.76902	0.73943	0.88341	0.87972	0.87439	0.28987	0.26098	0.23449
0.50	0.75000	0.72427	0.69738	0.85606	0.85768	0.85786	0.31553	0.28258	0.25258
0.55	0.69750	0.67449	0.65031	0.82584	0.83268	0.83835	0.33912	0.30214	0.26873
0.60	0.64000	0.61969	0.59823	0.79273	0.80468	0.81583	0.36044	0.31955	0.28286
0.65	0.57750	0.55987	0.54112	0.75674	0.77367	0.79026	0.37927	0.33466	0.29490
0.70	0.51000	0.49501	0.47898	0.71788	0.73965	0.76161	0.39542	0.34737	0.30481
0.75	0.43750	0.42512	0.41180	0.67614	0.70259	0.72986	0.40867	0.35755	0.31254
0.80	0.36000	0.35019	0.33957	0.63152	0.66248	0.69496	0.41881	0.36510	0.31805
0.85	0.27750	0.27021	0.26228	0.58402	0.61930	0.65689	0.42565	0.36992	0.32131
0.90	0.19000	0.18519	0.17993	0.53364	0.57304	0.61562	0.42897	0.37190	0.32231
0.95	0.09750	0.09512	0.09250	0.48038	0.52368	0.57110	0.42857	0.37095	0.32101
1.00	0.00000	0.00000	0.00000	0.42424	0.47121	0.52330	0.42424	0.36698	0.31740

TABLE II. ANALYTICAL SOLUTIONS TO ROTATING FGM ANNULAR DISKS.

r	σ_r			σ_θ			u		
	$\beta=0$	$\beta=0.25$	$\beta=0.5$	$\beta=0$	$\beta=0.25$	$\beta=0.5$	$\beta=0$	$\beta=0.25$	$\beta=0.5$
0.20	0.00000	0.00000	0.00000	1.00000	0.92494	0.85227	0.20000	0.18499	0.17045
0.24	0.14277	0.13214	0.12176	0.84348	0.78941	0.73599	0.19216	0.17771	0.16373
0.28	0.22380	0.20802	0.19245	0.74620	0.70595	0.66530	0.19014	0.17574	0.16182
0.32	0.27119	0.25300	0.23491	0.68006	0.64980	0.61848	0.19159	0.17689	0.16269
0.36	0.29835	0.27926	0.26012	0.63165	0.60914	0.58517	0.19517	0.17990	0.16520
0.40	0.31235	0.29325	0.27394	0.59390	0.57773	0.55990	0.20008	0.18403	0.16863
0.44	0.31720	0.29862	0.27970	0.56280	0.55206	0.53955	0.20576	0.18878	0.17256
0.48	0.31532	0.29759	0.27941	0.53593	0.52997	0.52226	0.21184	0.19381	0.17666
0.52	0.30822	0.29157	0.27436	0.51178	0.51014	0.50684	0.21804	0.19887	0.18074
0.56	0.29691	0.28147	0.26540	0.48935	0.49167	0.49251	0.22415	0.20379	0.18463
0.60	0.28205	0.26792	0.25311	0.46795	0.47397	0.47874	0.23000	0.20841	0.18822
0.64	0.26413	0.25137	0.23791	0.44712	0.45662	0.46515	0.23544	0.21263	0.19143
0.68	0.24348	0.23213	0.22006	0.42652	0.43930	0.45148	0.24036	0.21636	0.19418
0.72	0.22035	0.21043	0.19981	0.40590	0.42181	0.43753	0.24465	0.21952	0.19643
0.76	0.19492	0.18644	0.17729	0.38508	0.40396	0.42313	0.24822	0.22204	0.19813
0.80	0.16733	0.16029	0.15264	0.36392	0.38565	0.40819	0.25098	0.22388	0.19926
0.84	0.13769	0.13208	0.12595	0.34231	0.36676	0.39259	0.25285	0.22498	0.19978
0.88	0.10607	0.10189	0.09729	0.32018	0.34722	0.37628	0.25375	0.22531	0.19969
0.92	0.07256	0.06978	0.06672	0.29745	0.32697	0.35918	0.25362	0.22482	0.19896
0.96	0.03718	0.03581	0.03427	0.27407	0.30596	0.34125	0.25240	0.22349	0.19759
1.00	0.00000	0.00000	0.00000	0.25000	0.28413	0.32244	0.25000	0.22128	0.19557

REFERENCES

[1] S.P. Timoshenko, Strength of Materials, Part II. Advanced Theory and Problems, 3rd ed., New York: D. van Nostrand, 1956.

[2] S.P. Timoshenko and J.N. Goodier, Theory of Elasticity, 3rd ed. New York: McGraw-Hill, 1970.

[3] D.W.A. Rees, The Mechanics of Solids and Structures, New York: McGraw-Hill, 1990.

[4] A.P. Boresi, R.J. Schmidt, and O.M. Sidebottom, Advanced Mechanics of Materials, 5th ed. New York: Wiley, 1993.

[5] A.C. Ugural and S.K. Fenster, Advanced Strength and Applied Elasticity, 3rd ed. London: Prentice-Hall, 1995.

[6] A.M. Afsar and J. Go, "Finite element analysis of thermoelastic field in a rotating FGM circular disk," Applied Mathematical Modelling, vol. 34, pp. 3309-3320, 2010.

[7] M.N.M. Allam, A.M. Zenkour, and T.M.A. El-Azab, "Viscoelastic deformation of the rotating inhomogeneous variable thickness solid and annular disks," International Journal of Computational Methods in Engineering Science and Mechanics, vol. 8, pp. 313-322, 2007.

[8] T. Apatay and A.N. Eraslan, "Elastic deformation of rotating parabolic discs: Analytical solutions (in Turkish)," Journal of the Faculty of Engineering and Architecture of Gazi University, vol. 18, pp. 115-135, 2003.

[9] A.G. Arani, A. Loghman, A.R. Shajari, and S. Amir, "Semi-analytical solution of magneto-thermo-elastic stresses for functionally graded variable thickness rotating disks," Journal of Mechanical Science and Technology, vol. 24, pp. 2107-2118, 2010.

[10] H. Argeso, "Analytical solutions to variable thickness and variable material property rotating disks for a new three-parameter variation function," Mechanics Based Design of Structures and Machines, vol. 40, pp. 133-152, 2012.

[11] M. Asghari and E. Ghafoori, "A three-dimensional elasticity solution for functionally graded rotating disks," Composite Structures, vol. 92, pp. 1092-1099, 2010.

[12] A. Bagri and M.R. Eslami, "Generalized coupled thermoelasticity of functionally graded annular disk considering the Lord-Shulman theory," Composite Structures, vol. 83, pp. 168-179, 2008.

[13] M. Bayat, B.B. Sahari, M. Saleem, A.M.S. Hamouda, and J.N. Reddy, "Thermo elastic analysis of functionally graded rotating disks with temperature-dependent material properties: Uniform and variable thickness," International Journal of Mechanics and Materials in Design, vol. 5, pp. 263-279, 2009.

[14] M. Bayat, M. Saleem, B.B. Sahari, A.M.S. Hamouda, and E. Mahdi, "Analysis of functionally graded rotating disks with variable thickness," Mechanics Research Communications, vol. 35, pp. 283-309, 2008.

[15] P.M. Calderale, F. Vivio, and V. Vullo, "Thermal stresses of rotating hyperbolic disks as particular case of non-linearly variable thickness disks," Journal of Thermal Stresses, vol. 35, pp. 877-891, 2012.

[16] M. Damircheli and M. Azadi, "Temperature and thickness effects on thermal and mechanical stresses of rotating FG-disks," Journal of Mechanical Science and Technology, vol. 25, pp. 827-836, 2011.

[17] J.F. Durodola and O. Attia, "Deformation and stresses in functionally graded rotating disks," Composites Science and Technology, vol. 60, pp. 987-995, 2000.

[18] A.N. Eraslan, T. Apatay, and M. Gülgeç, "Elastic-plastic stress analysis of nonhomogeneous rotating solid discs (in Turkish)," Journal of the Faculty of Engineering and Architecture of Gazi University, vol. 23, pp. 627-635, 2008.

[19] A.N. Eraslan and T. Akis, "On the plane strain and plane stress solutions of functionally graded rotating solid shaft and solid disk problems," Acta Mechanica, vol. 181, pp. 43-63, 2006.

[20] A.N. Eraslan, "Stress distributions in elastic-plastic rotating disks with elliptical thickness profiles using Tresca and von Mises criteria," ZAMM Zeitschrift Fur Angewandte Mathematik und Mechanik, vol. 85, pp. 252-266, 2005.

[21] A.N. Eraslan, Y. Orcan, and U. Güven, "Elastoplastic analysis of nonlinearly hardening variable thickness annular disks under external pressure," Mechanics Research Communications, vol. 32, pp. 306-315, 2005.

[22] A.N. Eraslan and Y. Orçan, "A parametric analysis of rotating variable thickness elastoplastic annular disks subjected to pressurized and radially constrained boundary conditions," Turkish Journal of Engineering and Environmental Sciences, vol. 28, pp. 381-395, 2004.

[23] A.N. Eraslan and Y. Orçan, "Elastic-plastic deformation of a rotating solid disk of exponentially varying thickness," Mechanics of Materials, vol. 34, pp. 423-432, 2002.

[24] A.N. Eraslan and Y. Orçan, "On the rotating elastic-plastic solid disks of variable thickness having concave profiles," International Journal of Mechanical Sciences, vol. 44, pp. 1445-1466, 2002.

[25] A.N. Eraslan, "Elastoplastic deformations of rotating parabolic solid disks using Tresca's yield criterion," European Journal of Mechanics, A/Solids, vol. 22, pp. 861-874, 2003.

[26] A.N. Eraslan and H. Argeso, “Limit angular velocities of variable thickness rotating disks,” International Journal of Solids and Structures, vol. 39, pp. 3109-3130, 2002.

[27] B. Farshi, H. Jahed, and A. Mehrabian, “Optimum design of inhomogeneous non-uniform rotating discs,” Computers and Structures, vol. 82, pp. 773-779, 2004.

[28] J. Go, A.M. Afsar, and J.I. Song, “Analysis of thermoelastic characteristics of a rotating FGM circular disk by finite element method,” Advanced Composite Materials, vol. 19, pp. 197-213, 2010.

[29] A. Hassani, M.H. Hojjati, E. Mahdavi, R.A. Alashti, and G. Farrahi, “Thermo-mechanical analysis of rotating disks with non-uniform thickness and material properties,” International Journal of Pressure Vessels and Piping, vol. 98, pp. 95-101, 2012.

[30] A. Hassani, M.H. Hojjati, G. Farrahi, and R.A. Alashti, “Semi-exact elastic solutions for thermo-mechanical analysis of functionally graded rotating disks,” Composite Structures, vol. 93, pp. 3239-3251, 2011.

[31] S. Jafari, M.H. Hojjati, and A. Fathi, “Classical and modern optimization methods in minimum weight design of elastic rotating disk with variable thickness and density,” International Journal of Pressure Vessels and Piping, vol. 92, pp. 41-47, 2012.

[32] S.A.H. Kordkheili and R. Naghdabadi, “Thermoelastic analysis of a functionally graded rotating disk,” Composite Structures, vol. 79, pp. 508-516, 2007.

[33] G.J. Nie and R.C. Batra, “Stress analysis and material tailoring in isotropic linear thermoelastic incompressible functionally graded rotating disks of variable thickness,” Composite Structures, vol. 92, pp. 720-729, 2010.

[34] X. Peng and X. Li, “Thermal stress in rotating functionally graded hollow circular disks,” Composite Structures, vol. 92, pp. 1896-1904, 2010.

[35] N. Tutuncu and B. Temel, “An efficient unified method for thermoelastic analysis of functionally graded rotating disks of variable thickness,” Mechanics of Advanced Materials and Structures, vol. 20, pp. 38-46, 2013.

[36] N. Tutuncu and B. Temel, “A novel approach to stress analysis of pressurized FGM cylinders, disks and spheres,” Composite Structures, vol. 91, pp. 385-390, 2009.

[37] V. Vullo and F. Vivio, “Elastic stress analysis of non-linear variable thickness rotating disks subjected to thermal load and having variable density along the radius,” International Journal of Solids and Structures, vol. 45, pp. 5337-5355, 2008.

[38] F. Vivio and V. Vullo, “Elastic stress analysis of rotating converging conical disks subjected to thermal load and having variable density along the radius,” International Journal of Solids and Structures, vol. 44, pp. 7767-7784, 2007.

[39] L.H. You, J.X. Wang, and B.P. Tang, “Deformations and stresses in annular disks made of functionally graded materials subjected to internal and/or external pressure,” Meccanica, vol. 44, pp. 283-292, 2009.

[40] L.H. You, X.Y. You, J.J. Zhang, and J. Li, “On rotating circular disks with varying material properties,” ZAMM Zeitschrift Fur Angewandte Mathematik und Physik, vol. 58, pp. 1068-1084, 2007.

[41] X.Y. You, L.H. You, and J.J. Zhang, “Elastic-plastic analysis of annular disks with nonlinear strain-hardening under external pressure,” ZAMM Zeitschrift Fur Angewandte Mathematik Und Mechanik, vol. 85, pp. 202-210, 2005.

[42] A.M. Zenkour, “Stress distribution in rotating composite structures of functionally graded solid disks,” Journal of Materials Processing Technology, vol. 209, pp. 3511-3517, 2009.

[43] A.M. Zenkour, “Thermoelastic solutions for annular disks with arbitrary variable thickness,” Structural Engineering and Mechanics, vol. 24, pp. 515-528, 2006.

[44] Y.Z. Chen and X.Y. Lin, “Elastic analysis for thick cylinders and spherical pressure vessels made of functionally graded materials,” Computational Materials Science, vol. 44, pp. 581-587, 2008.