Numerical Solution of 2D SPL Heat Conduction Model for Analyzing Thermal Wave Oscillations

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***Abstract*—Thermal wave oscillations within a finite thin plate based on single-phase-lagging (SPL) heat conduction model is investigated. Results were obtained by Crank-Nicholson (CN) finite difference scheme. The stability of the numerical scheme has been discussed and observed that the solution is unconditionally stable. The whole analysis is presented in dimensionless form. A numerical example of particular interest has been studied and discussed in details.**

Keywords— SPL heat conduction model; thermal oscillation; CN finite difference scheme; unconditional stability

I. INTRODUCTION

Cattaneo [1] and Vernotte [2] removed the deficiency [3]-[6] occurs in the classical heat conduction equation based on Fourier's law and independently proposed a modified version of heat conduction equation by adding a relaxation term to represent the lagging behavior of energy transport within the solid, which takes the form

$$\tau \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -k\nabla T \qquad (1)$$

where k is the thermal conductivity of medium and τ is a material property called the relaxation time. This model characterizes the combined diffusion and wave like behavior of heat conduction and predicts a finite speed

$$c = \left(\frac{k}{\rho c_b \tau}\right)^{\frac{1}{2}} \qquad (2)$$

for heat propagation [7], where ρ is the density and c_b is the specific heat capacity. This model addresses short time scale effects over a spatial macroscale. Detailed reviews of thermal relaxation in wave theory of heat propagation were performed by Joseph and Preziosi [8] and Ozisik and Tzou [9]. The natural extension of CV model is

$$\mathbf{q}(\mathbf{r}, t+\tau) = -k\nabla T(\mathbf{r}, t) \qquad (3)$$

which is called the single-phase-lagging (SPL) heat conduction model [10]-[14]. According to SPL heat conduction model, there is a finite built-up time τ for onset of heat flux at $\mathbf{r}$, after a temperature gradient is imposed there i.e. τ represents the time lag needed to establish the heat flux (the result) when a temperature gradient (the cause) is suddenly imposed.

The single-phase-lagging heat conduction differs from the Fourier heat conduction, where, the maximum or minimum value of temperature will only be achieved at the boundary of the medium or at the initial instant if no heat source exists in the heat conduction medium while in single-phase-lagging heat conduction, the temperature of some inner regions in the medium may exceed the temperature at the boundary because of the presence of interference phenomenon [15]. Ordonez-Miranda and Alvarado-Gil [16] analytically studied the thermal wave oscillation based on hyperbolic heat conduction model and shown that the analysis of this phenomenon is useful in the determination of the thermal relaxation time which has elusive physical quantity. Thus to handle such type of problem, it is very desirable to construct high order algorithms for efficient computations.

Based on CV hyperbolic heat transport model, Ordonez-Miranda and Alvarado-Gil [16] shown that oscillations of amplitude and phase of the spatial component of the surface temperature are obtained in the high frequency regime for Dirichlet and Neumann boundary conditions. Dai et al. [17] extended the parabolic two step model for micro-heat transfer and analyzed the thermal oscillation in generalized N-carrier system by stable finite difference scheme. Lam and Fong [18] analytically compared the parabolic heat conduction equation based on Fourier's law and hyperbolic heat conduction equation based on CV constitutive relation. It was found that due to the effect of lagging parameter, the hyperbolic model predicts a wave front in the direction of propagation and causes a large concentration of energy in a localized area; whereas, the parabolic model predicts an instantaneous diffusion of heat into the medium. Based on the CV hyperbolic heat conduction model, Ordonez-Miranda and Alvarado-Gil [19] investigated the thermal wave transport in a layered system and found the analytical formulas to determine its thermal relaxation time as well as additional thermal properties for single semi-infinite layer and for a system of two finite layers in thermal contact. In the present work, we attempt to examine thermal wave oscillation in thin plate by computationally efficient high order accurate finite difference scheme.

In this paper, we present unconditionally stable accurate finite difference scheme for solving SPL heat conduction equation. Unconditional stability will give no restriction on the mesh ratio. The outline of the paper is as follows. SPL heat conduction model is given in Section 2. Section 3 deals solution of model. Stability of the numerical scheme is given in Section 4. Section 5 contains result and discussion. Conclusion is given in Section 6.

II. 2D SPL HEAT CONDUCTION MODEL

The combination of Fourier's law of heat conduction

$$q = -k\frac{\partial T}{\partial y} \qquad (4)$$

and law of conservation of energy

$$\rho c_b \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial y} + g^* \qquad (5)$$

provides the law of heat conduction as follows

$$\rho c_b \frac{\partial T}{\partial t} = k\nabla^2 T + g^* \qquad (6)$$

where g^* denotes the internal energy generation rate per unit volume inside the medium. In two dimension (6) can be written as

$$\rho c_b \frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^{*2}} + \frac{\partial^2 T}{\partial y^{*2}}\right) + g^* \qquad (7)$$

Equation (7) is the classical diffusion model which governs thermal energy transport in solids. By introducing dimensionless parameters

$$\theta = \frac{kcT}{\alpha I_r}, x = \frac{cx^*}{2\alpha}, y = \frac{cy^*}{2\alpha}, F_0 = \left(\frac{c^2}{2\alpha}\right)t$$

Equation (7) can be expressed in dimensionless form as

$$2\frac{\partial \theta}{\partial F_0} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + g \qquad (8)$$

where Fourier number F_0 represents dimensionless time. The CV constitutive relation (1) together with the energy conservation (5) gives the equation governing propagation of thermal energy

$$\frac{\partial T}{\partial t} + \tau\frac{\partial^2 T}{\partial t^2} = \alpha\left(\frac{\partial^2 T}{\partial x^{*2}} + \frac{\partial^2 T}{\partial y^{*2}}\right) + \frac{\alpha}{k}\left(g^* + \tau\frac{\partial g^*}{\partial t}\right) \qquad (9)$$

where α is the thermal diffusivity of the material and the relaxation time $\tau = \frac{\alpha}{c^2}$. On the left hand side of above equation, the second order time derivative term indicates that heat propagates as a wave with a characteristic speed given by equation (2) and the first order time derivative corresponds to a diffusive process, which damps spatially the heat wave. One can see that if energy travels at an infinite propagation speed (i.e. $c \to \infty$), then (9) reduces to the parabolic heat conduction equation (based on Fourier law). The equation (9) can be expressed in dimensionless form as

$$2\frac{\partial \theta}{\partial F_0} + \frac{\partial^2 \theta}{\partial {F_0}^2} = \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right) + \left(g + \frac{1}{2}\frac{\partial g}{\partial F_0}\right) \quad (10)$$

In present study, an isotropic thin plate, $0 \le x, y \le 1$, with uniform thickness and constant thermophysical properties, is assumed. Initially, the thin plate is at temperature $\theta(x, y, 0) = 0.01\sin\left(\frac{\pi x}{2}\right)$, which is a function of position within the thin plate and rate of change in temperature is θ_3. For time $F_0 > 0$, the following boundary conditions will be considered

$$\theta(x, y, 0) = 0.01\sin\left(\frac{\pi x}{2}\right), \frac{\partial \theta(x, y, 0)}{\partial F_0} = \theta_3 \quad (11)$$

$$\theta(0, y, F_0) = 0, \theta(1, y, F_0) = 0 \qquad (12)$$

$$\theta(x, 0, F_0) = 0, \theta(x, 1, F_0) = 0 \qquad (13)$$

For sake of convenience we assume that there is no internal heat source in the heat conduction medium.

III. SOLUTION

To establish numerical approximation $\Delta x = h = \frac{l}{N} > 0, \Delta y = k = \frac{l}{L} > 0$ and $\Delta F_0 = \frac{1}{M}$ be the grid size in space and time direction respectively. The grid points in the space interval $[0, l]$ are numbers $x_i = ih, i = 0,1,2...N; y_j = jk, j = 0,1,2...L$ and grid points in time interval $F_{0_n} = n\Delta F_0, n = 1,2,3...M$. The values of the functions θ at grid points are denoted by $\theta_{i,j}^n = \theta\left(x_i, y_j, F_{0_n}\right)$. To solve the above problem, (10)-(13), using the finite difference scheme, one may employ the Crank-Nicholson scheme for (10) as follows

$$2\frac{\theta_{i,j}^{n+1}-\theta_{i,j}^{n}}{\Delta F_0}+\frac{\theta_{i,j}^{n+1}-2\theta_{i,j}^{n}+\theta_{i,j}^{n-1}}{(\Delta F_0)^2}=$$

$$\frac{\theta_{i-1,j}^{n+1}-2\theta_{i,j}^{n+1}+\theta_{i+1,j}^{n+1}}{2h^2}+\frac{\theta_{i-1,j}^{n}-2\theta_{i,j}^{n}+\theta_{i+1,j}^{n}}{2h^2} \qquad (14)$$

It can be seen that the truncation error of the new scheme has an order of $(\Delta F_0)^2+h^2+k^2$ at all grid points $\theta\left(x_i, y_j, F_{0_{n+\frac{1}{2}}}\right); i=1,2..N;\ j=1,2,..L.$

Equation (14) together with (11)-(13) can be written in vector matrix form as follows

$$A\theta^1 = B\theta^0,\ \ n=0 \qquad (15)$$

$$A\theta^{n+1} = B\theta^n + C\theta^{n-1},\ \ n\geq 1 \qquad (16)$$

where $\theta^{n+1}=\left[\theta_1^{n+1},\theta_2^{n+1},.....\theta_{N-1}^{n+1},\theta_N^{n+1}\right]^t$, A, B and C are the matrices involved in the scheme.

Thus, from (15)-(16), we get dimensionless temperature θ. In present analysis, Mathematical software MATLAB 11.0 has been used to obtain dimensionless temperature.

IV. STABILITY OF NUMERICAL SCHEME

To examine the stability of the numerical scheme we first we re-write (15)-(16) in equivalently as [20, 21]

$$U^{n+1} = DU^n \qquad (17)$$

where

$$U^{n+1}=\begin{pmatrix}\theta^{n+1}\\ \theta^n\end{pmatrix},\ D=\begin{pmatrix}A^{-1}B & A^{-1}C\\ I & 0\end{pmatrix}_{N\times N}$$

This technique has reduced a three-level difference equation in time to a two level. Equation (17) will be stable when each eigenvalue of D has a modulus ≤ 1, i.e. $\rho(D)\leq 1$. From a computational point of view, the Eigen-values of D can be evaluated numerically. It is shown that the Eigen-values satisfy the stability condition. Our experience for solved examples shows that spectral radius of the corresponding D matrix stays less than 1, for various ΔF_0 and Δx, Δy.

In *Figs. 1-2*, the Eigen-values of the D matrix are sketched. We show numerically that the upper bound for the absolute value of D's Eigen-values is 1 for all temporal grid size (*Fig. 1*) and spatial grid size (Fig. 2). Thus, the scheme is not dependents on grid size hence unconditionally stable.

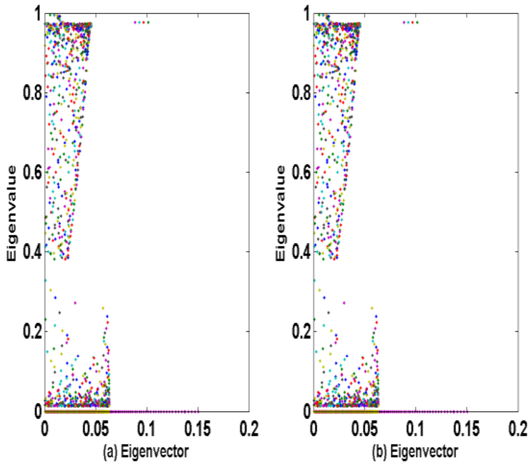

Fig. 1. *Variation of eigenvector with eigenvalue for* $N=1000$ *at (a)* $\Delta F_0=0.01$*; (b)* $\Delta F_0=0.001$.

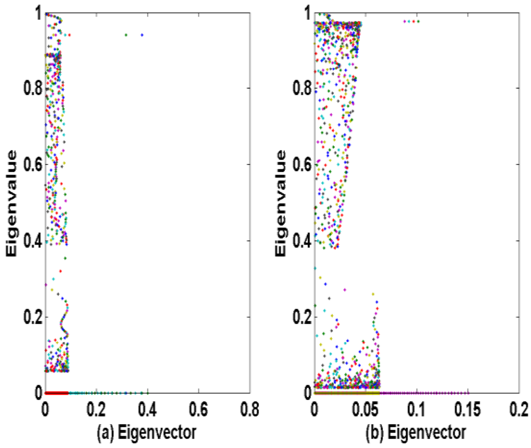

Fig. 2. *Variation of eigenvector with eigenvalue for* $\Delta F_0=0.001$ *at (a)* $N=250$*; (b)* $N=500$.

V. RESULTS AND DISCUSSION

This section presents complete analysis of thermal wave oscillations and observes the effect of Fourier number (F_0). The Figures presented in this study, only the parameters whose values different from the reference value are indicated. The selected reference values include $h=0.01,\ k=0.01,\ \Delta F_0=0.001$, $\theta_3=0$.

Figs. 3-6 show the dimensionless spatial temperature profiles at different heating periods. For small heating period, two sharp waves appear near the boundaries as both boundaries are insulated, as shown in *Fig. 3*. Due to lagging behavior, two thermal waves travel slowly toward the centre of thin plate and as energy moves inward, both waves' meets at the centre of thin plate at $F_0=0.025$ which creates a peak temperature at this location. The ``increase'' in peak temperature is due to constructive interference of somewhat elastic waves, is only momentary, and is unstable.

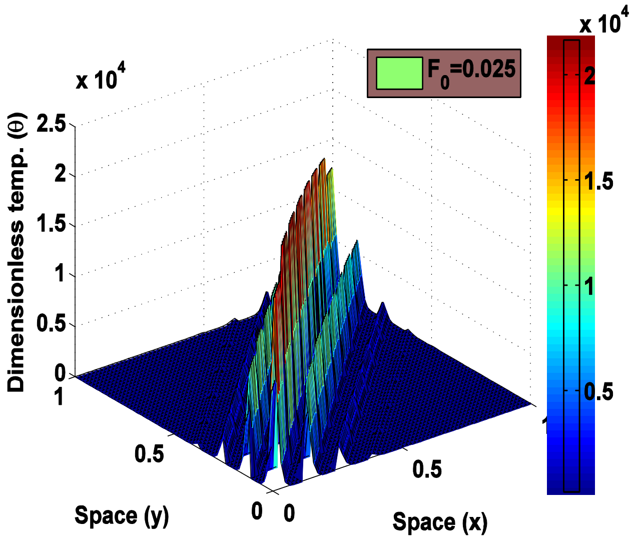

Fig. 3. *Spatial temperature profile for Fourier number* (F_0)*=0.025.*

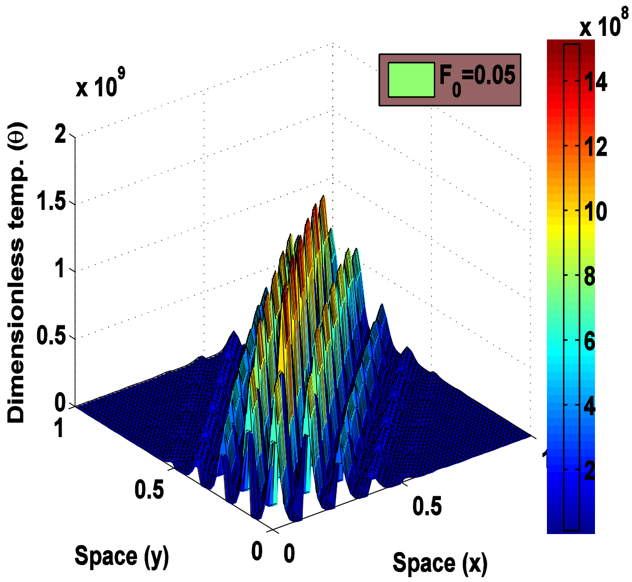

Fig. 4. *Spatial temperature profile for Fourier number* (F_0)*=0.05.*

From *Fig. 4* it is observed that temperature at midpoint of thin film decreases due to the destructive interference of thermal waves. A close examination of *Figs. 3* and *4* reveals that the dimensionless temperature increases with Fourier number as Fourier number is a measure of rate of heat conduction in comparison with the heat storage in the given volume element. Larger the Fourier number deeper the penetration of heat into a body over a given time.

Further, from *Figs. 5-6*, it is evident that, for large heating period, intensity of peak temperature increases, similar to the result given in *Fig. 4*. A close examination of *Figs. 5* and *6* reveals that intensity of peak temperature increases very rapidly as time period of heating increases and energy spread over a larger time interval.

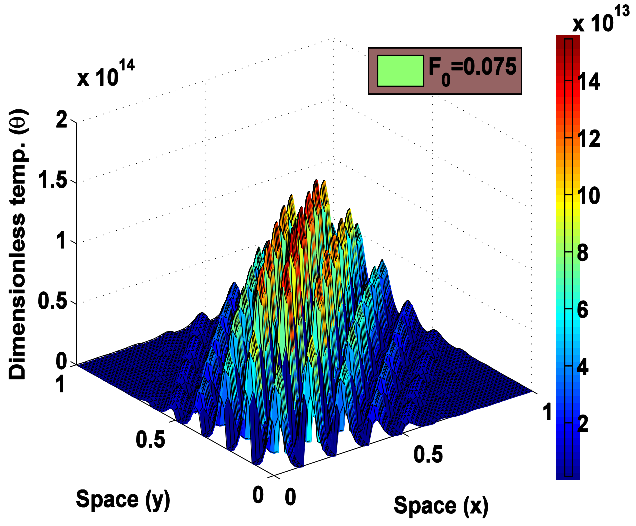

Fig. 5. *Spatial temperature profile for Fourier number* (F_0)*=0.075.*

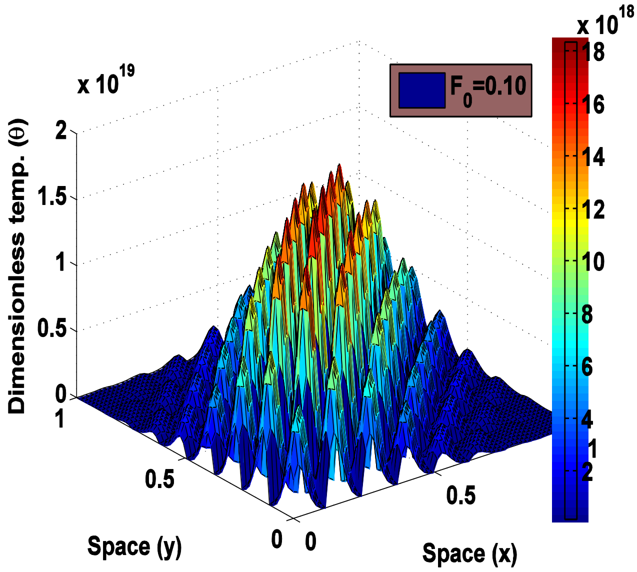

Fig. 6. *Spatial temperature profile for Fourier number* (F_0)*=0.1.*

VI. CONCLISION

A mathematical model of two-dimensional SPL heat conduction is solved by high order numerical techniques. The effect of spatial and temporal grid size on stability of numerical scheme has been observed and found that scheme is independent on grid size i.e. scheme is unconditionally stable. Due to the lagging behavior two waves travels towards each other and creates peak due to the constructive interference. The intensity of peak temperature increases with increase of Fourier number.

This technique is useful to measure the accurate analysis of sudden increase of temperature which is responsible for thermal damage. Further, this technique can be applied to solve SPL model of general body under most generalized boundary conditions.

VII. Nomenclature

A Matrix of order $N \times N$

B Matrix of order $N \times N$

C Matrix of order $N \times N$

c Thermal wave propagation speed (m/s)

c_b Specific heat capacity $(J/kg.K)$

f_r Reference heat flux $(\mathbf{q}/\mathbf{q}*)$

F_0 Fourier number $(c^2t/2\alpha)$

ΔF_0 Temporal grid size (m)

g Dimensionless internal heat source $(4\alpha g*/cf_r)$

k Thermal conductivity $(W/m.K)$

l Length of the medium (m)

L Dimensionless length of the medium $(2\alpha/l)$

$\mathbf{q}*$ Dimensionless heat flux $(\mathbf{q}/f_r)$

$\mathbf{r}$ Position vector

t Time (s)

T Temperature (K)

ΔT Temperature gradient (K/m)

x Dimensionless spatial coordinate $(cx*/2\alpha)$

y Dimensionless spatial coordinate $(cy*/2\alpha)$

Δx Spatial grid size

Δy Spatial grid size

α Thermal diffusivity (m^2/s)

θ Dimensionless Temperature $(kcT/\alpha f_r)$

ρ Density (kg/m^3)

τ Lagging parameter (s)

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Rferences

[1] C. Cattaneo, “Sur une forme de l'Equation de la chaleur elinant le paradox d'une propagation instantance,” C. R. Acad. Sci., vol. 247, pp. 431-433, 1958.

[2] M.P. Vernotee, “Les paradoxes de la theorie continue de l' equation de la chleur,” C. R. Acad. Sci., vol. 246, pp. 3154-3155, 1958.

[3] D.Y. Tzou, Macro-to-Microscale Heat Transfer: The Lagging Behavior, Taylor & Francis, Washington, 1996.

[4] D.G. Cahill, W.K. Ford, K.E. Goodson, G.D. Mahan, A. Majumdar, H.J. Maris, R. Merlin, and S.R. Phillpot, “Nanoscale thermal transport,” J. Appl. Phys., vol. 93, pp. 793-818, 2003.

[5] A.A. Joshi, and A. Majumdar, “Transient ballistic and diffusive phonon heat transport in thin films,” J. Appl. Phys., vol. 74, pp. 31-39, 1993.

[6] C.L. Tien, A. Majumdar, and F.M. Gerner, Microscale Energy Transport, Taylor & Francis, Washington, 1998.

[7] L.Q. Wang, X. Zhou, and X. Wei, Heat Conduction: Mathematical Models and Analytical solutions, Springer-Verlag, Berlin, 2008.

[8] D.D. Josheph, and L. Preziosi, “Heat waves,” Rev. Mod. Phys., vol. 61, pp. 41-73, 1989.

[9] M.N. Ozisik, and D.Y. Tzou, “On the wave theory in heat conduction,” ASME J. Heat Transfer, vol. 116(3), pp. 526-535, 1994.

[10] D.Y. Tzou, “Thermal shock phenomena under high rate response in solids,” Annual Review of Heat Transfer, vol. 4, pp. 111-185, 1992.

[11] D.Y. Tzou, “Shock wave formation around a moving heat in a solid with finite speed of heat propagation,” Int. J. Heat Mass Transfer, vol. 32, pp. 1979-1987, 1989.

[12] D.Y. Tzou, “Thermal shock wave induced by a moving crack,” ASME J. Heat Transfer, vol. 112, pp. 21-27, 1990.

[13] D.Y. Tzou, “Thermal shock waves induced by a moving crack-a heat flux formulation,” Int. J. Heat Mass Transfer, vol. 33, pp. 877-885, 1990.

[14] D.Y. Tzou, “On thermal shock waves induced by a moving heat source,” ASME J. Heat Transfer, vol. 111, pp. 232-238, 1989.

[15] M. Xu, J. Guo, L. Wang, and L. Cheng, “Thermal wave interference as the origin of the overshooting phenomenon in dual-phase-lagging heat conduction,” Int. J. Thermal Sciences, vol. 50, pp. 825-830, 2011.

[16] J. Ordonez-Miranda, J.J. Alvarado-Gil, “Thermal wave oscillations and thermal relaxation time determination in hyperbolic heat transport

model," Int. J. Thermal Sciences, vol. 48, pp. 2053-2062, 2009.

[17] W Dai, F. Zhu, and D.Y. Tzou, "A stable finite difference scheme for thermal analysis in an N-carrier system," Int. J. Thermal Sciences, vol. 48, pp. 1530-1541, 2009.

[18] T.T. Lam, and E. Fong, "Heat diffusion vs. wave propagation in solids subjected to exponentially-decaying heat sources: analytical solution," Int. J. Thermal Sciences, vol. 50, pp. 2104-2116, 2011.

[19] J. Ordonez-Miranda, and J.J. Alvarado-Gil, "Effective thermal properties of multilayered systems with interface thermal resistance in a hyperbolic heat transfer model," Int. J. Thermophys, vol. 31, pp. 900-925, 2000.

[20] A. Malek, and S.H. Momeni-Masuleh, "A mixed collocation finite difference method for 3D macroscopic heat transport problems," J. Compt. Appl. Math., vol. 217, pp. 137-147, 2008.

[21] A. Malek, Z.K. Bojdi, and P. Golbarg, "Solving fully three-dimensional microscale dual-phase-lag problem using mixed-collocation finite difference discritization," ASME J. Heat Transfer, vol. 134, pp. 1-6, 2012.