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Dimensionless Analysis of Axially Vibrating Beams

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***Abstract*— The main purpose of this study is to obtain dimensionless differential equation of motion for free vibration of the axially vibrating beams, to solve this dimensionless vibration problem for different boundary conditions and to obtain the natural frequencies for the first five modes. The dimensionless differential equation of motion of the beam is obtained and solved to obtain the frequency equation. An I-profile beam is chosen for the numerical analysis. The natural frequencies of the first five modes are obtained for the I-beam modeled as a fixed beam, a cantilever beam and a free beam. The frequency values are listed in the tables.**

Keywords—free axial vibration; dimensionless analysis; natural frequency; I-profile beam

I. INTRODUCTION

In practice, the representation of a beam by a discrete model is an idealized model; however, in fact, beams have continuously distributed mass and elasticity. Mostly, especially for the axially vibration, beams are modeled as continuous systems having infinite number of degreed of freedom and, usually, the dimensional equation of motion is taken into consideration in the vibration analysis of the beams [1-5].

In this study, the dimensionless equation of motion of an axially vibrating beam for free vibration is obtained and solved by the method of separation of variables [6]. The natural frequencies for the first five modes are obtained for three beam types, being fixed beam, cantilever beam and free beam. The axially vibrating beam considered in the study is assumed to be homogeneous and isotropic.

II. EQUATION OF MOTION FOR AN AXIALLY VIBRATING BEAM

A. *Dimensional Equation of Motion*

An axially vibrating beam, given in Fig. 1, with the distributed mass m, the length L, the modulus of elasticity E, the cross-section area A and the axial rigidity AE has a dimensional differential equation of motion for free vibration as [7]

$$\frac{\partial^2 \mathrm{u(x,t)}}{\partial x^2} - \frac{m}{\mathrm{AE}}\frac{\partial^2 \mathrm{u(x,t)}}{\partial t^2} = 0 \qquad (1)$$

where u(x,t) is the displacement function of the beam in terms of both displacement x and time t. Application of the separation of variables method to (1) as in the form of (2) is commonly used in vibration analysis of beams.

$$u(x,t) = X(x).T(t) = X(x).[A.\sin(\omega t) + B.\cos(\omega t)] \qquad (2)$$

In (2), X(x) is the eigenfunction named as shape function, T(t) is time function, ω is the eigenvalue of the solution named as natural frequency and A, B are the integration constants.

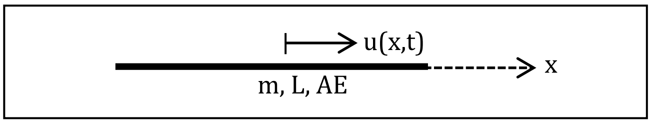

Fig. 1. *An axially vibrating beam with the distributed mass m, the length L, the modulus of elasticity E, the cross-section area A and the axial rigidity AE.*

The derivatives used in (1) can, therefore, be written as

$$\frac{\partial^2 u(x,t)}{\partial x^2} = u''(x,t) = X''(x).[A.\sin(\omega t) + B.\cos(\omega t)] = X''(x).T(t) \qquad (3)$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \ddot{u}(x,t) = X(x).(-\omega^2)[A.\sin(\omega t) + B.\cos(\omega t)] = -\omega^2.X(x).T(t) \qquad (4)$$

where ($^{\prime\prime}$) and ($\ddot{}$) denote the second order derivative due to x and t, respectively. Substitution of (3) and (4) in (1) gives the governing equation of motion in the dimensional form as

$$X''(x).T(t) + \frac{m\omega^2}{\mathrm{AE}}X(x).T(t) = 0$$

$$X''(x) + \frac{m\omega^2}{\mathrm{AE}}X(x) = 0 \quad 0 \le x \le L \qquad (5)$$

B. *Dimensionless Equation of Motion*

Taking z=x/L as the dimensionless displacement variable with x=z.L, ∂x=∂z.L and $\partial x^2=\partial z^2.L^2$, the dimensionless differential equation of motion for axial vibration of a beam is obtained from (1) as

$$\frac{\partial^2 \mathrm{u(z,t)}}{\partial z^2 L^2} - \frac{m}{\mathrm{AE}}\frac{\partial^2 \mathrm{u(z,t)}}{\partial t^2} = 0 \qquad \frac{\partial^2 \mathrm{u(z,t)}}{\partial z^2} - \frac{mL^2}{\mathrm{AE}}\frac{\partial^2 \mathrm{u(z,t)}}{\partial t^2} \qquad (6)$$

Substituting the successive differentiations of the dimensionless displacement function u(z,t) in (7) obtained, again, by using the method of separation of variables into (6) give (8) for general solution of dimensionless equation of motion.

$$u(z,t) = Z(z).T(t) \tag{7}$$

$$\frac{\partial^2 u(z,t)}{\partial z^2} = u''(z,t) = Z''(z).T(t)$$

$$\frac{\partial^2 u(z,t)}{\partial t^2} = \ddot{u}(z,t)$$

$$= Z(z).(-\omega^2)[A.\sin(\omega t) + B.\cos(\omega t)]$$
$$= -\omega^2.Z(z).T(t)$$

$$Z''(z) + \frac{mL^2\omega^2}{\text{AE}}Z(z) = 0$$

$$for\ \alpha^2 = \frac{mL^2\omega^2}{\text{AE}} \qquad Z''(z) + \alpha^2 Z(z) = 0 \tag{8}$$

The characteristic equation and the solution of (8) is given as follows as D being d/dz:

$$D^2 + \alpha^2 = 0 \quad \rightarrow \quad D_{1,2} = \pm i\alpha \tag{9}$$

$$Z(z) = C_1.\sin(\alpha z) + C_2.\cos(\alpha z) \qquad 0 \le z \le 1 \tag{10}$$

(10) gives the dimensionless shape function of the axially vibrating beam due to the dimensionless displacement variable, z. Therefore, from (7), the dimensionless displacement function of the axially vibrating beam has the form of (11).

$$u(z,t) = [C_1.\sin(\alpha z) + C_2.\cos(\alpha z)].T(t) \tag{11}$$

III. Bondary Conditions

Two boundary conditions have to be written for each of the three beams considered in this study since two integration constants (C_1, C_2) are obtained in the solution of second order differential equation of motion. The dimensional boundary conditions written for the fixed and the free ends of axially vibrating beam are given, respectively, as [8]

for $0 \le x \le L$

$$u(x,t) = 0 \quad N(x,t) = AE\frac{\partial u(x,t)}{\partial x} = AEu'(x,t) = 0 \tag{12}$$

where N(z,t) being the axial force. Thus, the dimensionless boundary conditions for the same ends are obtained from (12) as in (13).

for $0 \le z \le 1$

$$u(z,t) = 0 \quad N(z,t) = AE\frac{\partial u(z,t)}{\partial z.L} = \frac{AE}{L}u'(z,t) = 0 \tag{13}$$

A. Fixed Beam

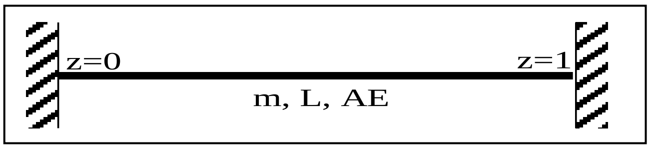

Fig. 2. *Fixed beam with the distributed mass m, the length L, the modulus of elasticity E, the cross-section area A and the axial rigidity AE.*

The dimensional boundary conditions of the fixed beam given in Fig. 2 can be written from (13) as

$$u(z=0,t) = 0 \quad and \quad u(z=1,t) = 0 \tag{14}$$

Thus, one gets the same natural angular frequency equation, that is valid in the dimensional analysis also, using (14) for the fixed beam as in the following.

$$\omega = n_i \frac{\pi}{L}\sqrt{\frac{AE}{\text{m}}} \tag{15}$$

where n_i is the number of mode.

B. Cantilever Beam

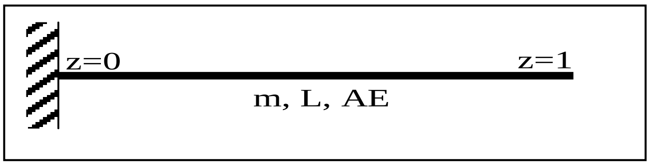

Fig. 3. Cantilever *beam with the distributed mass m, the length L, the modulus of elasticity E, the cross-section area A and the axial rigidity AE.*

The dimensional boundary conditions of the cantilever beam given in Fig. 3 can be written from (13) as

$$u(z=0,t) = 0 \quad and \quad u'(z=1,t) = 0 \tag{16}$$

Thus, one gets the same natural angular frequency equation, that is valid in the dimensional analysis also, using (16) for the cantilever beam as in the following.

$$\omega = \frac{(2n_i - 1)\pi}{2L}\sqrt{\frac{AE}{\text{m}}} \tag{17}$$

where n_i is the number of mode.

C. Free Beam

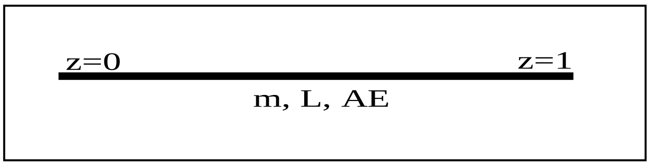

Fig. 4. Free *beam with the distributed mass m, the length L, the modulus of elasticity E, the cross-section area A and the axial rigidity AE.*

The dimensional boundary conditions of the free beam given in Fig. 4 can be written from (13) as

$$u'(z=0,t) = 0 \quad and \quad u'(z=1,t) = 0 \tag{18}$$

Thus, one gets the same natural angular frequency equation, that is valid in the dimensional analysis also, using (14) for the free beam as in the following.

$$\omega = n_i \frac{\pi}{L}\sqrt{\frac{AE}{\text{m}}} \tag{19}$$

where n_i is the number of mode.

IV. Natural Frequencies

After the natural frequency equations valid in both the dimensional and the dimensionless vibration analysis are obtained for the first five vibration modes of, respectively, the fixed, the cantilever and the free beams. The IPB 1000 profile is chosen with distributed mass of m=0,32 kNsec^2/m^2, elasticity modulus of

E=210000000 kN/m^2 and cross-section area of A=0,04 m^2. The free vibration analysis is made for the beam lengths, L, of 1, 3, 5 and 8 meters. The first five mode natural frequencies of the fixed, the cantilever and the free beams are given in, respectively, Tables I, II and III.

TABLE I. NATURAL FREQUENCIES OF THE FIXED BEAM

Modes	Natural Frequensies (rad/sec)			
	L=1 m	*L=3 m*	*L=5 m*	*L=8 m*
ω_1	16095,872624	5365,290875	3219,174525	2011,984078
ω_2	32191,745248	10730,581750	6438,349050	4023,968156
ω_3	48287,617872	16095,872624	9657,523575	6035,952234
ω_4	64383,490496	21461,163500	12876,698100	8047,936312
ω_5	80479,363120	26826,454373	16095,872624	10059,920390

TABLE II. NATURAL FREQUENCIES OF THE CANTILEVER BEAM

Modes	Natural Frequensies (rad/sec)			
	L=1 m	*L=3 m*	*L=5 m*	*L=8 m*
ω_1	8047,936312	2682,645437	1609,587262	1005,992039
ω_2	24143,808936	8047,936311	4828,761786	3017,976117
ω_3	40239,681560	13413,227185	8047,936310	5029,960195
ω_4	56335,554184	18778,518059	11267,110834	7041,944273
ω_5	72431,426808	24143,808933	14486,285358	9053,928351

TABLE III. NATURAL FREQUENCIES OF THE FREE BEAM

Modes	Natural Frequensies (rad/sec)			
	L=1 m	*L=3 m*	*L=5 m*	*L=8 m*
ω_1	16095,872624	5365,290875	3219,174525	2011,984078
ω_2	32191,745248	10730,581750	6438,349050	4023,968156
ω_3	48287,617872	16095,872624	9657,523575	6035,952234
ω_4	64383,490496	21461,163500	12876,698100	8047,936312
ω_5	80479,363120	26826,454373	16095,872624	10059,920390

V. CONCLUSIONS

In this study, free axial vibration analysis of a beam with different supporting conditions is made. The governing differential equation of motion is solved, firstly, in dimensional form and the frequency equations of the beams with different support conditions are obtained. Secondly, the equation of motion in dimensionless form is solved and the same frequency equations are obtained for the considering beams. It is concluded, therefore, that for the single span beams, as considered in this study, there is no need to solve the equation of motion in dimensionless form for the free vibration of axially vibrating beam since same frequency equation that is related to the length of beam, L, is obtained in both dimensional and dimensionless vibration analysis.

It can be seen from the Tables I, II and III that the frequency values decreases as the length of beam increases since L is at the denominator of the frequency equation.

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