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Fixed Point Results With Rational Expressions On Fuzzy Mappings

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***Abstract*—The present paper we prove some fixed point theorems and common fixed point results for fuzzy mappings over a complete metric space which include the rational expressions in these types results.**

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Introduction

In 1965 the fuzzy sets was introduced by Zadeh [21]. After that a lot of work has been done regarding fuzzy sets and fuzzy mappings. The concept of fuzzy mappings was first introduced by Heilpern [10], he proved fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of the fixed point theorem for multi valued mappings of Nadler [15], Vijayraju and Marudai [19] generalized the Bose and Mukherjee's [2] fixed point theorems for contractive types fuzzy mappings. Marudai and Srinivasan [14] derived the simple proof of Heilpern's [10] theorem and generalization of Nadler's [15] theorem for fuzzy mappings.
Bose and Sahani [3], Butnariu [4,5,6], Chang and Huang [7], Chang [8], Chitra [9], Som and Mukharjee [18] studied fixed point theorems for fuzzy mappings. Lee and Cho [12] described a fixed point theorem for contractive type fuzzy mappings which is generalization of Heilpern's [10] result. Lee, Cho, Lee and Kim [13] obtained a common fixed point theorem for a sequence of fuzzy mappings satisfying certain conditions, which is generalization of the second theorem of Bose and Sahini [3].
Recently, Rajendran and Balasubramanian [17] worked on fuzzy contraction mappings. More recently Vijayraju and Mohanraj [20] obtained some fixed point theorems for contractive type fuzzy mappings which are generalization of Beg and Azam [1], fuzzy extension of Kirk and Downing [11] and which obtained by the simple proof of Park and Jeong [16].
In the present paper we are proving some fixed point and common fixed point theorems in fuzzy mappings containing the rational expressions.

Preliminaries

To prove the results we need following definitions and assumptions:-
Fuzzy Mappings: Let X be any metric linear space and d be any metric in X. A fuzzy set in X is a function with domain X and values in [0, 1]. If A is a fuzzy set and x ∈ X, the function value A(x) is called the grade of membership of x in A. The collection of all fuzzy sets in X is denoted by Ғ(x).
Let A ∈ Ғ(x).and α ∈ [0, 1].The set α –level set of A, denoted by A_α
$A_\alpha = \{x: A(x) \geq \alpha\}$ if α ∈ [0, 1],

$$A_0 = \overline{\{x : A(x) > 0\}}, \text{whenever } \bar{B} \text{ is closuer of } B$$

Now we distinguish from the collection Ғ(x) a sub collection of approximate quantities, denoted W(x).

Definition 2.1: A fuzzy subset A of X is an approximate quantity iff its α-level set is a compact subset (non fuzzy) of X for each α ∈ [0, 1], and $\sup_{x\in X} A(x)=1$

When A ∈ W(x) and $A(x_0)$ = 1 for some x_0 ∈ W(x), we will identify A with an approximation of x_0 .Then we shall define a distance between two approximate quantities.
Definition 2.2: Let A, B ∈ W(x), α ∈ [0,1], define

$$p_\alpha(A,B) = \inf_{x\in A_\alpha, y\in B\alpha} d(x,y), D_\alpha(A,B)$$

$$= dist(A_\alpha, B_\alpha), d(A,B) = \sup_\alpha D_\alpha(A,B)$$

Where dist. is Hausdorff distance. The function p_α is called α-spaces, and a distance between A and B. It is easy to see that p_α is non decreasing function of α.
We shall also define an order of the family W(x), which characterizes accuracy of a given quantity.

Definition 2.3: Let A, B ∈ W(x). An approximate quantity A is more accurate then B, denoted by $A \subset B$, iff A(x) ≤ B(x), for each x ∈ X.
Now we introduce a notion of fuzzy mapping, i.e. a mapping with value in the family of approximate quantities.

Definition 2.4: Let X be an arbitrary set and Y be any metric linear space. F is called a fuzzy mapping iff F is mapping from the set X into W(Y), i.e., F(x) ∈ W(Y) for each x ∈ X

A fuzzy mapping F is a fuzzy subset on X × Y with membership function F(x, y).The function value F(x, y) is grade of membership of y in F(x).
Let A ∈ Ғ(X), B ∈ Ғ(Y) the fuzzy set $F^{-1}(B)$ in Ғ(X), is defined as
$F^{-1}(B)(x) = \sup_{y\in Y} (F(x,y) \cap B(y))$ where $x \in X$ First of all we shall give here the basic properties of α-space and α-distance between some approximate quantities.

Lemma 1: Let x ∈ X, A ∈ W(X), and {x} be a fuzzy set with membership function equal a characteristic function of set {x}. If {x} is subset of a then $p_\alpha(x, A) = 0$ for each α ∈ [0, 1].

Lemma 2: $p_\alpha(x,A) \le d(x,y) + p_\alpha(y,A)$ for any x, y ∈ X.

Lemma 3: If {x_0} is subset of A, then $p_\alpha(x_0, B) \le D_\alpha(A, B)$ for each B ∈ W(X).

Lemma 4[12]: Let (X, d) be a complete metric linear space, T be a fuzzy mapping from X into W(X) and x_0 ∈ X, then there exists x_1 ∈ x such that {x_1} ⊂ T{x_0}

Lemma 5[13]: Let A, B ∈ W(X) then for each {x}⊂A, there exists {y} ⊂ B such that D ({x}, {y}) ≤ D (A, B)

Let X be a non empty set and I = [0, 1].A fuzzy set of X is an element of I^X. For A, B ∈ I^X we denote A⊆B if and only if A(x) ≤ B(x) for each x ∈ X.

Definition (2.5)[15]: An intuitionist fuzzy set (i-fuzzy set) a of X is an object having the form $A=\langle A^1, A^2\rangle$, where A^1, A^2 ∈ I^X and $A^1(x) + A^2(x) \le 1$ for each x∈X. We denote by IFS(X) the family of all i-fuzzy sets of X

Definition (2.6)[14]: Let x_α be a fuzzy point of X. We will say that $\langle x_\alpha, 1-x_\alpha\rangle$ is an i-fuzzy point of x and it will be denoted by [x_α].

In particular [x] $=\langle\{x\},1-\{x\}\rangle$ will be called an i-point of X.

Definition (2.7)[15]: Let A, B ∈ IFS(X). Then A⊂B if and only if $A^1 \subset B^1$ and $B^2 \subset A^2$

Remark 2: Notice [x_α] ⊂ A if and only if $x_0 \subset A^1$

Let (X, d) be a metric linear space. The α- level set of A is denoted by

$$A_\alpha=\{x : A(x)\ge\alpha\}\ if\ \alpha\in]0,1]\ and\ A_0=\overline{\{x : A(x)>0\}},$$

Where $\bar{B}$ *denotes the clouser of (non fuzzy set) B*

Heilpern [10] called a fuzzy mapping from the set of X into a family W(X) ⊂ I^X defined as A ∈ W(X) if and only if A_α is compact and convex in X for each α ∈]0,1] and

Sup {Ax: x ∈ X} = 1. In this context we give the following definitions.

Definition (2.8)[6]: Let X be a metric space and α ∈ [0, 1]. Consider the following family

$$W_\alpha(X)=\{A\in I^X : A_\alpha \text{ is nonempty ,compact and convex}\}$$

Now we define the family if i-fuzzy sets of X as follows:

$$\Phi_\alpha(X)=\{A\in IFS(X): A^1\in W_\alpha(X)\},$$

it is clear that $\alpha\in I, W(X)\subset\Phi_\alpha(X)$.

Definition (2.9)[14]: Let x_α be a fuzzy point of X. we will say that x_α is a fixed fuzzy point of the fuzzy mapping F over X if x_α ⊂ F(x) (i.e. the fixed degree of x is at least α). In particular and according to [10], if {x} ⊂ F(x), we say that x is a fixed point of F.

Main Results

Theorem 1: Let α∈]0, 1] and (X, d) be a complete metric space. Let F be continuous fuzzy mapping from X into $W_\alpha(X)$ satisfying the following condition:

$D_\alpha(F(x),F(y))\le K(M(x,y))$, For all $x, y \in X$ with $x\ne y$,

and

$$M(x,y)=\phi\left\{\begin{array}{l} d(x,y).p_\alpha(x,Fy),\ p_\alpha(y,Fy).p_\alpha(x,Fy),\\ \dfrac{d(x,y)+p_\alpha(x,Fy)}{1+d(x,y).p_\alpha(x,Fy)},\\ \dfrac{p_\alpha(x,Fx)+p_\alpha(x,Fy)}{1+p_\alpha(x,Fx).p_\alpha(x,Fy)},\\ \dfrac{p_\alpha(y,Fy)+p_\alpha(x,Fy)}{1+p_\alpha(y,Fy).p_\alpha(x,Fy)}\end{array}\right\}$$

for all K ∈ (0,1]. Then there exists x ∈ X such that x_α is a fixed fuzzy point of F iff x_0, x_1 ∈X such that $x_1 \in F(x_0)_\alpha$ with $\sum_{n=1}^{\infty}k^n d(x_0,x_1)<\infty$ .In particular if α = 1 then x is a fixed point of F.

Proof: If there exists x ∈ X such that x_α is fixed fuzzy point of F, i.e. $x_\alpha \subset F(x)$ then $\sum_{n=1}^{\infty}k^n d(x,x) = 0$. Let x_0 ∈ K and suppose that there exists $x_1 \in (F(x_0))_\alpha$ such that $\sum_{n=1}^{\infty}k^n d(x_0,x_1)<\infty$.

Since $(F(x_1))_\alpha$ is a nonempty compact subset of X, then there exists $x_2 \subset (F(x_1))_\alpha$, such that $d(x_1,x_2)=p_\alpha(x_1,F(x_1))\le D_\alpha(F(x_0),F(x_1))$ By induction we construct a sequence {x_n} in X such that $x_n \subset (F(x_{n-1}))_\alpha$,

and $d(x_n,x_{n+1})\le D_\alpha(F(x_n),F(x_{n-1}))$.Since K is given to be the non-decreasing, so

$$d(x_n,x_{n+1})\le K\{M(x,y)\}$$

$$=K\phi\left\{\begin{array}{l} d(x_n,x_{n-1}).p_\alpha(x_n,F(x_{n-1})),\ p_\alpha(x_{n-1},F(x_{n-1})).p_\alpha(x_n,F(x_{n-1})),\\ \dfrac{d(x_n,x_{n-1})+p_\alpha(x_n,F(x_{n-1}))}{1+d(x_n,x_{n-1})p_\alpha(x_n,F(x_{n-1}))},\ \dfrac{p_\alpha(x_n,F(x_n))+p_\alpha(x_n,F(x_{n-1}))}{1+p_\alpha(x_n,F(x_n))p_\alpha(x_n,F(x_{n-1}))},\\ \dfrac{p_\alpha(x_{n-1},F(x_{n-1}))+p_\alpha(x_n,F(x_{n-1}))}{1+p_\alpha(x_{n-1},F(x_{n-1}))p_\alpha(x_n,F(x_{n-1}))}\end{array}\right\}$$

$$=K\phi\left\{\begin{array}{l} d(x_n,x_{n-1}).d(x_n,x_n),(x_{n-1},x_n).d(x_n,x_n),\dfrac{d(x_n,x_{n-1})+d(x_n,(x_n)}{1+d(x_n,x_{n-1})d(x_n,(x_n))},\\ \dfrac{d(x_n,x_{n+1})+(x_n,x_n)}{1+d(x_n,x_{n+1})d(x_n,x_n)},\dfrac{d(x_{n-1},x_n)+d(x_n,x_n)}{1+d(x_{n-1},x_n)\ d(x_n,x_n)}\end{array}\right\}$$

$$=K\phi\left\{\begin{array}{l} d(x_n,x_{n-1}).d(x_n,x_n),(x_{n-1},x_n).d(x_n,x_n),d(x_n,x_{n-1}),\\ \dfrac{d(x_n,x_{n-1})}{1},\dfrac{d(x_n,x_{n+1})}{1},\dfrac{d(x_{n-1},x_n)}{1}\end{array}\right\}$$

$$= K\left\{d(x_n, x_{n-1})\right\}$$

$$= K(p_\alpha d(x_{n-1}, F(x_{n-1}) \le K\left[D_\alpha\left(F(x_{n-1}, F(x_{n-2}))\right]\right.$$

$$\le K\left(Kd(x_{n-1}, x_{n-2})\right) ....K^n d(x_0, x_1)$$

$$\Rightarrow d(x_n, x_{n+m}) \le d(x_n, x_{n+1}) + ... + d(x_{n+m-1}, x_{n+m})$$

$$\le K^n\, d(x_0, x_1) + ... + K^{n+m-1} d(x_0, x_1)$$

$$= \sum_{k=n}^{k=n+m-1} K^k d(x_0, x_1)$$

Since $\sum_{n=1}^{\infty} k^n d(x_0, x_1) < \infty$ it follows that there exists u such that d(x_n, x_{n+m}) < u ∊X. Therefore the sequence {x_n} is a Cauchy sequence in X. So by completeness of X, {x_n} converges to x ∊ X. By the help of lemma 1 and 2 we have

$$p_\alpha(x, F(x)) \le d(x, x_n) + p_\alpha(x_n, F(x))$$

$$\le d(x, x_n) + D_\alpha(F(x_{n-1}), F(x))$$

$$\le d(x, x_n) + K\, d(x_{n-1}, x)$$

Consequently, $p_\alpha\left(x, F(x)\right) = 0,\ and\ by\ \ lemma 1\quad x_\alpha \in F(x)$.

Clearly x_α is a fixed fuzzy point of the fuzzy mapping F over X. In particular if α = 1 then x is a fixed point of F.

Now we will generalize this theorem for common fixed point.

Theorem 2: Let α∊]0, 1] and (X, d) be a complete metric space. Let T and S be continuous fuzzy mappings from X into W_α (X) and F: X→ W_α (X) be a mapping such that

$I.\quad F(X) \subset S(X) \bigcap T(X)$

$II.\ \{S, F\}$ and $\{T, F\}$ are

$R-$weakly commuting mappings.

$III.\ D_\alpha\,(Fx, Fy) \le K\left[M(x, y)\right]$

$\forall\, x, y \in X \quad$ with $x \ne y$,

where $M(x, y)$ is defined as

$$M(x, y) = K\,\phi\begin{Bmatrix} D_\alpha(Sx, Ty).D_\alpha(Fx, Ty), D_\alpha(Ty, Fy). \\ D_\alpha(Fx, Ty), \dfrac{D_\alpha(Sx, Ty) + D_\alpha(Fx, Ty)}{1 + D_\alpha(Sx, Ty) D_\alpha(Fx, Ty)}, \\ \dfrac{D_\alpha(Sx, Fx) + D_\alpha(Fx, Ty)}{1 + D_\alpha(Sx, Fx) D_\alpha(Fx, Ty)} \end{Bmatrix}$$

Where K is non decreasing function such that $K : [0, \infty) \to [0, \infty)$, $K(0) = 0$ and $K(t) < t\ \ \forall t \in (0, \infty)$, then $\exists x \in X$ such that x_α is common fixed fuzzy point of S, T and F if and only if $x_0, x_1 \in X$ such that $\sum_{n=1}^{\infty} K^n d(x_0, x_1) < \infty$. In perticuler if $\alpha = 1$, then x is common fixed point of S, T and F.

Proof: Let for x_0 ∊ X there exists x_1 and x_2 such that x_1 ∊ $(S(x_1))_\alpha$ ⊂ $(F(x_0))_\alpha$

And x_2 ∊ $(T(x_2))_\alpha$ ⊂ $(F(x_1))_\alpha$ .By induction one can construct a sequence { x_n } in X such that

$$x_{2n+1} \in \left(Sx_{2n+1}\right)_\alpha \subset \left(F(x_{2n}\right)_\alpha.$$

and $x_{2n+2} \in \left(Tx_{2n+2}\right)_\alpha \subset \left(F(x_{2n+1}\right)_\alpha$.

Since K is given to be non-decreasing. So

$$d(x_n, x_{n+1}) \le D_\alpha(F(x_{n-1}), F(x_n)) \le K.M(x_{n-1}, x_n)$$

$$= K\,\phi\begin{Bmatrix} D_\alpha(Sx_{n-1}, Tx_n).D_\alpha(Fx_{n-1}, Tx_n), \\ D_\alpha(Tx_n, Fx_n), D_\alpha(Fx_{n-1}, Tx_n), \\ \dfrac{D_\alpha(Sx_{n-1}, Tx_n) + D_\alpha(Fx_{n-1}, Tx_n)}{1 + D_\alpha(Sx_{n-1}, Tx_n) D_\alpha(Fx_{n-1}, Tx_n)}, \\ \dfrac{D_\alpha(Sx_{n-1}, Fx_{n-1}) + D_\alpha(Fx_{n-1}, Tx_n)}{1 + D_\alpha(Sx_{n-1}, Fx_{n-1}) D_\alpha(Fx_{n-1}, Tx_n)} \end{Bmatrix}$$

$$= K\ \phi\begin{Bmatrix} d(x_{n-1}, x_n).d(x_n, x_n), d(x_n, x_{n+1}). \\ d(x_n, x_n), \dfrac{d(x_{n-1}, x_n) + (x_n, x_n)}{1 + d(x_{n-1}, x_n) d(x_n, x_n)}, \\ \dfrac{d(x_{n-1}, x_n) + d(x_n, x_n)}{1 + (x_{n-1}, x_n)(x_n, x_n)} \end{Bmatrix}$$

$$= K\left\{d(x_n, x_{n-1})\right\}$$

Therefore $d(x_n, x_{n+1}) \le Kd(x_{n-1}, x_n)$

$$= KD_\alpha\left(F(x_{n-2}), F(x_{n-1})\right)$$

$$\le K^2 d(x_{n-1}, x_{n-2})$$

....

$$< K^n\ d(x_0, x_1)$$

$$\Rightarrow d(x_n, x_{n+m}) \le d(x_n, x_{n+1}) + ... + d(x_{n+m-1}, x_{n+m})$$

$$\le K^n\ d(x_0, x_1) + ... + K^{n+m-1} d(x_0, x_1)$$

$$= \sum_{j=n}^{k=n+m-1} K^j d(x_0, x_1)$$

Since $\sum_{n=1}^{\infty} k^n d(x_0, x_1) < \infty$ it follows that there exists u such that d(x_n, x_{n+m}) < u $\in$ X. Therefore the sequence {x_n} is a Cauchy sequence in X. So by completeness of X, {x_n} converges to x $\in$ X. And $(Sx_{2n+1})_\alpha$, (Tx_{2n+2}) also converges on X.

Since $\{S,F\}$ and $\{T,F\}$ are R-weakly commuting mappings. So

$$p_\alpha(x, F(x)) \le d(x, x_n) + p_\alpha(x_n, F(x))$$
$$\le d(x, x_n) + D_\alpha(F(x_{n-1}), F(x))$$
$$\le d(x, x_n) + K\, d(x_{n-1}, x)$$

Consequently, $p_\alpha\left(x, F(x)\right) = 0$,

and by $lemma\,1$ $x_\alpha \subset F(x)$

Clearly x_α is a common fixed fuzzy point of the fuzzy mapping F, S and T over X. In particular if α = 1 then x is a common fixed point of F, S and T.

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