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Analytical and Numerical Solutions to Rotating Variable Thickness Disks for a New Thickness Profile

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***Abstract*—Analytical and numerical solutions to rotating variable thickness disks are obtained. A new one-parameter exponential model is used to express the variation of disk thickness. It is shown by taking the limit that the present analytical solution reduces to the well-known homogeneous thickness solution. Furthermore, analytical and numerical solutions are brought into view together to allow comparison and further verification. The results of the solutions are presented in tables and figures to provide benchmark data for interested readers.**

Keywords*— *Rotating disks; Variable thickness; von Mises criterion; Exponential thickness variation

I. Introduction

Research on the prediction of stress and deformation in rotating or stationary disks under different loading conditions and comprising different materials is unending because of the importance of these basic structures in various branches of engineering. Plane stress analytical solutions for rotating solid and annular disk problems in the elastic state of stress have been available for many years in standard textbooks: [16]; [17]; [3]; [18]; [15]. Solutions involving thickness variability, partially plastic stress states, and material nonhomogeneity relevant to this investigation may be found in the most recent articles by [1], [2], [4], [6], [7], [8], [9], [10], [11], [12], [13], [14], [19], [20], [21], [22].

In this work, analytical and numerical solutions are obtained for rotating variable thickness homogeneous solid and annular disks. The thickness variaton is described by

$$h(r) = h_0 \, exp\left[\frac{\beta(r-a)}{b-a}\right] \tag{1}$$

where β is a parameter, r the radial coordinate, a and b inner and outer radii of the disk and h_0 is the value of disk thickness at $r = a$. Although this variation is used in FGM pressure chember studies of Chen and Lin [5], it is new in disk studies. The variation of the disk thickness along the radial direction in case of a solid disk is presented in Fig. 1.

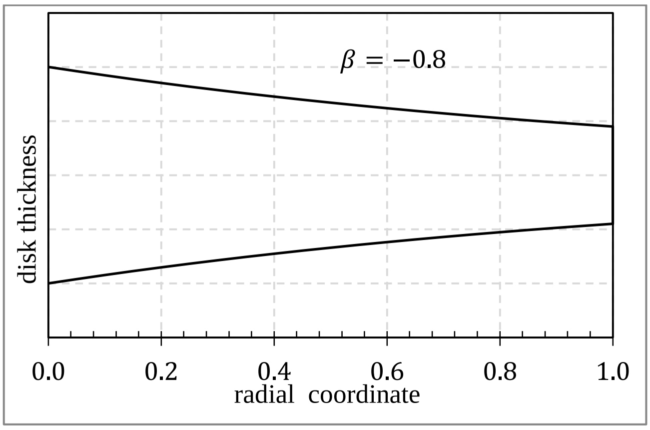

Figure 1. Variation of disk thickness along the radial direction.

II. Theory

A. Basic Equations

Thin disk and hence a state of plane stress is assumed. The thickness $h(r)$ of the disk varies in the radial direction according to (1). Dimensionless disk thickness $h = h/h_0$ and dimensionless and normalized variables are used in the model.

The strain displacement relations

$$\bar{\epsilon}_r = \frac{d\bar{u}}{d\bar{r}} \quad and \quad \bar{\epsilon}_\theta = \frac{\bar{u}}{\bar{r}} \tag{2}$$

the equation of equilibrium

$$\frac{d}{d\bar{r}}\left(\bar{h}\bar{r}\bar{\sigma}_r\right) - \bar{h}\bar{\sigma}_\theta + \bar{h}\Omega^2\bar{r}^2 = 0. \tag{3}$$

The compatibility relation

$$\frac{d}{d\bar{r}}(\bar{r}\bar{\epsilon}_\theta) - \bar{\epsilon}_r = 0, \tag{4}$$

and the equations of the generalized Hooke's law

$$\bar{\epsilon}_r = \bar{\sigma}_r - \nu\bar{\sigma}_\theta \tag{5}$$

$$\bar{\epsilon}_\theta = \bar{\sigma}_\theta - \nu\bar{\sigma}_r \tag{6}$$

constitute the basic equations of the variable thickness model in their dimensionless forms [17]. In these equations, $\bar{\epsilon}_r = \epsilon_r E/\sigma_0$ and $\bar{\epsilon}_\theta = \epsilon_\theta E/\sigma_0$ represent the normalized strains, $\bar{r} = r/b$ the dimensionless radial coordinate, $\bar{u} = uE/b\sigma_0$ the dimensionless radial displacement, $\bar{\sigma}_r = \sigma_r/\sigma_0$ and $\bar{\sigma}_\theta = \sigma_\theta/\sigma_0$ the dimensionless stresses, ν the Poisson's ratio, $\Omega = \omega b\sqrt{\rho/\sigma_0}$ the dimensionless angular speed, ω the angular speed, ρ the mass density, and σ_0 the yield limit. Thereafter overbars will not be used for simplicity. We now introduce a stress function of the form

$$F(r) = hr\sigma_r\,, \tag{7}$$

and get from the equation of equilibrium (3)

$$\sigma_\theta = \Omega^2 r^2 + \frac{1}{h}\frac{dF}{dr}. \tag{8}$$

Thus, the expressions for the elastic strains take the forms

$$\epsilon_r = \frac{F}{hr} - \nu\left(\Omega^2 r^2 + \frac{1}{h}\frac{dF}{dr}\right), \tag{9}$$

and

$$\epsilon_\theta = \Omega^2 r^2 + \frac{1}{h}\frac{dF}{dr} - \nu\frac{F}{hr}. \tag{10}$$

B. The Governing Equation

The governing differential equation for variable thickness disk is obtained upon substitution of (9) and (10) into the compatibility relation,(4). The result, after some algebra is

$$\frac{d^2F}{dr^2} + \left(\frac{1}{r} - \frac{\beta}{1-a}\right)\frac{dF}{dr} - \left(\frac{1}{r} - \frac{\beta\nu}{1-a}\right)\frac{F}{r} = -exp\left[\frac{\beta(r-a)}{1-a}\right](3+\nu)r\Omega^2\,. \tag{11}$$

III. Analytical and Numerical Solutions

A. Analytical Solution

The governing equation, (11), is a second order, nonhomogeneous, linear ordinary differential equation with variable coefficients. The general solution is obtained by the power series method. The solution can be put into the form

$$F(r) = C_1F_1(r) + C_2F_2(r) + F_P(r), \tag{12}$$

where C_1 and C_2 are arbitrary integration constants and

$$F_1(r) = r + \frac{\beta(1-\nu)}{3(1-a)}r^2 + \frac{\beta^2(1-\nu)(2-\nu)}{24(1-a)^2}r^3 + \frac{\beta^3(1-\nu)(2-\nu)(3-\nu)}{360(1-a)^3}r^4 + \frac{\beta^3(1-\nu)(2-\nu)(3-\nu)(4-\nu)}{8640(1-a)^4}r^5 + \frac{\beta^3(1-\nu)(2-\nu)(3-\nu)(4-\nu)(5-\nu)}{302400(1-a)^5}r^6 + \cdots\,, \tag{13}$$

$$F_2(r) = \ln r\left[\frac{\beta^2\nu(1+\nu)}{(1-a)^2}r + \frac{\beta^3\nu(1+\nu)(1-\nu)}{3(1-a)^3}r^2 + \frac{\beta^4\nu(1+\nu)(1-\nu)(2-\nu)}{24(1-a)^4}r^3 + \frac{\beta^5\nu(1+\nu)(1-\nu)(2-\nu)(3-\nu)}{360(1-a)^5}r^4 + \frac{\beta^6\nu(1+\nu)(1-\nu)(2-\nu)(3-\nu)(4-\nu)}{8640(1-a)^6}r^5 + \cdots\right] - \frac{2}{r} - \frac{2\beta(1+\nu)}{1-a} - \frac{\beta^2(1+2\nu)}{(1-a)^2}r - \frac{\beta^3(3+4\nu-9\nu^2-4\nu^3)}{9(1-a)^3}r^2 - \frac{\beta^4(24+26\nu-97\nu^2-2\nu^3+25\nu^4)}{288(1-a)^4}r^3 + \cdots\,, \tag{14}$$

$$F_P(r) = -exp\left[-\frac{\beta a}{1-a}\right](3+\nu)\Omega^2 r^3\left[\frac{1}{8} + \frac{\beta(11-\nu)}{120(1-a)}r + \frac{\beta^2(104-15\nu+\nu^2)}{2880(1-a)^2}r^2 + \frac{\beta^3(1000-179\nu+20\nu^2-\nu^3)}{100800(1-a)^3}r^3 + \cdots\right] \tag{15}$$

In the case of a rotating solid disk, the stresses must be finite at the center, hence $C_2 = 0$. The outer boundary is free of traction, and as a result, $\sigma_r(1) = 0$. Then, we find from (12)

$$C_1 = -\frac{F_P(1)}{F_1(1)} \tag{16}$$

It should be noted that, for a solid disk

$$\sigma_r(0) = \lim_{r\to 0}\left(\frac{F}{r}\right) = \frac{dF}{dr}\,. \tag{17}$$

The boundary conditions for a rotating annular disk are $\sigma_r(a) = \sigma_r(1) = 0$, which leads to

$$C_1 = \frac{F_2(a)F_P(1) - F_2(1)F_P(a)}{F_1(a)F_2(1) - F_1(1)F_2(a)}; \quad C_2 = -\frac{F_1(a)F_P(1) - F_1(1)F_P(a)}{F_1(a)F_2(1) - F_1(1)F_2(a)} \tag{18}$$

Note that for $\beta = 0$

$$F(r) = C_1 r + \frac{C_2}{r} - \frac{1}{8}(3+\nu)\Omega^2 r^3, \tag{19}$$

and, as a result, the solution of uniform thickness homogeneous disk is verified [18]

$$u = C_1 r + \frac{C_2}{r} - \frac{1}{8}(1-\nu^2)\Omega^2 r^3 \tag{20}$$

B. Numerical Solution

The governing equation is put into the form

$$\frac{d^2F}{dr^2} = -\left(\frac{1}{r} - \frac{\beta}{1-a}\right)\frac{dF}{dr} + \left(\frac{1}{r} - \frac{\beta\nu}{1-a}\right)\frac{F}{r} - exp\left[\frac{\beta(r-a)}{1-a}\right](3-\nu)r\Omega^2 \tag{21}$$

If we let $\varphi_1 = F$, and $\varphi_2 = dF/dr$, then by differentiating

$$\frac{d\varphi_1}{dr} = \varphi_2,$$
$$\frac{d\varphi_2}{dr} = -\left(\frac{1}{r} - \frac{\beta}{1-a}\right)\varphi_2 + \left(\frac{1}{r} - \frac{\beta\nu}{1-a}\right)\frac{\varphi_1}{r} - exp\left[\frac{\beta(r-a)}{1-a}\right](3-\nu)r\Omega^2. \tag{22}$$

In this way, the governing equation is transformed into an initial value problem (IVP) consisting of two dependent variables. This IVP can accurately be integrated by using a state of the art ODE solver, starting with the initial conditions: φ_1^0, and φ_2^0. Since $\varphi_1 = F = hr\sigma_r$, for both solid and annular disks $\varphi_1^0 = 0$ and $\varphi_1(1) = 0$, but φ_2^0 is not known. This unknown initial value can be determined by shooting iterations. The condition that should be satisfied is

$$\varphi_1(1) = 0.$$

Hence, iterations begin with an initial estimate φ_2^0 and continue until $\varphi_1(1) = 0$ is satisfied.

At the k^{th} iteration, the IVP described by (22) is solved 3 times: starting with φ_2^{k-1} to obtain $\Phi_1 = \varphi_1(1)$, with $\varphi_2^{k-1} + \Delta\varphi$ to obtain $\Phi_2 = \varphi_1(1)$, and finally with $\varphi_2^{k-1} - \Delta\varphi$ to obtain $\Phi_3 = \varphi_1(1)$, where $\Delta\varphi$ is a small increment of the order of $\sim 10^{-3}$. A better approximation to φ_2^0 is then acquired from Newton's formula

$$\varphi_2^0 = \varphi_2^k = \varphi_2^{k-1} - \frac{2\Delta\varphi\,\Phi_1}{\Phi_2 - \Phi_3}\ . \qquad (23)$$

When the iterations converge, the IVP system in (22) is solved once more with the converged φ_2^0 value in order to determine the stress and deformations. The Runge-Kutta-Fehlberg fourth-fifth order integration method is used with tight tolerances for the integration of the IVP.

The advantages of this method are the accuracy, stability, and rapid rate of convergence. With a reasonable initial estimate φ_2^0 only a few iterations are performed to reach convergence.

IV. Presentation of Results

In the following calculations $\nu = 0.3$. The von Mises yield criterion given by

$$\sigma_Y = \sqrt{\sigma_r^2 - \sigma_r\sigma_\theta + \sigma_\theta^2}$$

is used to determine the elastic limit of the disk [6]. As σ_r and σ_θ are dimensionless, the elastic limit corresponds to $\sigma_Y = 1$. In all the figures solid lines belong to analytical and dots to numerical solutions.

A. *Solid Disk*

It is apparent from existing literature that, if the thickness of the disk gets thinner from the center to the edge, the strength of the disk increases because of the reduction of stresses. For this reason negative values of the thickness parameter β are used in variable thickness calculations. For $\beta = 0$, the elastic limit angular speed has been determined earlier as $\Omega = 1.55700$. Analytical calculations are carried out for $\beta = -0.25$ and $\beta = -0.5$ at this limit. The nonzero integration constants are determined to be $C_1 = 0.886952$ and 0.786864 corresponding to $\beta = -0.25$ and $\beta = -0.5$, respectively. The results of these analytical calculations together with the results of the numerical solutions are put forward in Fig. 2 and Fig. 4. Furthermore, selected numbers from the analytical solutions are presented in Table 1.

B. *Annular Disk*

The elastic limit angular speed of a uniform thickness homogeneous annular disk of $a = 0.2$ is $\Omega = 1.09632$. Calculations are performed at this angular speed. The integration constants are determined as $C_1 = 0.467962$, $C_2 = 0.00956$ corresponding to $\beta = -0.25$ and $C_1 = 0.426189$, $C_2 = 0.00921$ corresponding to $\beta = -0.5$. The analytical and numerical profiles are drawn in Fig. 5 and Fig. 7. Table 2 tabulates some analytical results pertinent to these calculations.

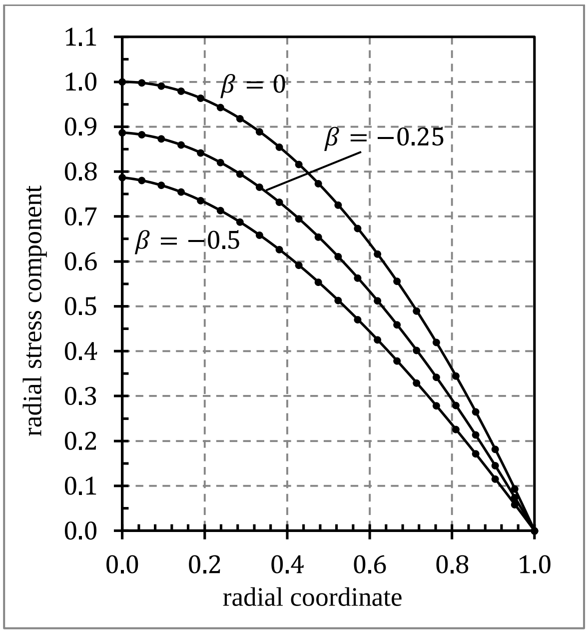

Figure 2. Variation of radial stress in a rotating variable thickness solid disk for different values of $\boldsymbol{\beta}$.

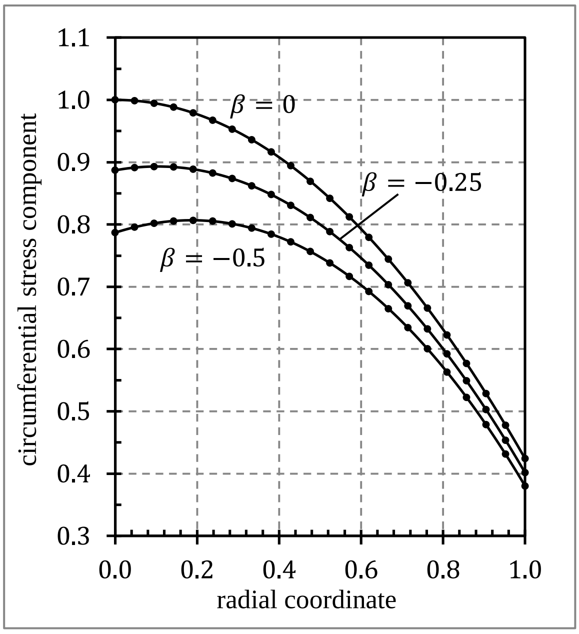

Figure 3. Variation of circumferential stress in a rotating variable thickness solid disk for different values of $\boldsymbol{\beta}$.

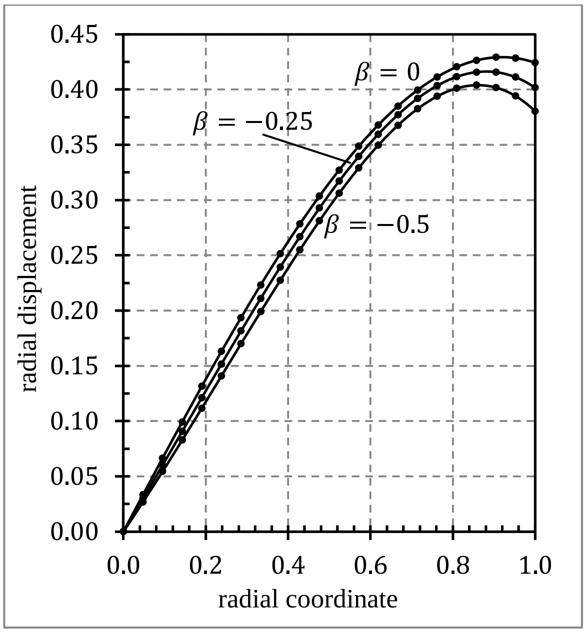

Figure 4. Variation of radial displacement in a rotating variable thickness solid disk for different values of $\boldsymbol{\beta}$.

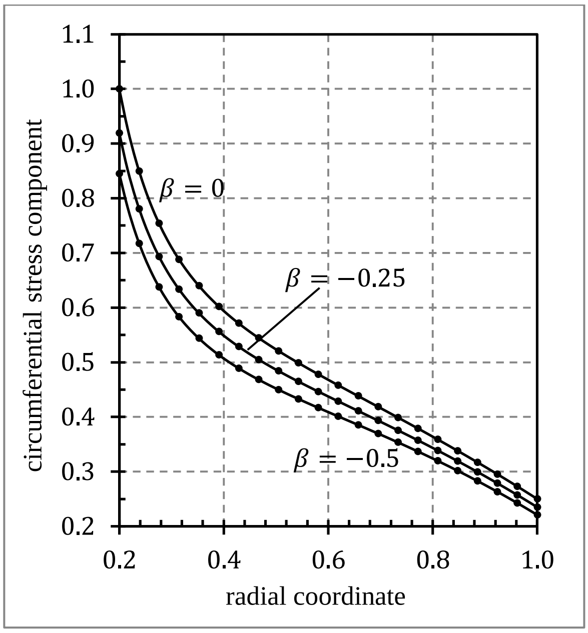

Figure 6. Variation of circumferential stress in a rotating variable thickness annular disk of $\boldsymbol{a = 0.2}$ for different values of $\boldsymbol{\beta}$.

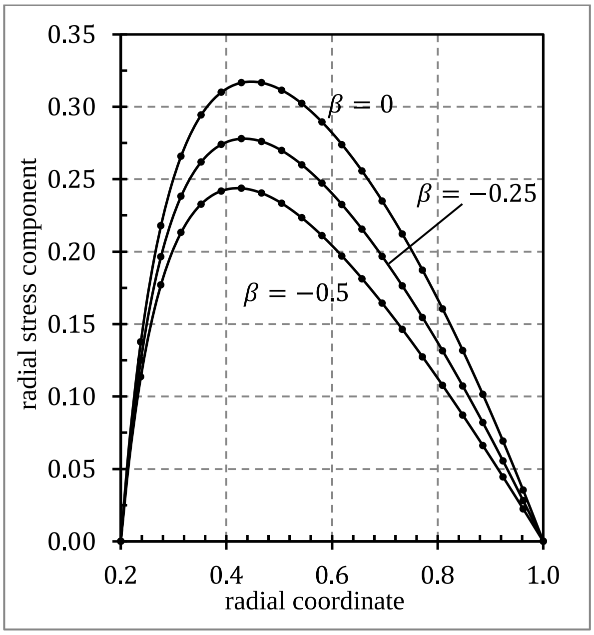

Figure 5. Variation of radial stress in a rotating variable thickness annular disk of $\boldsymbol{a = 0.2}$ for different values of $\boldsymbol{\beta}$.

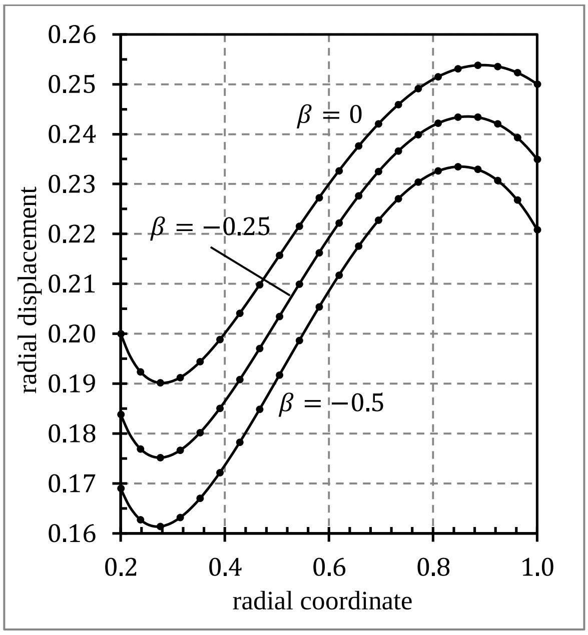

Figure 7. Variation of radial displacement in a rotating variable thickness annular disk of $\boldsymbol{a = 0.2}$ for different values of $\boldsymbol{\beta}$.

Table 1. Analytical solutions to rotating variable thickness solid disks

r	σ_r			σ_θ			u		
	$\beta=0$	$\beta=-0.25$	$\beta=-0.5$	$\beta=0$	$\beta=-0.25$	$\beta=-0.5$	$\beta=0$	$\beta=-0.25$	$\beta=-0.5$
0.00	1.00000	0.88695	0.78686	1.00000	0.88695	0.78686	0.00000	0.00000	0.00000
0.05	0.99750	0.88189	0.77984	0.99856	0.89145	0.79600	0.03497	0.03134	0.02810
0.10	0.99000	0.87198	0.76813	0.99425	0.89309	0.80238	0.06972	0.06315	0.05719
0.15	0.97750	0.85735	0.75198	0.98705	0.89184	0.80595	0.10407	0.09520	0.08705
0.20	0.96000	0.83811	0.73163	0.97697	0.88769	0.80667	0.13779	0.12725	0.11744
0.25	0.93750	0.81441	0.70732	0.96402	0.88059	0.80447	0.17069	0.15907	0.14807
0.30	0.91000	0.78635	0.67927	0.94818	0.87053	0.79930	0.20256	0.19039	0.17866
0.35	0.87750	0.75407	0.64769	0.92947	0.85748	0.79110	0.23318	0.22094	0.20888
0.40	0.84000	0.71767	0.61280	0.90788	0.84140	0.77982	0.26235	0.25044	0.23839
0.45	0.79750	0.67728	0.57478	0.88341	0.82226	0.76538	0.28987	0.27859	0.26683
0.50	0.75000	0.63300	0.53384	0.85606	0.80005	0.74774	0.31553	0.30507	0.29380
0.55	0.69750	0.58495	0.49014	0.82584	0.77472	0.72682	0.33912	0.32958	0.31888
0.60	0.64000	0.53324	0.44388	0.79273	0.74624	0.70257	0.36044	0.35176	0.34165
0.65	0.57750	0.47797	0.39521	0.75674	0.71460	0.67492	0.37927	0.37128	0.36163
0.70	0.51000	0.41925	0.34430	0.71788	0.67975	0.64380	0.39542	0.38778	0.37836
0.75	0.43750	0.35717	0.29130	0.67614	0.64166	0.60914	0.40867	0.40088	0.39131
0.80	0.36000	0.29185	0.23637	0.63152	0.60031	0.57088	0.41881	0.41021	0.39998
0.85	0.27750	0.22338	0.17963	0.58402	0.55566	0.52894	0.42565	0.41535	0.40379
0.90	0.19000	0.15185	0.12124	0.53364	0.50769	0.48325	0.42897	0.41592	0.40219
0.95	0.09750	0.07736	0.06132	0.48038	0.45634	0.43374	0.42857	0.41148	0.39458
1.00	0.00000	0.00000	0.00000	0.42424	0.40161	0.38034	0.42424	0.40161	0.38034

Table 2. Analytical solutions to rotating variable thickness annular disks

r	σ_r			σ_θ			u		
	$\beta=0$	$\beta=-0.25$	$\beta=-0.5$	$\beta=0$	$\beta=-0.25$	$\beta=-0.5$	$\beta=0$	$\beta=-0.25$	$\beta=-0.5$
0.20	0.00000	0.00000	0.00000	1.00000	0.91913	0.84486	0.20000	0.18383	0.16897
0.24	0.14277	0.12964	0.11767	0.84348	0.77527	0.71260	0.19216	0.17673	0.16255
0.28	0.22380	0.20174	0.18173	0.74620	0.68625	0.63111	0.19014	0.17521	0.16145
0.32	0.27119	0.24269	0.21702	0.68006	0.62614	0.57645	0.19159	0.17707	0.16363
0.36	0.29835	0.26510	0.23535	0.63165	0.58249	0.53709	0.19517	0.18107	0.16794
0.40	0.31235	0.27558	0.24290	0.59390	0.54874	0.50693	0.20008	0.18643	0.17362
0.44	0.31720	0.27790	0.24318	0.56280	0.52113	0.48246	0.20576	0.19262	0.18018
0.48	0.31532	0.27430	0.23832	0.53593	0.49742	0.46157	0.21184	0.19926	0.18724
0.52	0.30822	0.26623	0.22965	0.51178	0.47615	0.44290	0.21804	0.20606	0.19448
0.56	0.29691	0.25464	0.21808	0.48935	0.45638	0.42555	0.22415	0.21279	0.20167
0.60	0.28205	0.24018	0.20422	0.46795	0.43746	0.40889	0.23000	0.21925	0.20858
0.64	0.26413	0.22331	0.18851	0.44712	0.41894	0.39249	0.23544	0.22525	0.21500
0.68	0.24348	0.20437	0.17127	0.42652	0.40049	0.37602	0.24036	0.23064	0.22076
0.72	0.22035	0.18361	0.15276	0.40590	0.38186	0.35923	0.24465	0.23528	0.22565
0.76	0.19492	0.16124	0.13317	0.38508	0.36286	0.34194	0.24822	0.23901	0.22951
0.80	0.16733	0.13741	0.11266	0.36392	0.34335	0.32400	0.25098	0.24171	0.23216
0.84	0.13769	0.11223	0.09134	0.34231	0.32323	0.30528	0.25285	0.24323	0.23342
0.88	0.10607	0.08582	0.06934	0.32018	0.30240	0.28568	0.25375	0.24345	0.23309
0.92	0.07256	0.05827	0.04673	0.29745	0.28078	0.26512	0.25362	0.24223	0.23102
0.96	0.03718	0.02964	0.02359	0.27407	0.25832	0.24353	0.25240	0.23945	0.22700
1.00	0.00000	0.00000	0.00000	0.25000	0.23496	0.22084	0.25000	0.23496	0.22084

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