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The Condition For A Genetic Algebra To Be A Special Train Algebra

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***Abstract*—This paper discusses the structure of non-associative commutative algebra over real or complex numbers associated with genetics. .In particular we discuss the relation between genetic algebra and special train algebra. Genetic algebra and special train algebra are subclass of baric algebra (algebra which contains nontrivial homomorphism to its field). We obtain that each special train algebra is genetic but the converse is not true. Furthermore,with the condition that every principal powers of ideal (kernel homomorphismof baric) is also an ideal of genetic algebra, we have equivalency between genetic algebra and special train algebra.**

***Keywords*—baric algebra, genetic algebra, special train algebra.**

I. Introduction

In1939, Etherington[1] introduced the algebra concepts of genetics algebra as a commutative algebra, non-associative and finite dimensional over real or complex numbers. One of the concept introduced is baric algebra if it admits a nontrivial algebra homomorphism ω into the coefficient field., i.e.baric algebra is the algebra with a one-dimensional representation.One of the subclass of baric algebra is special train algebra. Furthermore, Schafer [4] in 1949 introduced the concept of genetic algebra and showed that every special train algebra is the genetic algebra but not vice-versa. In 1971, Gonshor [2] defined the concept of genetic algebra more operational and showed that the definition given is equivalent to the definition of genetic algebra defined by Schafer. Moreover, Wörz-Busekkros [5] in 1980 introduced a more common genetic algebra, i.e genetic algebra over expansion of the Gonshor's field. In this paper we wil luse the definition of genetic algebra based on the definition by Wörz-Busekkros. in [5], Wörz-Busekkros have shown that each special train algebra is a genetic algebra, but genetic algebra not necessarily be a special train algebra because the principal powers of ideal (kernel homomorphism of baric) is not necessarily an ideal.Therefore, we will give a condition that make a genetic algebra becoming a special train algebra.

II. GENETIC ALGEBRA

Genetic algebra is a subclass of baric algebra. Hence, before defining the genetic algebra we define a baric algebra.

Definition 1 Algebra *A* over of the field *F* ($\mathbb{R}$ or $\mathbb{C}$) is called *baric algebra* if it admits a non-trivial homomorphismω: $A \to F$. The homomorphism ω is called the weight function (or baric function).

The following is an example of algebra which is baric algebra and non baric algebra.

Example 2 Let $A = \langle a_1, a_2, a_3 \rangle_{\mathbb{R}}$ is a3-dimensional commutative algebra with the multiplication table as follows:

	a_1	a_2	a_3
a_1	a_1+a_2	a_2	a_2
a_2	a_2	a_2	a_2
a_3	a_2	a_2	a_2+a_3

Define ω_1: $A \to \mathbb{R}$ with $\omega_1(a_1)=1$ and $\omega_1(a_2)= \omega_1(a_3)=0$ and define ω_2: $A \to \mathbb{R}$ with $\omega_2(a_3)=1$ and $\omega_2(a_1)= \omega_2(a_2)=0$. Obviously we have that $\omega_1 \neq \omega_2$ and its easy to see that ω_1, ω_2 was a homomorphism.

Example 3 Let *A* algebra over the field $\mathbb{R}$, ω: *A*→ $\mathbb{R}$ is a homomorphism and multiplication on *A* is define as:

$$xy = \omega(x)y - x\omega(y)$$

For each $x, y \in A$, $\omega(x)\omega(y) = \omega(xy) = \omega(x)\omega(y) - \omega(x)\omega(y) = 0$. Since $\omega(x), \omega(y) \in \mathbb{R}$ then $\omega(x) = 0$ or $\omega(y) = 0$,but $\omega(x^2) = \omega(x)\,\omega(x) = 0$ then $\omega(x) = 0$for each $x \in A$. As a consequence *A* is not a baric algebra because it only contain a trivial homomorphism
ω: *A*→ $\mathbb{R}$.
Based on this concept of baric algebra, Schafer [4] was defined the genetic algebra . In this paper we use the genetic algebra equivalent definition that is introduced by Gonshor [2] and was expanded byWörz-Busekkros [5].

Definition 4 Let *A* be a commutative algebra over *F*. *A* is called a genetic algebra if algebra A_L (*A* is Gonshor genetic over *L*, *L* is suitable algebraic extension of *F*), admit a basis $c_0, c_1, \dots, c_n, c_0 \in A$ where the multiplication constants λ_{ijk}defined by :

$$c_i c_j = \sum_{k=0}^{n} \lambda_{ijk} c_k$$

We have the following properties:

$(i)\ \lambda_{000} = 1,$

$(ii)\ \lambda_{0jk} = 0, \qquad k < j,$

$(iii)\ \lambda_{ijk} = 0, \qquad i, j > 0, k \le max(i, j).$

The bases of *A* is called the canonical base. Furthermore, $\lambda_{000} = 1$, $\lambda_{011}, \dots, \lambda_{0nn}$ is called the *train roots* of *A*.

Furthermore, a genetic algebra can be characterized by a chain of its ideal with some additional properties.

Theorem 5 *Let A algebra over F with dimension* $m + 1$.*The following assertions are equivalent:*
(1) A genetic
A had the ideal π *dimension m with* $A^2 \not\subseteq \pi$ *and* A_L *chain of the ideals:*

$$A_L > \pi_L = \pi_1 > \cdots > \pi_m > \pi_{m+1} = \langle 0 \rangle$$

Top of the field extension L of F in accordance with the $dim\, \pi_i = m + 1 - i \quad i = 1, 2, \dots, m$ *and* $\pi_i \pi_j \subseteq \pi_l, \ l = max(i, j) + 1, \ i, j = 1, 2, \dots, m.$

Proof (⇒) Let *A* be a genetic algebra, thus *A* is a baric algebra. As a consequence *A* has a *m*-dimensional ideal $\pi = \ker\omega$ and $A^2 \not\subseteq \pi$. By definition , there are the expansion field *L* of *F* accordingly so that$A_L$has canonical basis$c_0, c_1, \dots, c_m$. Put$\pi_i = \langle c_i, c_{i+1}, \dots, c_m \rangle_L$, $i = 1, 2, \dots, m$thusdim$\,\pi_i = m + 1 - i$. Since *A* genetic then $c_i c_j = \sum_{k=0}^{m} \lambda_{ijk} c_k$ applicable $\lambda_{ijk} = 0, \ k \le \max(i, j), \ i, j > 0$.

Let $\max(i, j) = i$ then $k \le i$, consequently $k < i + 1 = l$. Since $\pi_i = \langle c_i, c_{i+1}, \dots, c_m \rangle$ and $\pi_j = \langle c_j, c_{j+1}, \dots, c_m \rangle$, then $\pi_l = \langle c_{i+1}, c_{i+2}, \dots, c_m \rangle$. Note that,

$$c_i c_j = \sum_{k=0}^{m} \lambda_{ijk} c_k = 0, k \le i$$

Which means that $c_i c_j \in \pi_l, \ k \ge l$. So, $\pi_i \pi_j \subseteq \pi_l, \ l = \max(i, j) + 1, \ i, j = 1, 2, \dots, m.$
(⇐)Suppose (2) hold. Choose a basis$\{c_1, \dots, c_n\}$of$\pi_1$so that $c_j \in \pi_j, c_j \in \pi_{j+1}, j = 1, 2, \dots, m$ and $c_0 \in A, \omega(c_0) = 1, \ \omega(c_j) = 0$. Since$\omega$a *weight* homomorphism, then $1 = \omega(c_0{}^2) = \omega(\sum_{k=0}^{m} \lambda_{00k} c_k) = \lambda_{000}\omega(c_0) + \cdots + \lambda_{00m}\omega(c_m)$.
We obtain $\lambda_{000} = 1$ .Furthermore, since $\pi_j = \langle c_j, c_{j+1}, \dots, c_m \rangle, j = 1, 2, \dots, m$ ideal of A_L thus $c_0 c_j = \sum_{k=0}^{m} \lambda_{0jk} c_k \in \pi_j$. Therefore, it should be$\lambda_{0jk} = 0, k < j$. Furthermore, since $\pi_i \pi_j \subseteq \pi_l, \ l = \max(i, j) + 1, \ i, j = 1, 2, \dots, m$, then$\lambda_{ijk} = 0, \ k \le \max(i, j)$.Thus, it can be concluded that *A* is genetic.■

Corollary 6 *Let A be a genetic algebra top of L* field extension *of F* with weight homomorphism ω, then $\pi = ker\,\omega$ is nilpotent.

Proof By the Theorem 5,There area sequence of ideal:

$$A_L > \pi_L = \pi_1 > \cdots > \pi_m > \pi_{m+1} = \langle 0 \rangle$$

Where *L* is a suitable extension field of *F* and $\pi_i \pi_j \subseteq \pi_l, \ l = \max(i, j) + 1, \ i, j = 1, 2, \dots, m$. Since $\pi = \pi_1 \cap A$ then by induction on *i* it will be shown that the *principal powers of* π satisfy the relation:
$\pi^i = \pi^{i-1}\pi \subseteq \pi_{i-1}\pi_1 \subseteq \pi_i$.
For*i* = 1, trivial. For *i* = 2, $\pi^2 = \pi\pi \subseteq \pi_1\pi_1 \subseteq \pi_2$. Suppose properly for *i* = *k*, then we have relation$\pi^k = \pi^{k-1}\pi \subseteq \pi_{k-1}\pi_1 \subseteq \pi_k$. Consequently, for *i* =*k* +1 we obtain:
$\pi^{k+1} = \pi^k\pi = (\pi^{k-1}\pi)\pi \subseteq (\pi_{k-1}\pi_1)\pi \subseteq \pi_k\pi \subseteq \pi_k\pi_1 \subseteq \pi_{k+1}$.
Thus $\pi^{m+1} \subseteq \pi_{m+1} = \langle 0 \rangle$, i.e. πnilpotent. ■

III. SPECIAL TRAIN ALGEBRA

We will further discuss a special train algebra and its properties related to the genetic algebra.

Definition 7. Baric algebra *A* with homomorphism *weight* ω is called special train algebra *if* π = *ker* ω is nilpotent and all *principal powers* $\pi^i$of *A* that defined by $\pi^1 = \pi$, $\pi^i = \pi^{i-1}\pi$ for $i = 2,3, \dots$, is ideal of *A.*

One of the main characteristics of the special train algebra to be genetic algebraas stated in the following theorem. Our proof follows [5].

Theorem 8. *Let A baric algebra over F with homomorphism weight* ω*.* The following statementis equivalent*:*

(1) A special train algebra.

(2) A genetic and all *principal power of ker* ω *is ideal of A.*

Before proving Theorem 8, we will discusse the left linear transformation of algebra *A*.

Let $c_1, c_2, \dots, c_n$is *F*-basis of *A*with the multiplication:

$$c_i c_j = \sum_{k=1}^{n} \gamma_{ijk}\, c_k, \qquad i,j = 1, \dots, n$$

For each $x \in A$, left linear transformation $\mathcal{L}_x: A \longrightarrow A$ with the association of $c \mapsto xc$, $c \in A$ and $x = \sum_{i=1}^{n} \varepsilon_i c_i$ is represented by the matrix:

$$L_x = \begin{pmatrix} \sum_{i=1}^{n} \varepsilon_i \gamma_{i11} & \cdots & \sum_{i=1}^{n} \varepsilon_i \gamma_{in1} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} \varepsilon_i \gamma_{i1n} & \cdots & \sum_{i=1}^{n} \varepsilon_i \gamma_{inn} \end{pmatrix} = \left(\sum_{i=1}^{n} \varepsilon_i \gamma_{ijk} \right)^{T}_{j,k=1,\dots,n}$$

Matrix corresponding to the linear transformation left $\mathcal{L}_{c_1}, \dots, \mathcal{L}_{c_n}$expressed by:

$$\mathcal{T}_1 = \left(\gamma_{1jk}\right)^T, \dots, \mathcal{T}_n = \left(\gamma_{njk}\right)^T$$

Consequently $L_x$can be written as:

$L_x = \sum_{i=1}^{n} \varepsilon_i \mathcal{T}_i$.

Example 9. Let *A* the genetic algebra over *F*, then $A_L$has a canonical basis $c_0, c_1, \dots, c_m$, $c_0 \in A$with the multiplication tables $c_i c_j = \sum_{k=0}^{n} \lambda_{ijk}\, c_k$, $i,j = 0, \dots, n$ and $\lambda_{000} = 1, \lambda_{0jk} = 0, k < j$ and $\lambda_{ijk} = 0, k \leq \max(i,j)$, $i,j > 0$. Provided that each $\mathcal{L}_x$ with $x = \sum_{i=0}^{n} \varepsilon_i c_i \in A$ corresponds to the matrix:

$$L_x = \sum_{i=0}^{m} \varepsilon_i \mathcal{T}_i$$

where

$$\mathcal{T}_0 = \begin{pmatrix} 1 & & & 0 \\ \lambda_{001} & \lambda_{011} & & \\ \vdots & \vdots & \ddots & \\ \lambda_{00m} & \lambda_{01m} & \cdots & \lambda_{0mm} \end{pmatrix}$$

and

$\mathcal{T}_i$

$$= \begin{pmatrix} 0 & & & & & \\ \vdots & \ddots & & & & \\ \lambda_{i0i} & 0 & 0 & & & \\ \lambda_{i0(i+1)} & \cdots & \lambda_{ii(i+1)} & 0 & & \\ \vdots & & \vdots & \vdots & \ddots & \\ \lambda_{i0m} & \cdots & \lambda_{iim} & \cdots & \lambda_{i(m-1)m} & 0 \end{pmatrix}, \qquad i = 1, \cdots, m$$

.

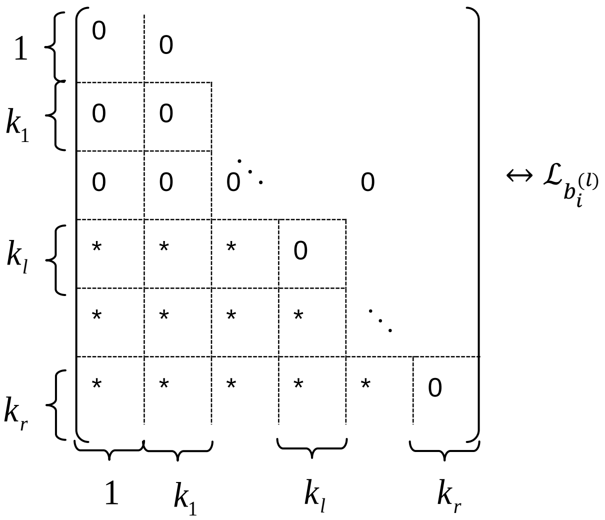
Example 10. Let $A = \langle c_0, c_1 \rangle_{\mathbb{R}}$ with themultiplicationtables:

	c_0	c_1
c_0	$c_0 + c_1$	-2c_1
c_1	-2c_1	-c_1

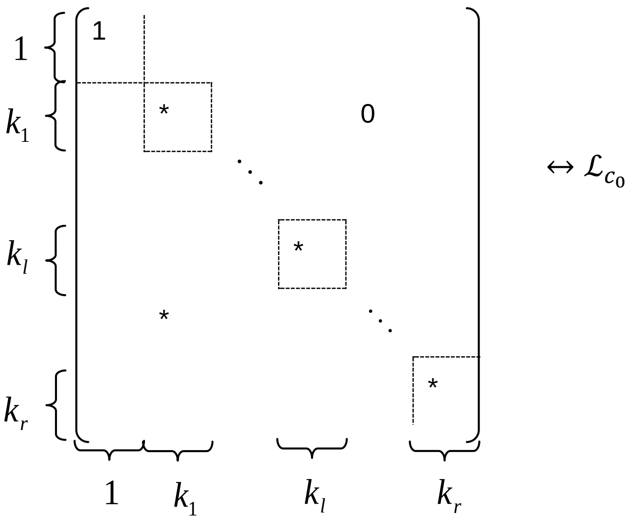
Thus left transformation $\mathcal{L}_{c_0}$ dan $\mathcal{L}_{c_1}$is represented by the matrix:

$L_{c_0} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ and $L_{c_1} = \begin{pmatrix} 0 & 0 \\ 2 & -1 \end{pmatrix}$.

Proof of Theorem 8 ($\Rightarrow$) Let *A* is train algebra and $\pi = \ker \omega$ is nilpotent of index *r* and the principal powers $\pi = \pi^1$ are ideal of *A*. It will be shown that *A* have a canonical basis over a suitable extension field *L* of *F*. Let $c_0 \in A$ with $\omega(c_0) = 1$. Furthermore, let $b_1^{(1)}, \dots, b_{k_1}^{(1)}, \dots, b_1^{(r)}, \dots, b_{k_r}^{(r)}$ be basis of π^1 so that $b_1^{(1)}, \dots, b_{k_l}^{(l)} \in \pi^1$ but not in $\pi^{l+1}, l = 1, \dots, r-1$. Since $\pi^l \pi \subseteq \pi^{l+1}$ left transformation $\mathcal{L}_{b_1^{(l)}}, \dots, \mathcal{L}_{b_{k_l}^{(l)}}$ induces a trivial mapping on the factor space $\pi^l / \pi^{l+1}, l = 1, \dots, r$. By basis $c_0, b_1^{(1)}, \dots, b_{k_r}^{(r)}$, left transformation $\mathcal{L}_{b_1^{(l)}}, \dots, \mathcal{L}_{b_{k_l}^{(l)}}$ is represented by a lower triangular matrices of the form:



Since $\pi^1, \pi^2, \ldots, \pi^r$ are ideal of *A*, the left transformation $\mathcal{L}_{c_0}$ is represented by a lower block triangular matrix:



Let *L* be an extension field of *F*containing the splitting field of the characteristic polynomial of $\mathcal{L}_{c_0}$. We choose the bases:

$$c_0^{(l)} + \pi^{l+1}, \ldots, c_{k_l}^{(l)} + \pi^{l+1}$$

of $\pi^l/\pi^{l+1}, l = 1, \ldots, r$ such that the linear mapping$\mathcal{L}_{c_0^{(l)}}$induced by$\mathcal{L}_{c_0}$in $\pi^l/\pi^{l+1}$are represented by lower triangular matrices. Consequently, $c_0, c_1^{(1)}, \ldots, c_{k_1}^{(1)}, \ldots, c_1^{(r)}, \ldots, c_{k_r}^{(r)}$ is a basis of A_L with respect to $\mathcal{L}_{c_0}$ with is represented by a lower triangular matrices and with respect to which the linear transformation $\mathcal{L}_{b_i^{(l)}}, i = 1, \ldots, k_1,\ l = 1, \ldots, r$ that also represented by lower block triangular matrices.

We will show that$c_0, c_1^{(1)}, \ldots, c_{k_1}^{(1)}, \ldots, c_1^{(r)}, \ldots, c_{k_r}^{(r)}$ form a canonic basis of A_L. With this aim we enumerate these basis elements in the given order from 0 to m. The multiplication table of $A_L$with respect to this basisis: $c_i c_j = \sum_{k=0}^{n} \lambda_{ijk} c_k,\ \ i,j = 1, \ldots, m.$

Since ω is algebra homomorphism, and $\omega(c_0) = 1$, $\omega(c_i) = 0$, then $\lambda_{000} = 1$. Since $c_0 \in A$ related linear transformations $\mathcal{L}_{c_0}$ which represented by lower triangular matrices, then $\lambda_{0jk} = 0,\ k < j$.

Furthermore, because each$c_i, i = 1, \ldots, m$is a linear combination of$b_1, \ldots, b_m$, $c_i = \sum_{h=1}^{m} \sigma_{hi} b_h$. For $i,j = 1, \ldots, m$ we have:

$$c_i c_j = \sum_{k=0}^{n} \lambda_{ijk} c_k$$

and also

$$c_i c_j = \sum_{h=1}^{m} \sigma_{hi} b_h\, c_j = \sum_{h=1}^{m} \sigma_{hi} \mathcal{L}_{b_h} c_j$$

Let $c_j \in \pi^l$,but $c_j \notin \pi^{l+1}$, then$\mathcal{L}_{b_h} c_j \in \pi\pi^l \subseteq \pi^{l+1}$. If we compare the coefficients, we obtain$\lambda_{ijk} = 0,\ k \le j$. Since *A* is commutative we have also$\lambda_{ijk} = 0,\ k \le i$. So for $i,j > 0$ we found that $\lambda_{ijk} = 0,\ k \le \max(i,j)$. As a consequence *A*is genetic.

($\Leftarrow$) Suppose (2) hold. By Corollary 6, $\ker \omega = \pi$ is nilpotent. Furthermore, because the *principal powers* ofπ is ideal of *A* then *A* is a special train algebra. ■

Corrolary 11. Each special train algebra is genetic.■

There is a genetic algebra which is not a special train algebra, as follows

Example 12. Let *A* algebra over $\mathbb{R}$ with a basis$\{c_0, c_1, \ldots, c_5\}$ and the multiplication table:

	c_0	c_1	c_2	c_3	c_4	c_5
c_0	c_0	$\frac{1}{2}c_1$	$\frac{1}{4}c_3$	0	0	0
c_1		$\frac{1}{4}c_2$	$\frac{1}{8}c_4$	0	0	0
c_2			$\frac{1}{16}c_5$	0	0	0
c_3				0	0	0
c_4					0	0
c_5						0

Then $\{c_0, c_1, \ldots, c_5\}$ is canonic basis of *A*. Hence *A* is a genetic. Furthermore,clearly that *A* is also baric algebra where the weight homomorphism ω defined by $\omega(c_0) = 1$ and $\omega(c_i) = 0,\ i = 1,2,3,4,5.$

The principal powers of ker$\omega = \pi$ are.:

$\pi = \langle c_1, \ldots, c_5 \rangle, \pi^2 = \langle c_2, c_4, c_5 \rangle,\ \pi^3 = \langle c_4, c_5 \rangle,\ \pi^4 = \langle 0 \rangle.$

However,$c_0 c_2 = \frac{1}{4} c_3 \notin \pi^2$. Thus π^2 not an ideal of *A* and *A* is not aspecial train.

By experience in Example 12 above, we will give the condition to π^i *to be* an ideal of *A*.

Theorem 13. *Let A is genetic with basis* $c_0, \dots, c_m$ *in A and multiplication* $c_i c_j = \sum_{k=0}^{n} \lambda_{ijk} c_k$ *and* $\pi = \ker \omega$ *(ω homomorphism weight), then*

(1) π^i *ideal of A* if and only if $c_0 \pi^i \subseteq \pi^i, i \in \mathbb{N}$.

(2) If the multiplication constants $\lambda_{ijk} \neq 0$ *for* $k = maks(i,j) + 1,\ i,j > 0$, *then* $\pi^i = \langle c_i, \dots, c_m \rangle, i = 1, \dots, m$ *and* π^i *is ideal of A.*

Proof (1) $(\Rightarrow)$ *Let* π^i ideal of*A*, it is obvious that $c_0 \pi^i \subseteq \pi^i, i \in \mathbb{N}$.

$(\Leftarrow)$ Let $c_0 \pi^i \subseteq \pi^i, i \in \mathbb{N}$. Using the induction on *i* we will be shown π^i is an ideal of *A*. For $i = 1$, $\pi^1 = \langle c_1, \dots, c_m \rangle = \ker \omega$. Thus$\pi^1$ is an ideal of *A*. Let for *i=k* , $k \geq 1$ i.e. π^k is an ideal of *A*. By definition $\pi^{k+1} = \pi^k \pi$. Since π^k is an ideal, then $c_i \pi^k \subseteq \pi^k$. Consequently, $c_i \pi^{k+1} = c_i \pi^k \pi \subseteq \pi^k \pi = \pi^{k+1}$. Thus, π^{k+1} is an ideal of *A*. ■

(2) We will show by induction on *i* that $\pi^i = \langle c_i, \dots, c_m \rangle, i = 1, \dots, m$ and π^i an ideal of *A*. For $i = 1$, $\pi^1 = \langle c_1, \dots, c_m \rangle = \ker \omega$. Thus π^1 is an ideal of *A*. Let for $i \geq 1$, $\pi^i = \langle c_i, \dots, c_m \rangle$, π^i is an ideal of *A*. By definition $\pi^{i+1} = \pi^i \pi$. Take $c_i \in \pi^i$ and $c_j \in \pi$ then for $i,j > 0$:

$$c_i c_j = \lambda_{ij0} c_0 + \lambda_{ij1} c_1 + \dots + \lambda_{iji} c_i + \lambda_{ij(i+1)} c_{(i+1)} + \dots + \lambda_{ijm} c_m$$

Since *A* genetic, then $\lambda_{ijp} = 0$ for $p \leq maks\,(i,j)$. Let $maks\,(i,j) = i$ hence:

$$c_i c_j = \lambda_{ij(i+1)} c_{(i+1)} + \dots + \lambda_{ijm} c_m$$
$$c_{(i+1)} c_j = \lambda_{(i+1)j(i+2)} c_{(i+2)} + \dots + \lambda_{(i+1)im} c_m$$
$$\vdots$$
$$c_{(m-1)} c_j = \lambda_{(m-1)jm} c_m$$
$$c_m c_j = 0$$

Thus acquired, $\pi^{i+1} = \langle c_{(i+1)}, \dots, c_m \rangle$. Consequently, $c_i \pi^{i+1} \subseteq \pi^{i+1}, i > 0$.

Furthermore,

$$c_0 c_{(i+1)} = \lambda_{0(i+1)(i+1)} c_{(i+1)} + \dots + \lambda_{ijm} c_m \in \pi^{i+1}$$
$$c_0 c_{(i+2)} = \lambda_{0(i+2)j(i+2)} c_{(i+2)} + \dots + \lambda_{(i+1)im} c_m \in \pi^{i+1}$$
$$\vdots$$
$$c_0 c_{(m-1)} = \lambda_{0(m-1)(m-1)} c_{(m-1)} \in \pi^{i+1}$$
$$c_0 c_m = \lambda_{0mm} c_m \in \pi^{i+1}$$

Thus $c_0 \pi^{i+1} \subseteq \pi^{i+1}$. Therefore,$\pi^{i+1}$ is an ideal of *A*.■

IV. CONCLUSIONS

Based on the discussion, it is generally found that special train algebra is a genetic algebra. Furthermore,with the ideal condition (kernel homomorphism of weight) such that each principal powers are ideal, then we have that the genetic algebra is special train algebra.

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