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One Some Properties Of GPSSE-Rings

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***Abstract*—The aim of this paper is to investigate the concept of semiperfect *GP*-injective rings with essential socle, (*GPSSE*-rings for short) and study some generalizations of *PF*-rings by means of generalized principally injective rings. This paper is concluded by giving an example proving that there is no relation between right *CSSES* and right *GPSSE*-rings, that is the class of right *CSSES*-rings is not contained in the class of *GPSSE*-rings and vice versa.**

Keywords*— *PF-rings; GPSSE-rings; QF-rings; CSSES-rings.

I. Introduction

A ring *R* such that every faithful right *R*-module generates the category Mod-*R* of right *R*-modules is called right pseudo-Frobenius, briefly right *PF*. These rings were introduced by Azumaya [1] as a generalization of quasi-Frobenius rings. It is well known result of Osofsky [2] that *R* is right *PF* if and only if *R* is semiperfect, right self-injective with right socle essential as a right ideal in *R*. An important source of semiperfect rings is given by the theorem of Osofsky [2] which asserts that a left self-injective cogenerator ring (= a left *PF*-ring) is semiperfect and has finitely generated essential left socle. Recall that a module is *CS* (or extending), if every submodule is essential in a direct summand. This simple property is satisfied by every (quasi-) injective module. It is obvious that *R* is a left *PF*-ring if and only if it is left self-injective and left Kasch, where the latter condition just means that every simple left *R*-module is isomorphic to a minimal left ideal. From Osofsky's theorem it is also follows that a left *PF*-ring is right Kasch and so it is to ask whether a left self-injective right Kasch ring is left *PF*. This question is still open but in order to obtain a positive answer it would be enough to prove that *R* has essential left socle, because it has already been shown in [3] that these rings are semiperfect. This result was extended in [4], where it was shown that if *R* is left *CS* and the dual of every simple right *R*-module is simple, then *R* is semiperfect with $Soc({}_RR)= Soc(R_R) \leq_e R$.

Throughout this paper all rings *R* considered are associative with unity and all *R*-modules are unital. A submodule *K* of *M* is *essential* in *M*, denoted by $K \leq_e M$ if $K \cap L=0$ for every proper submodule *L* of *M* (i.e., in case for every submodule *L* of *M*, $K \cap L=0$ implies $L=0$). Let *M* be a right *R*-module, then $Z(M)= \{x \in M \mid xI=0$ for some essential right ideal *I* in $R\}$ is the right singular submodule of *M*. If $Z(M)=M$ (or $Z(M)=0$), then *M* is called *singular* (or *nonsingular*) module. A ring *R* is called *right singular* (or *right non-singular*) ring if $Z(R_R)=R$ (or $Z(R_R)=0$). Let *N* be any submodule of the module *M, N* is said to be small in *M*, denoted by $N \ll M$, if $N+K=M$ for any proper submodule *K* of *M* (i.e., in case for every submodule *K* of *M*, $N+K=M$ implies $K=M$), and it is said to be a *non-small module* if it is not a small module. It is known that a module *M* is small if it is small in its injective hull. Let *M* be a left *R*-module. Then the radical of M is given by;

$$Rad(M) = \cap \{K \leq M \mid K \text{ is maximal in } M\}$$
$$= \Sigma \{L \leq M \mid L \text{ is small in } M\}.$$

See [5, Proposition 9.13] for the proof. Let *R* be a ring, then the radical $Rad({}_RR)$ of ${}_RR$ is a (two-sided) ideal of *R*, [5, Proposition 9.14]. This ideal of *R* is called the *Jacobson radical* of *R*, and it usually abbreviate $J=J(R)= Rad({}_RR)$. If *M* is a left *R*-module, then the socle of *M* is given by;

$$Soc(M) = \Sigma \{K \leq M \mid K \text{ is minimal in } M\}$$
$$= \cap \{L \leq M \mid L \text{ is essential in } M\}.$$

The reader is referred to [5, Proposition 9.7] for the proof. Analogously, one can define the left and right socle for a ring *R*, (i.e., $Soc({}_RR)$ and $Soc(R_R)$). Right annihilators will be denoted as;

$$r(Y) = r_X(Y) = \{x \in X \mid yx = 0 \text{ for all } y \in Y\},$$

with a similar definition of left annihilators, $l_X(Y)=l(Y)$. For the unexplained terminology and undefined notations used in this paper, the reader is referred to [5-12].

Consider the following conditions for a right *R*-module *M* :

(C_1) Every submodule of *M* is essential in a direct summand.

(C_2) For any submodule *A* of *M* is isomorphic to a direct summand of *M* is itself a direct summand.

(C_3) For any direct summands M_1, M_2 with $M_1 \cap M_2=0$, $M_1 \oplus M_2$ is also direct summand of *M*.

A submodule *C* of *M* is called a *complement* of *K* in *M* if there exist submodules *C* of *M* maximal with respect to $K \cap C= 0$. Thus $K \leq_e M$ if and only if 0 is a complement of *K*. Let *A* and *P* be submodules of *M*, then *P* is called a *supplement* of *A* if it is minimal with the property $A+P=M$. The module *M* is called *CS-module* if it satisfies (C_1). *CS-module* is also said to be *extending module* in the context. Every injective module is *CS*. The ring *R* is called *right-CS ring* (resp. *left-CS ring*) if the right *R*-module R_R (resp. *left R-module* ${}_RR$) is *CS*-module, and similarly for the other conditions it has been defined for modules. *M* is said to be continuous if it is *CS* and (C_2). *M* is called quasi-continuous if it satisfies (C_1) and (C_3). With this terminology, It is well known that every continuous module is quasi-continuous, for more on this (see, [10]).

Now before recording some well-known results, classes of rings that arises in the next theorem should be introduced. Call a ring *R* semiregular if *R/J* is Von Neumann regular and idempotents lift modulo *J*.

Lemma 1.1 [11, Theorem 1.25] *Let M_R be a continuous module with S=End(M). Then*:

(1) *S is semiregular and* $J(S)=\{\alpha \in S \mid Ker(\alpha) \leq_e M\}$

(2) *R/J(S) is right continuous.*

(3) *If M is actually quasi-injective, S/J(S) is right self-injective.*

Corollary 1.1

(1) *If R is right continuous ring, then* $J(R)=Z(R_R)$.

(2) *If R is left continuous ring, then* $J(R)=Z({}_RR)$.

Proof

['(1). Being R right continuous by Lemma 1.1(1) $J(R)=\{x \in R \mid r(x) \leq_e R_R\}$. Hence $J(R)=Z(R_R)$.

(2). It is similar to the proof of (1) by symmetry. □

A ring *R* is called *local ring* if *R/J* is a division ring, equivalently if *R* has a unique maximal right (left) ideal and *R* is called *semilocal* ring if *R/J* is semisimple Artinian. An idempotent *e* is called primitive if and only if *e* cannot be written as direct sum of two nonzero idempotents if and only if for any idempotent *f*, the equivalencies *f=ef=fe* imply *e=f* if and only if *eR* is indecomposable right *R*-module. The ring *eRe* is local ring if and only if *e* is local idempotent.

Theorem 1.1 [Krull-Schmidt Theorem [5, 12.9]] Let $M_1 \oplus M_2 \oplus \dots \oplus M_n = A \oplus X$ *for modules* $M_1, M_2, \dots M_n$, *A, X with* $End(A_R)$ *local ring. Then, for some j, A is isomorphic to a direct summand of* M_j. *Thus, if each* M_i *is indecomposable, then* $M_j = A$.

A module *M* is called *semiperfect* if *M* is projective and every homomorphic image of *M* has a projective cover. That is, there is an epimorphism $p:P \to M$ where *P* is projective and *Ker*(*p*) is small in *P*. Note that every semiperfect ring *R* is semiperfect right *R*-module R_R.

The following characterization of semiperfect and Artinian rings will be used frequently.

Lemma 1.2

(1) *If R is a semiperfect ring, then* $r(J)=Soc({}_RR)$, *and* $l(J)=Soc(R_R)$.

(2) *If R is a right Artinian ring, then* $Soc(R_R) \leq_e R_R$.

Proof

(1). $JSoc({}_RR)=0$ implies $Soc({}_RR)\subseteq r(J)$ and $Soc(R_R)J=0$ implies $Soc(R_R)\subseteq l(J)$.

Now since *R* is a semiperfect, *R/J*(*R*) is semisimple, then *r*(*J*) is semisimple left module over the semisimple ring *R/J*(*R*) by the operation $\bar{r}t = rt$ where $\bar{r} \in R/J(R)$, $t \in r(J)$. Therefore, it is semisimple right R-module. Hence $r(J)\subseteq Soc({}_RR)$. Thus $r(J)= Soc({}_RR)$. Similarly *l*(*J*) is semisimple right module over the semisimple ring *R/J*(*R*) and so it is semisimple left *R*-module. Hence $l(J) \subseteq Soc(R_R)$. Thus $l(J)=Soc(R_R)$.

(2). Assume *R* is right Artinian ring and so it has a minimal right ideal *K*. Suppose $Soc(R_R)\cap L=0$ for all right ideals *L* of *R*. Since *K* is contained in Land $K\subseteq Soc(R_R)$, therefore, $Soc(R_R)\cap L \neq 0$. A contradiction. Hence $Soc(R_R)\leq_e R_R$. □

If *R* is a ring, a module M_R is called *right principally injective* (*P-injective*) if every *R*-homomorphism $\gamma : aR \to M$, $a \in R$, extends to $R \to M$, equivalently if $\gamma = m.$ is multiplication by some element $m \in M$. Every injective module is *P*-injective, and a ring *R* is regular if and only if every right *R*-module is *P*-injective. In fact, if the map $ar \to r+r(a)$ from $aR \to R/r(a)$ is given by left multiplication by $b + r(a)$, then $aba = a$.

A ring *R* is called *right principally injective* (or *right P-injective*) if R_R is a *P*-injective module. Thus every right self-injective ring is right *P*-injective. Moreover, neither converse is true: Every regular ring is both right and left *P*-injective, so there are *P*-injective rings that are not right self-injective.

Lemma 1.3 [13, Lemma 1.1] *The following conditions are equivalent for a ring R*:

(1) *R is right P-injective.*

(2) $lr(a)= Ra$ *for all* $a \in R$.

(3) $r(a) \subseteq r(b)$, *where* $a, b \in R$, *implies that* $Rb \subseteq Ra$.

(4) $l[bR \cap r(a)] = l(b) + Ra$ *for all* $a, b \in R$.

(5) *If* $\gamma: aR \to R$, $a \in R$, *is R-homomorphism, then* $\gamma(a) \in Ra$.

Proof

(1) ⇒ (2). Always $Ra \subseteq lr(a)$. If $b \in lr(a)$ then $r(a) \in r(b)$, so $\gamma : aR \to R$ is well defined by $\gamma(ar) = br$. Thus $\gamma = c.$ for some $c \in R$ by (1), whence $b=\gamma(a)= ca \in Ra$.

(2) ⇒ (3). If $r(a) \subseteq r(b)$ then $b\in lr(a)$, so $b \in Ra$ by (2).

(3) ⇒ (4). Let $x \in l[bR \cap r(a)]$. Then $r(ab) \subseteq r(xb)$, so $xb = rab$ for some $r \in R$

by (3). Hence $x - ra \in l(b)$, proving that $l[bR \cap r(a)]\subseteq l(b) + Ra$. The other inclusion always hold.

(4) ⇒ (5). Let $\gamma:aR \to R$ be *R-homomorphism*, and write $\gamma(a)=d$. Then

$r(a) \subseteq r(d)$, so $d \in lr(a)$. But $lr(a)=Ra$ [*take b=1* in (4)], so $d \in Ra$.

(5) ⇒ (1). let $\gamma:aR \to R_R$. By (5) write $\gamma(a)=ca$, $c\in R$. Then $\gamma=c.$, proving (1). □

Lemma 1.4 [11, Proposition 5.10] *Every right P-injective ring is a right* C_2*- ring.*

Proof

If T is a right ideal of R and $T \cong eR$, where $e^2=e \in R$, then $T=aR$ for some $a \in R$ and T is projective. Hence $r(a) \leq dR_R$, say $r(a)=fR$, where $f^2=f \in R$. Hence $Ra = lr(a) = R(1-f) \leq {}_dR R$, and so $T= aR \leq dR_R$. □

A right R-module M is called *torsionless* if M is embedded in a direct product of copies of R (if and only if M is embedded in a free right R-module). For any right ideal T of R, R/T is *torsionless* as a right R-module if and only if $rl(T)=T$. Hence A cyclic right R-module is torsion less if and only if R/T is *torsionless* as a right R-module for any right ideal T of R. Hence every right R-module is torsionless if and only if R is right cogenerator. A right R-module M is called faithful if $\eta(M)=0$, where $\eta(M)=\{r \in R \mid mr = 0 \text{ for all } m \in M\}$.

A ring R such that every faithful right R-module generates the category Mod-R of right R-modules is called *right pseudo-Frobenius* (or *right PF-rings*). These rings were introduced by Azumaya [1] as a generalization of quasi-Frobenius rings. A right R-module M is called *Kasch module* if every simple right R-module can be embedded in M. A ring R is called *right Kasch ring* (or simply *right Kasch*) if every simple right R-module K embeds in R_R, equivalently if R_R cogenerates K or every maximal right ideal is a right annihilator. Every semisimple Artinian ring is right and left Kasch, and a local ring R is right Kasch if and only if $Soc(R_R)\neq 0$ because R has only one simple right module.

II. GPSSE-Rings

Let M be a right R-module. Recall that M said to be generalized principally injective module *GP-injective module* if for any $0\neq a \in R$, there exists a positive integer n such that $a^2\neq 0$ and any right R-homomorphism from a^nR to M extends to a homomorphism from R to M. A ring R is said to be *right GP-injective ring* if the right R-module R_R is *GP*-injective module or if for any $0\neq a^n \in R$, there exists $n > 0$ such that $a^n\neq 0$ and $Ra^n=lr(a^n)$. Analogously, one defines left *GP*-injective rings. It is clear that every right *P*-injective ring is right *GP*-injective.

Recall that M is said to be *GPSSE-Module* if M is a projective, semiperfect, *GP*-injective module with $Soc(M)$ essential in M. If the right R-module R_R is *GPSSE*-module, R is called *right GPSSE-ring*. Following Page and Zhou [14], a ring R is called *right* (*left*) *AP-injective* if every principal left (right) ideal is a direct summand of a right (left) annihilator. Clearly, every right (left) *P*-injective ring is right (left) *AP*-injective. It is proved in [15, Lemma 8] that if R is a von Neumann regular ring, then every right R-module is *GP*-injective.

Lemma 2.1 [16, Theorem 2.5] *Assume that R is a right GPSSE-ring. Then R is a left and right Kasch.*

Lemma 2.2 *Let e and f be local idempotents in a GPSSE-ring. If eR and fR contains isomorphic simple right ideals, then eR and fR are isomorphic.*

Proof

Let K be a simple right ideal in eR and $K \xrightarrow{\alpha} fR$ be a monomorphism. Let $a \in K$ be a nonzero element. Then there exists a positive integer n such that $a^n \neq 0$ and any map from a^nR to R extends to an endomorphism of R. Then $K=a^nR$ and so α extends to an endomorphism β of R, and therefore, α is a left multiplication by $\beta(1)$. Let $\beta(1)=b$. Since $\beta(a)= \beta(1)ea =f\beta(1)ea=(fbe)a$ and so defining a new map $aR=a^nR \xrightarrow{\gamma} R$ by $\gamma(ar)=(fbe)ar$ and then extend γ to R, as a result it is assuming that $b=fbe \in fRe$. Hence $\beta(eR) \subseteq fR$. Now $0\neq\alpha(K)=bK \subseteq bSoc(R_R) \subseteq bSoc({}_RR)$. By hypothesis $r(J)=Soc({}_RR)$ and $l(J)=Soc(R_R)$ and $J \subseteq lr(J)=l(Soc({}_RR))$. This and $bSoc({}_RR)\neq 0$ show that b is not in $l(Soc({}_RR))$. Hence b is not in J, so $beR= bR$ is not contained in fJ. Then $bR=fR$ since fR is local module with unique submodule fJ. Since $\beta_{|K}$ is monic and K is essential submodule of eR, the restriction $\beta_{|eR}$ of β on eR is monic, therefore $\beta_{|eR}$ is an isomorphism from eR onto fR. □

Lemma 2.3 [16, Theorem 2.3] *Let R be right Kasch, right GP -injective.*

1. *For any $a \in R$, if Ra is a minimal left ideal, then aR is a minimal right ideal.*
2. *$Soc({}_RR)=Soc(R_R)$ is essential in ${}_RR$.*
3. *$J(R)=r(S)=rl(J)$, where $S=Soc(R_R)=Soc({}_RR)$.*
4. *$l(J)$ is essential left ideal in R.*
5. *$J(R)=Z(R_R)=Z({}_RR)$.*

By combine the proceeding results and to obtain the following Theorem.

Theorem 2.1 *Let R be a right GP SSE-ring. Then*

1. *R is right and left Kasch.*
2. *For any $a \in R$, if Ra is a minimal left ideal, then aR is a minimal right ideal.*
3. *$Soc({}_RR)=Soc(R_R)$ is essential in R_R and ${}_RR$.*
4. *$J(R)=r(S)=rl(J)$, where $S=Soc(R_R)=Soc({}_RR)$.*
5. *$l(J)$ is essential left ideal in R.*
6. *$J(R)=Z(R_R)=Z({}_RR)$.*

Proof

(1) Clear from Lemma 2.1.
(2), (3), (4), (5), and (6) clear from Lemma 2.3. □

Corollary 2.1 [16, Lemma 2.2] *Let R be a right GP-injective ring.*

1. *For any $x \in R$, if xR is a minimal right ideal, then Rx is a minimal left ideal.*
2. *$Soc(R_R) \subseteq Soc({}_RR)$.*

This paper is concluded by saying that there is no relation between right *CSSES* and right *GPSSE*-rings, that is the class of right *CSSES*-rings is not contained in the class of *GPSSE*-rings and vice versa. The following examples are mentioned to clarify these:

Example 2.1 Consider the ring R as in [17, Example 3.1]. Then R is a right *CSSES*-ring. However, by Corollary 2.1(2) R is not right *GP*-injective since $Soc(R_R) \not\subset Soc(_RR)$. Hence R is not right *GPSSE*-ring.

Example 2.2 [11, Example 2.5] Let F be a field and assume that $\partial : F \to \bar{F} \subseteq F$ is an isomorphism given by $a \to \bar{a}$, where the subfield $\bar{F} \neq F$ (i.e., $\partial(F) \neq F$) and $2 \leq dim(_{\bar{F}}F) < \infty$ is finite. Let $R=F\{1, t\}=\{a_0 1 + a_1 t \mid a_0, a_1 \in F, t^2=0\}$ denote the left vector space on basis $\{1, t\}$, and make R into an F-algebra by defining $t^2=0$ and $ta = \bar{a}t$ for all $a \in F$. Then $J(R)=Rt=Ft$ is the only proper left ideal of R and so R is local, $R/J \cong F$. Let $a \in R$, then:

Case(i) if $0 \neq a \neq t$, $Ra = R$ and hence $r(a)=0$ and $lr(a)=l(0)=R=Ra$.

Case(ii) if $0 \neq a=t$, then $Rt \subseteq lr(t)$. Thus either $Rt=lr(t)$ or $lr(t)=R$ which does not occur and in both cases $lr(a)=Ra$ for all $a \in R$. Hence R is right *P*-injective ring by Lemma 1.3 (2) and so it is right *GP*-injective. R is left and right Artinian by [11, Example 2.5(5)(a)] and therefore, it is semiperfect ring with $Soc(R_R) \leq_e R_R$ by Lemma 1.2 (2). R is not right continuous ring. Indeed, if R were right continuous then, being local, it would be right uniform. But if X and Y are nonzero $\bar{F}$-subspace of F with $X \cap Y=0$ then $P=Xt$ and $Q=Yt$ are nonzero right ideals with $P \cap Q=0$. Moreover, R is right C_2-ring by Lemma 1.4. Thus R is not right *CS*-ring and hence R is right *GPSSE*-ring but not right *CSSES*-ring.

It is proved by Rutter [18, Example 2] that the ring R as in the following Example 6.8 is not left *P*-injective. It is given a short proof that R is not left *P*-injective ring as in the following example.

Example 2.3 [19, Example 1] Let K be a field and L be a proper subfield of K such that $\rho : K \to L$ is an isomorphism (e.g., let $K=F(y_1, y_2, \ldots)$ with F a field $\rho(y_i)=y_{i+1}$ and $\rho(c)=c$ for all $c \in F$ [20]). Let $K[x_1, x_2; \rho]$ be the ring of twisted right polynomials over K where $kx_i=x_i\rho(k)$ for all $k \in K$ and for $i=1,2$.

Set $R=K[x_1,x_2;\rho]/(x_1^2,x_2^2)=K+x_1K+x_2K+x_1x_2K$.

Then:

(1) R is a right *CSSES*-ring.

(2) R is a left *GPSSE*-ring but neither left *GPF*-ring nor left *PF*–ring.

(3) R is a left and right Kasch.

(4) R is a right continuous ring.
(5) R is not *QF*-ring.

Proof

(1). Firstly, it should be shown that R is right *CSSES*-ring. Note that x_1x_2L is minimal left ideal of R and x_1x_2K is a left ideal and a minimal right ideal of R. Also $x_1x_2L \subseteq x_1x_2K$ but $x_1x_2K \not\subset x_1x_2L$. Hence it is easy to check that x_1x_2K is contained in every right ideal of R. Hence R is right uniform and so right *CS*-ring. Now proving that $x_1x_2K \cap I \neq 0$ for every left ideal I of R. For if, I is a nonzero left ideal of R, let $0 \neq a=k_0 + x_1k_1 + x_2k_2 + x_1x_2k_3 \in I$. The case $k_0 \neq 0$ and $k_1=k_2=k_3=0$ is not possible. Thus if, $k_0 \neq 0$ and $k_1 \neq 0$, then $0 \neq x_1x_2a=x_1x_2k_0 \in I \cap (x_1x_2K)$. Without loss of generality, it is assuming that $k_0=0$ and $k_1 \neq 0$. Then $0 \neq x_2a=x_1x_2k_1 \in I \cap (x_1x_2K)$. Therefore, $Soc(R_R)=x_1x_2K$ is essential as a right and left ideal of R. The left socle is;

$$Soc(_RR) = \sum_{\substack{k_i=1 \\ or\ k_i \notin L}} (x_1x_2Lk_i)$$

Now to proving $Soc(_RR)$ is essential left ideal: Let $_RI$ be any left ideal in R and $0 \neq a= k_0 + x_1k_1 + x_2k_2 + x_1x_2k_3 \in {_RI}$.

The case $k_0 \neq 0$ and $k_1=k_2=k_3=0$ is not possible. Without loss of generality, it is assuming that $k_0 \neq 0$ and $k_1 \neq 0$. Then $x_1x_2a=x_1x_2k_0 \in I$. Since I is left ideal and $Rx_1x_2=x_1x_2L$, $Rx_1x_2a=Rx_1x_2k_0=x_1x_2Lk_0 \subseteq I$. But $x_1x_2Lk_0 \subseteq Soc(_RR)$. Hence $I \cap Soc(_RR)=0$.

Now it should be proved that $Soc(_RR)$ is essential right ideal: Let I_R be any right ideal in R and $0 \neq a = k_0 + x_1k_1 + x_2k_2 + x_1x_2k_3 \in I_R$.

The case $k_0 \neq 0$ and $k_1=k_2=k_3=0$ is not possible. Without loss of generality, it is assuming that $k_0 \neq 0$ and $k_1 \neq 0$. Then $ax_1x_2=x_1x_2\rho^2(k_0) \in I_R$. Since $x_1x_2\rho^2(k_0) \in x_1x_2L \subseteq Soc(_RR)$, $I_R \cap Soc(_RR) \neq 0$. Thus $Soc(_RR)$ is essential as a right ideal in R. Since for any $k \in K$, $Lk \subseteq K$, $x_1x_2Lk \subseteq x_1x_2K=Soc(R_R)$ and so $Soc(_RR) \subseteq Soc(R_R)$. Since $x_1x_2K=Soc(R_R)$ is contained in every nonzero right ideal of R, $Soc(R_R) \subseteq Soc(_RR)$. So $Soc(R_R)=x_1x_2K \subseteq Soc(_RR)$. Therefore $Soc(_RR)=Soc(R_R)$. Hence right and left socles are equal. Note also that $I=x_1K + x_2K + x_1x_2K$ is the unique maximal right ideal of R in which $I^4=0$ and so $J(R)=I$. Thus $R/J \cong K$ is semisimple. Therefore, R is semiperfect by [11, Theorem B.9]. Hence R is right *CSSES*-ring.
(2). First it is shown that R is left *GP*-injective as it is shown in [19, Example 1]. For any $0 \neq a \in R$, write $a=k_0 + x_1k_1 + x_2k_2 + x_1x_2k_3$ where $k_i \in K$ for $i=0, 1, 2, 3$. As for the other three cases.

Case (i). $k_0 \neq 0$. Then a is a unit of R, so $aR=rl(a)$.
Case (ii). $k_0=0$ but $k_1k_2 \neq 0$.

(a) If $k_1k_2^{-1} \notin L$, then $a^2=x_1x_2[\rho(k_1)k_2+\rho(k_2)k_1] \neq 0$ and

$rl(a^2)=r(x_1K+x_2K+x_1x_2K)=x_1x_2K=a^2R$.

(b) If $k_1k_2^{-1} \in L$, then

$l(a)=l(x_1k_1+x_2k_2+x_1x_2k_3)$

$=\{x_1k'_1+x_2k'_2+x_1x_2k'_3 \in R \mid \rho(k'_1)=-\rho(k'_2)k_1k_2^{-1}\}$

and hence

$rl(a)=r(\{x_1k'_1+x_2k'_2+x_1x_2k'_3 \in R \mid \rho(k'_1)$

$=-\rho(k'_2)\,k_1k_2^{-1}\})$

$=\{x_1k''_1+x_2k''_2+x_1x_2k''_3 \in R \mid k''_1k_2=k_1k''_2\}$

Note that $x_1x_2=a[x_1\rho(k_2^{-1})] \in aR$ and that, when $k''_1k_2=k_1k''_2$,

$ak^{-1}{}_1k''_1=x_1k''_1+x_2k_2k^{-1}{}_1k''_1+x_1x_2k_3k^{-1}{}_1k''_1$

$=x_1k_1+x_2k^{-1}{}_1k_1k''_2+x_1x_2k_3k^{-1}{}_1k''_1$

$=x_1k''_1+x_2k''_2+x_1x_2k_3k^{-1}{}_1k''_1$

Since $a\,k^{-1}{}_1k''_1 \in aR$ and $x_1x_2k_3k^{-1}{}_1k''_1 \in aR$, then $x_1k''_1+x_2k''_2 \in aR$. So $rl(a) \subseteq aR$.

Hence $rl(a)=aR$.

Case (iii). $k_0=0$ and $k_1k_2=0$.

(a) $k_1=k_2=0$ and $k_3 \neq 0$. Then

$rl(a)=rl(x_1x_2k_3)=r(x_1K+x_2K+x_1x_2K)=x_1x_2K=aR$.

(b) $k_1 \neq 0$ and $k_2=0$. Then

$rl(a)=rl(x_1k_1+x_1x_2k_3)=r(x_1K+x_1x_2K)=x_1K+x_1x_2K$.

Since $x_1=a[k^{-1}{}_1-x_2\rho(k^{-1}{}_1)k_3k^{-1}{}_1] \in aR$ and

$x_1x_2=a[x_2\rho(k^{-1}{}_1)] \in aR$, then $rl(a)=aR$.

(c) $k_1=0$ and $k_2 \neq 0$. This is similar to (b) of Case (iii). Therefore, R is a left GP-injective ring and hence it is left $GPSSE$-ring.

Second it is shown that R is not left P-injective. Since $K \neq L$, take $k \in K \setminus L$ and

let $a=x_1k+x_2 \in R$. It is shown that $aR \neq rl(a)$. In fact,

$l(a)=l(x_1k+x_2)=x_1x_2K$,
$rl(a)=r(x_1x_2K)=x_1K+x_2K+x_1x_2K$, and
$aR=aK+x_1x_2K$.

However, $aR \subseteq rl(a)$ since $x_1+x_2 \in rl(a)$ but $x_1+x_2 \notin aR$. By Lemma 1.3, R is not left P-injective. Hence it is not left GPF-ring and so is not left PF-ring.

(3). It is easy to check that, for any left and right ideals I_1 and I_2 of R, respectively, $r(I_1) \neq 0$ and $l(I_2) \neq 0$. Hence R is left and right Kasch by [11, Proposition 1.44].

(4). Clear from (1) and (3) since every left (right) Kasch ring satisfies right (left) (C_2) condition by [9, Lemma 2.22(2)].

(5). By (2), R is not left P-injective. Thus it is not left self-injective and so not QF-ring by using [17, Theorem 2.30(2)]. □

In [19, Proposition 2] it is proved in a complicated way that the ring in Example 2.3 is not left AP-injective. It is given a short proof of that in the following proposition.

Proposition 2.1 *Let R be a ring as in Example 2.3. Then R is not left AP- injective.*

Proof Suppose to the contrary that R is left AP-injective. Then every principal right ideal aR is direct summand of a right annihilator. For any $a \in R$ there exists a subset X of R such that $r(X)=(aR)\oplus L$ for some right ideal L of R. It is proved in Example 2.3 that R is right uniform ring. Therefore L must be zero submodule and $r(X)=aR$. This leads us to being R left P-injective. This contradicts [18, Example 2] or Example 2.3(2). Hence R is not left AP-injective. □

Finally by Examples 2.1, 2.2, and 2.3 the following inclusions are strict:
{QF-rings} $\subset$ {right PF-rings} $\subset$ {right GPF-rings} $\subset$ {right $GPSSE$-rings}. However, {right $GPSSE$-rings} $\not\subset$ {right $CSSES$-rings} and {right $CSSES$-rings} $\not\subset$ {right $GPSSE$-rings}.

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