Journal of Multidisciplinary Engineering Science and Technology (JMEST)
ISSN: 3159-0040
Vol. 2 Issue 6, May - 2015

Some Computations of Polynomial Identities on Weyl Algebra

Sanzhar Orazbayev[1], Yernaz Satybaldiyev[2]
Kazakh-British Technical University
Almaty, Kazakhstan
[1]sanzhar.orazbayev@gmail.com, [2]satybaldiyev.yernaz@gmail.com

***Abstract*—We report some graph-theoretic based computations used for finding polynomial identities on the Weyl algebra.**

Keywords—Graph Theory, Weyl Algebra, Path Decomposition

I. INTRODUCTION

The n -the Weyl algebra A_n is an associative algebra (over a field k) with $2n$ generators $x_1,\dots,x_n,\ \partial_1,\dots,\partial_n$ subject to relations

$$[x_i,x_j]=[\partial_i,\partial_j]=0,[\partial_i,x_j]=\delta_{i,j},$$

where $[a,b]=ab-ba$ is commutator and $\delta_{i,j}$ is the Kronecker delta. In this note we consider some further computations of the technique developed in [3] for finding polynomial identities on the subspace

$$A_n^{(1,1)}=\langle x_i\partial_j \mid i,j\in\{1,\dots,n\}\rangle$$

of the Weyl algebra A_n.

Let s_m be a skew-symmetric polynomial over an associative noncommutative algebra A

$$s_m(X_1,X_2,...,X_m):=\sum_{\sigma\in S_m} sgn(\sigma)X_{\sigma(1)}X_{\sigma(2)}\cdots X_{\sigma(m)},$$

where $sgn(\sigma)$ is the sign of a permutation. We say that s_m is a (standard) polynomial identity on A if $s_m=0$ for all $X_1,\dots,X_m\in A$.

Amitsur-Levitzki theorem [1] states that s_{2n} is a minimal polynomial identity on the matrix algebra, for any $n\times n$ matrices $A_1,\dots,A_{2n}$ the identity $s_{2n}(A_1,\dots,A_{2n})=0$ always holds. It is known that this fact can be proved using Euler tours in digraphs [4, 2].

Approach using decompositions of digraphs has been addressed in [3] to study polynomial identities on the subspace $A_n^{(1,1)}$ of the Weyl algebra. From [3] it is known that $s_{2n}=0$ is an indentity for $n=1,2,3$ and it is not an identity for $n\geq 4$. Here we summarize some computations for $n=4,5$ as well as implementation aspects.

II. MAIN PROPERTIES

We use technique developed in [3], for more details we refer to that paper.

Let G be a digraph with n vertices and (directed) edges $(e_1,\dots,e_m)$. Consider any decomposition of G into edge-disjoint paths $P=\{P_1,\ P_2,\ ...,P_k\}$ with sets of sources I and sinks J; every path P_i here is viewed as a permutation $(l_1,l_2,...,\ l_i)$ which is the sequence of edges $e_{l_1},e_{l_2},...,e_{l_i}$.

For permutations σ_1, σ_2, ..., σ_m, define the shuffle set $sh(\sigma_1,\sigma_2,\ ...,\ \sigma_m)$ as the set of permutations on $\cup_{i=1}^m\sigma_i$ such that the order of each σ_i is respected. Define

$$E(P):=\sum_{\sigma\in Sh(P_1,P_2,...,P_k)} sgn(\sigma)$$

Proposition 2.1. *The following formula holds for* $E_G(I)$,

$$E_G(I)=\sum_{P:I\to J}E(P)$$

where the sum is taken over all k-decompositions with sources I and sinks J.

Connection to the Weyl algebra is established as follows.

Theorem 2.1.[[3]] *Let* $w_1,w_2,...,w_m\in A_n^{(1,1)}$ *be monomials. Then*

$$s_m(w_1,w_2,...,w_m)=\sum_I E_G(I)\prod_{i\in I}x_i\prod_{j\in J}\partial_j$$

where the sum runs through all possible multisets of decomposition sources I and digraph G with n vertices has m edges represented by $w_1,w_2,...,w_m$ (i.e. if $w_l=x_{i_l}\partial_{j_l}$, then there is an edge (i_l,j_l) in G)

III. COMPUTING THE SHUFFLE SUM

Fast computation of the shuffle sum in Proposition 2.1 is established via the following formula.

Lemma 3.1. *Let* x_i *be (disjoint) permutations such that* $\bigcup_{i=1}^m x_i=\{1,2,...,n\}$ and by $[x_i]$ denote the number of elements in x_i. Define

$$q(x_1, x_2, ..., x_m) = \sum_{\sigma \in sh(x_1, x_2, ..., x_m)} sgn(\sigma)$$

Then

$$q(x_1, x_2, ..., x_m) = 0, \textit{ if there are } i \textit{ and } j, \textit{ such that } i \neq j \textit{ with } [x_i] \textit{ and } [x_j] \textit{ both odd}$$

$$q(x_1, x_2, ..., x_m) = w(x_1, x_2, ..., x_m) \frac{(\sum_{i=1}^{m} [\frac{[x_i]}{2}])!}{\prod_{i=1}^{m} ([\frac{[x_i]}{2}]!)},$$

$$\textit{otherwise};$$

$$\textit{where } w(x_1, x_2, ..., x_m) = sgn(x_1 \cup x_2 \cup ... \cup x_m)$$

$$\textit{i.e. for concatenation of permutations } x_1, \ldots, x_m;$$

$$q \textit{ does not change for any permutation of } x_1, x_2, ..., x_m.$$

Proof. First we prove that $w(x_1, x_2, ..., x_m)$ does not change for any permutation of $x_1, \ldots, x_m$ if no two x_i have odd length. If we swap x_1 and x_2, then the difference in a number of inversions is equal to $[x_1][x_2]$, which is even. By these swap operations we can obtain all permutations of $x_1, x_2, ..., x_m$.

Let us prove next formulas by induction. We have

i) $q(\{a\}, \{b\}) = 0$

ii) $q(\{a\}) = 1$

iii) $q(\{a, b\}) = w(\{a, b\})$

We compute the recurrence relation assuming that $x_i = a_i x_i'$, i.e. a_i is the first element of x_i. Therefore

$$q(x_1', x_2, ..., x_m) = \sum_{i=1}^{m} w(a_i, x_1, x_2, ..., x_i', ..., x_m) \times q(a_i, x_1, x_2, ..., x_i', ..., x_m) \quad (1)$$

1) There are some i and j, such that $[x_i]$ and $[x_j]$ are odd. Let us say that $i = 1$ and $j = 2$. If the first element is not from x_i or x_j, then by induction their sign sum is 0, because they have at least two permutations with odd length. So from (1) we have

$$\begin{aligned} q(x_1, x_2, ..., x_m) &= w(a_1, x_1', x_2, ..., x_m)(q(x_1', x_2, ..., x_m)) \\ &+ w(a_2, x_1, x_2', ..., x_m)(q(x_1, x_2', ..., x_m)) \end{aligned} \quad (2)$$

$[x_1]$ is odd, then it is clear that $[\frac{[x_1]}{2}] = [\frac{[x_1']}{2}]$, same as for x_2. Then

$$q(x_1', x_2, ..., x_m) = q(x_1, x_2', ..., x_m)$$

We can say that $w(a_1, x_1', x_2, ..., x_m) = w(x_1, x_2, ..., x_m)$, because $x_1 = a_1 x_1'$ and so difference in inversion count from $(x_1 x_2 \ldots x_n)$ to $(a_2 x_1 x_2' \ldots x_n)$ is exactly $[x_1]$ which is odd, therefore

$$w(a_1, x_1', x_2, ..., x_m) = -w(a_2, x_1, x_2', ..., x_m)$$

From (2), $q(x_1, x_2, ..., x_m) = 0$

2) There is only one permutation from $[x_1, x_2, ..., x_m]$ having odd number of elements and suppose it is x_1. If the first element is not taken from the x_1 then by induction hypothesis $q(x_1, x_2, ..., x_i', ..., x_m) = 0$. Therefore, from equation (1) we have

$$\begin{aligned} q(x_1, x_2, ..., x_m) &= w(a_1, x_1', x_2, ..., x_m)(q(x_1', x_2, ..., x_m)) \\ &= w(x_1, x_2, ..., x_m) \frac{(\sum_{i=1}^{m} [\frac{[x_i]}{2}])!}{\prod_{i=1}^{m} ([\frac{[x_i]}{2}]!} \end{aligned} \quad (3)$$

$$([\frac{[x_1]}{2}] = [\frac{[x_1']}{2}])$$

3) There are no permutations of $[x_1, x_2, ..., x_m]$ having odd number of elements. As we showed before $w(x_1, x_2, ..., x_m) = w(x_1, x_2, ..., x_i', ..., x_m)$ since the difference in inversion count is even. Also $\sum_{i=1}^{m} [\frac{[x_i]}{2}] = (\frac{[x_1]}{2} + ... + \frac{[x_j']}{2} + ... + \frac{[x_m]}{2}) + 1$ for some j. So equation (1) can be showed in thin form:

$$\begin{aligned} q(x_1, x_2, ..., x_m) &= w(x_1, x_2, ..., x_m)(\sum_{i=1}^{m} \frac{((\sum_{j=1}^{m} [\frac{[x_j]}{2}]) - 1)!}{\prod_{j=1}^{m} ([\frac{[x_j]}{2}]!/[\frac{[x_i]}{2}])} \\ &= w(x_1, x_2, ..., x_m) \frac{((\sum_{i=1}^{m} [\frac{[x_i]}{2}]) - 1)!}{\prod_{i=1}^{m} ([\frac{[x_i]}{2}]!)} (\sum_{i=1}^{m} \frac{1}{1/[\frac{[x_i]}{2}]}) \\ &= w(x_1, x_2, ..., x_m) \frac{((\sum_{i=1}^{m} [\frac{[x_i]}{2}]) - 1)!}{\prod_{i=1}^{m} ([\frac{[x_i]}{2}]!)} (\sum_{i=1}^{m} [\frac{[x_i]}{2}]) \\ &= w(x_1, x_2, ..., x_m) \frac{(\sum_{i=1}^{m} [\frac{[x_i]}{2}])!}{\prod_{i=1}^{m} ([\frac{[x_i]}{2}]!)} \end{aligned}$$

IV. IMPLEMENTATION AND RESULTS

We generate all the digraphs with given number of vertices and edges. Only balanced digraphs are considered, i.e. where indegree and outdegree of each vertex are equal. Our implementation workes well up to $n=5$ and $m=13$, where n is number of vertices and m is number of edges. In our computations we obtain the following experimental results:

- $n=4$ and $n=8$, s_n is not 0 for the following digraph

[[1, 1], [1, 2], [1, 3], [2, 1], [2, 2], [3, 3], [3, 4], [4, 1]];
(here $[i,j]$ is an edge $i \to j$ and hence s_n is considered on the operators $x_i\partial_j$.)

- $n=4$ and $n=9$, s_n is not 0 for the following digraph

[[1, 1], [1, 2], [1, 3], [2, 1], [2, 2], [2, 3], [3, 1], [3, 4], [4, 2]];

- $n=4$ and $n=10$, s_n is 0 for all digraphs;
- $n=5$ and $n=10$, s_n is not 0 for the following digraph

[[1, 1], [1, 2], [1, 3], [1, 4], [2, 1], [2, 2], [3, 1], [4, 4], [4, 5], [5, 1]];

- $n=5$ and $n=11$, s_n is not 0 for the following digraph

[[1, 1], [1, 2], [1, 3], [1, 4], [2, 1], [2, 2], [2, 3], [3, 1], [3, 5], [4, 1], [5, 2]];

- $n=5$ and $n=12$, s_n is not 0 for the following digraph

[[1, 1], [1, 2], [1, 3], [1, 4], [2, 1], [2, 2], [2, 3], [3, 1], [3, 2], [4, 4], [4, 5], [5, 1]];

- $n=5$ and $n=13$, s_n is not 0 for the following digraph

[[1, 1], [1, 2], [1, 3], [1, 4], [1, 5], [2, 1], [2, 2], [2, 4], [3, 1], [3, 2], [4, 1], [4, 3], [5, 1]];

In particular, s_{10} is an identity for $n=4$ and there is no identity s_m for $n=5$ if $m \le 13$.

References

[1] A. Dzhumadil'daev, D. Yeliussizov, Path Decompositions of Digraphs and Their Applications to Weyl Algebra, Advances in Applied Mathematics, 2015

[2] S. A. Amitsur and J. Levitzki, Minimal Identities for Algebras, Proceedings of the American Mathematical Society 1, 1950

[3] R. G. Swan, An application of graph theory to algebra, Proceedings of the American Mathematical Society 14, 1963

[4] B. Bollob'as, Modern graph theory, Vol. 184, Springer, 1998