Reduction of the Navier-Stocks Equation to the 'Natural' Three-Velocity Form

Alexei M. Frolov

Department of Applied Mathematics
University of Western Ontario, London, Ontario N6H 5B7, Canada
Email: afrolov@uwo.ca

Abstract

We develop a new approach which is applied to simplify the Navier-Stocks equation(s). Our approach is based on the derivation of the governing equations for the divergence and rotor (or curl) of the liquid/gas velocity $\mathbf{v}$. These values $D = div\mathbf{v}$ and $\vec{\omega}$ are considered to be the leading field variables (or fields) in our approach. Two arising equations for D and $\vec{\omega}$ are linear and they are equivalent to the incident Navier-Stocks equation, but they are significantly simpler and allow one to determine analytical/numerical solutions of this equation and investigate the properties of these solutions.

Keywords–divergence, rotor, vorticit

1 Introduction

In this communication we discuss the general Navier-Stocks equation(s) which describes all possible motions in the regular liquid and/or gas. Everywhere below the liquid/gas are assumed to be truly non-relativistic with the known equation(s) of state. The explicit form of the Navier-Stocks equation for such a liquid/gas is (see, e.g., [2])

$$\rho\Big[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\Big] = -\nabla p + \eta\Delta\mathbf{v} + \Big(\zeta + \frac{1}{3}\eta\Big)\nabla(\nabla\cdot\mathbf{v}) \quad (1)$$

where the vector $\mathbf{v} = (v_x, v_y, v_z)$ is the velocity of the liquid/gas, ρ is the local density and p is the local pressure. All these values are the functions of the spatial point $\mathbf{r} = (x, y, z)$ and time t. The coefficients η and ζ are the usual and second viscosities of the liquid/gas, respectively. The operator $\nabla = \Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\Big)$ is the gradient operator, or $\nabla-$operator. In old textbooks this operator was also called the Hamilton operator. Note that the ∇ operator is, in fact, a vector-operator which transforms under rotations as a covariant vector. Therefore, if ∇ is applied to a vector $\mathbf{b}(\mathbf{r})$, then, in the general case, one finds three possible combinations of ∇ and $\mathbf{b}$: scalar, vector and tensor (see, e.g., [3], [4]). The arising scalar is called the divergence of the vector $\mathbf{b}$, i.e. $div\mathbf{b} = \frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial z}$. The vector which arise during application of ∇ to the vector $\mathbf{b}(\mathbf{r})$ is called the rotor rot (or $curl$) of the vector $\mathbf{b}$, i.e. $\nabla\times\mathbf{b} = rot\mathbf{b} = \Big(\frac{\partial b_z}{\partial y} - \frac{\partial b_y}{\partial z}, \frac{\partial b_x}{\partial z} - \frac{\partial b_z}{\partial x}, \frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y}\Big)$. The third combination of the ∇ operator and vector $\mathbf{b}$ is the traceless 3×3 symmetric tensor $\hat{t} = \nabla\otimes\mathbf{b}$ which has the following matrix elements

$$(\nabla\otimes\mathbf{b})_{ik} = (\hat{t})_{ik} = \frac{1}{2}\Big(\frac{\partial b_i}{\partial x_k} + \frac{\partial b_k}{\partial x_i}\Big) - \frac{1}{3}\Big(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial z}\Big)\delta_{ik} \quad (2)$$

where $i = x, y, z$ and $k = x, y, z$. The choice $\mathbf{b} = \mathbf{r} + \mathbf{u}(\mathbf{r})$ leads to the strain vector known in the theory of elasticity, or infinitesimal strain theory [1]. Another possible choice for the vector $\mathbf{b}$ is $\mathbf{b} = \mathbf{v}$. It is used in hydrodynamics of the viscous liquids/gases (see, e.g., [2]). In our analysis below the traceless tensor $(\nabla\otimes\mathbf{v})$ is not used.

As follows from the explicit form of Eq.(1) the Navier-Stocks equation is a linear equations upon spatial and time derivatives of the velocity $\mathbf{v}$. Some terms of this equation contain quadratic powers of the velocity $\mathbf{v}$ and its spatial derivatives. Note that the Navier-Stocks equation, Eq.(1), can be reduced to another equivalent form

$$\frac{\partial\mathbf{v}}{\partial t} + \frac{1}{2}\nabla v^2 - \mathbf{v}\times(\nabla\times\mathbf{v}) = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{v} + \Big(\mu + \frac{1}{3}\nu\Big)\nabla(\nabla\cdot\mathbf{v}) \quad (3)$$

where $\nu = \frac{\eta}{\rho}$ is the first kinematic viscosity, while $\mu = \frac{\zeta}{\rho}$ is the second kinematic viscosity of the liquid (or gas). Let us introduce another vector $\vec{\omega} = \nabla\times\mathbf{v}$ which plays an important role in hydrodynamics. This vector is called the vorticity. By using this vorticity vector one reduces the Navier-Stocks equation, Eq.(3), to a different form which is often used in applications

$$\frac{\partial\mathbf{v}}{\partial t} + \frac{1}{2}\nabla v^2 - \mathbf{v}\times\vec{\omega} = -\frac{1}{\rho}\nabla p + \Big(\mu + \frac{4}{3}\nu\Big)\Delta\mathbf{v} + \Big(\mu + \frac{1}{3}\nu\Big)\nabla\times\vec{\omega} \quad (4)$$

To transform Eq.(3) to the form of Eq.(4) we have used the following formulas

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{2}\nabla v^2 - \mathbf{v} \times \vec{\omega} \quad and$$
$$\nabla(\nabla \cdot \mathbf{v}) = \Delta\mathbf{v} + \nabla \times (\nabla \times \mathbf{v}) = \Delta\mathbf{v} + \nabla \times \vec{\omega} \tag{5}$$

The explicit derivation of analytical and/or numerical solution of the Navier-Stocks equation is a very difficult problem, which has been solved only in a few cases with the use of a number of fundamental simplifications in the incident problem. General solutions of the Navier-Stocks equation has never been found in closed analytical forms. Such a solution essentially means the explicit expression for the liquid/gas velocity $\mathbf{v}(t, \mathbf{r})$ in each point of the time-spatial continuum which is occupied by the moving liquid/gas. Formally, the Navier-Stocks equation, Eq.(5), is a quadratic equation upon $\mathbf{v}$ which also contains various spatial derivatives of $\mathbf{v}$. This substantially complicates search of solutions of Eq.(5) written in closed, analytical forms.

In this study we formulate another approach to the Navier-Stocks equation. This original approach is based on an obvious fact that the Navier-Stocks equation is a linear equation upon spatial (and time) derivatives of the velocity $\mathbf{v}$. This leads to an idea to use these derivatives as the top priority values and re-write the original equation(s) as linear equations for the derivatives of the velocity vector $\mathbf{v}$. If we know the explicit expressions for the divergence and rotor/curl of the vector $\mathbf{v}$ we can always uniformly reconstruct the velocity vector $\mathbf{v}$ itself. This follows from the fundamental theorem of vector calculus (see, e.g., [3]). Discussion of this theorem with a lot of interesting details can be found in \$ 19 of [3]. For finite volumes we also need to know the numerical values of the normal components of such a vector at each point of the boundary surface [3]. By applying this theorem to the moving liquid/gas we replace the original Navier-Stocks equation by the two equations: one equation for the divergence $D = div\mathbf{v}$, which is a true scalar, and another vector-equation for the rotor $rot\mathbf{v} = \vec{\omega}$, where $\vec{\omega}$ is the vorticity vector. These two equations for D and $\vec{\omega}$ are simpler than the incident Navier-Stocks equation. An obvious success of this approach follows from the explicit form of the Navier-Stocks equation which is a linear equation upon spatial and time derivatives of the liquid's velocity $\mathbf{v}$. In reality, we can try to determine solutions of these (new) equations and investigate their properties, rather than to operate with the Navier-Stocks equation. It appears that this approach has quite a number of advantages in applications to actual problems from hydro- and aerodynamics. The goal of this study is to develop this approach and apply it to the Navier-Stocks equation. Basic equations of this approach are derived in the following Section.

2 Basic equations

In this Section by using Eq.(4) derived above we obtain the basic equations of our approach. To derive these equations we need to apply the first-order differential operators *rot* (or *curl*) and *div* to the both sides of Eq.(4). The unknown vector $\mathbf{v}$ is replaced by the combination of a transverse axial vector $\vec{\omega}$ and one scalar D. For simplicity, we shall assume that the kinematic viscosities ν and μ in Eq.(4) are the constants, i.e. they do not depend upon any of the spatial coordinates. This assumption allows one to avoid operations with extremely complex expressions with contains many additional spatial derivatives of the ν and μ values. By calculating *rot* from both sides of Eq.(4) one finds the following vector-equation for the vorticity $\vec{\omega}$

$$\frac{\partial \vec{\omega}}{\partial t} = \nabla \times (\mathbf{v} \times \vec{\omega}) + \nu\Delta\vec{\omega} =$$
$$(\vec{\omega} \cdot \nabla)\mathbf{v} - (\nabla \cdot \mathbf{v})\vec{\omega} - (\mathbf{v} \cdot \nabla)\vec{\omega} + \tag{6}$$
$$\frac{1}{\rho^2}(\nabla\rho \times \nabla p) + \nu\Delta\vec{\omega} \quad ,$$

which describes time-evolution of the vorticity vector $\vec{\omega}$. To derive an analogous equation for the divergence of the velocity vector, i.e. for $D = \nabla \cdot \mathbf{v}$, we introduce the scalar function $s(\mathbf{r})$ which is the density-distribution function of the internal sources of the liquid/gas located inside of the moving liquid and/or gas. It is shown below that at each point of the moving liquid we have $\nabla \cdot \mathbf{v} = s$, and, therefore, $D = s$. Now, by using the explicit notation for this function we obtain the following scalar equation for the divergence D

$$\frac{\partial D}{\partial t} = -\frac{1}{2}\Delta v^2 - \mathbf{v} \cdot (\nabla \times \vec{\omega}) + \omega^2 -$$
$$\frac{1}{\rho}\Delta p + \frac{1}{\rho^2}(\nabla\rho \cdot \nabla p) + \Big(\mu + \frac{4}{3}\nu\Big)\Delta D \quad , \tag{7}$$

The solution of Eqs.(6) - (7) allows one to determine the rotor (or curl) and divergence of the velocity vector $\mathbf{v}$ in each local point with the Cartesian coordinates $\mathbf{r}$. The next step is to reconstruct the velocity vector $\mathbf{v}$ by using the known values of D and $\vec{\omega}$. Since we know its rotor $\vec{\omega}$ and divergence $div\mathbf{v} = D$, then as it follows from the vector calculus, we can always determine the velocity vector $\mathbf{v}$ itself. Let us assume that the velocity vector is represented as a sum of three different velocities, i.e. $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$, where the velocity $\mathbf{v}_1$ has non-zero divergence and zero rotor, while the velocity $\mathbf{v}_2$ has zero divergence and non-zero rotor. Formally, it is sufficient to represent the vector $\mathbf{v}$ in the form $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, but for actual fluids/gases one always finds additional problems related with the boundary conditions, or velocity components at the surfaces of the moving liquid.

It can be shown that to solve this problem completely we need to know three components of the liquid/gas velocity at each point of such a boundary, e.g., one normal and two tangential components of the vector $\mathbf{v}$. As follows from \$ 19 of [3] there are two additional conditions for the vorticity vector $\vec{\omega}$, source function s, or divergence D of the moving liquid/gas, and numerical value of the integral which includes the normal derivative of the liquid/gas velocity. The first of these conditions $\nabla \cdot \vec{\omega} = 0$ is always obeyed in the moving liquid/gas (it follows from the definition of vorticity). The second condition is formulated in the form

$$\int div\mathbf{v}dV = \oint(\mathbf{v}\cdot\mathbf{n})dS \quad , \quad or$$
$$\int DdV = \oint v_n dS \tag{8}$$

where v_n is the normal component of the velocity at the border of a finite volume occupied by the liquid/gas and $\mathbf{n}$ is the vector of outer normal to the surface S of the volume V.

Thus, we have replaced one unknown vector $\mathbf{v}$ by the new transverse vector $\vec{\omega}$ (since $div\vec{\omega} = 0$) and one scalar D. In respect to this one three-dimensional equation for $\mathbf{v}$ is replaced by a system of equations for one scalar D and one two-dimensional transverse vorticity vector $\vec{\omega}$ (since $div\vec{\omega} = 0$). The equations arising for D and $\vec{\omega}$ are simpler and they are directly related to the physics of moving liquid/gas. The advantage of this approach is obvious, since the two known motions of the moving liquid (potential motion and rotational motion) are already separated into different equations. This approach is a complete analogy with the Maxwell equations where each of the vector of electromagnetic field $\mathbf{E}$ and $\mathbf{H}$ is replaced by its divergence and rotor/curl. This analogy can be traced even further, since $\mathbf{E}$ is a true vector (or polar vector), while $\mathbf{H}$ is a pseudo vector (or axial vector). In the case of Navier-Stocks equation the scalar $D = div\mathbf{v}$ is a true scalar, while the vorticity $\vec{\omega} = rot\mathbf{v}$ is an axial vector. This fact explains the explicit form of right-hand sides of both equation (see Eq.(6) and Eq.(7)). This approach is correct and general, but some people still consider it as a formal procedure which has no close connections with the physics of moving liquid/gas. To avoid discussions of these pure formal questions we have developed another approach which is based on the same ideas, but it is physically transparent. In this approach the velocity of the moving liquid in each local point is represented as a sum of three terms, but in some cases such sum contains only two terms and even one term, e.g., in the case of potential flow. This approach is discussed in the next Section.

3 Three-term velocity representation

In the previous Section we have shown that the Navier-Stocks equation can be re-written into a different form by introducing the transversal vorticity axial-vector $\vec{\omega}$ and true scalar D which is, in fact, the divergence of the velocity $\mathbf{v}$ of the moving fluid/gas. Formally, the same result can be derived by representing the velocity of the liquid/gas as a sum of a few (two or three) terms. Each of these terms describe one of the well known liquid motions, e.g., potential flow, or rotational flow, or transit flow. Combinations of such flows allows one to describe any possible motion of the actual liquids and/or gases. The goal of this Section is to prove that statement.

First, let us discuss a question about multi-term representation of the velocity in the moving moving liquid/gas. Suppose we have a vector function $\mathbf{f}(\mathbf{r})$ defined in each local point $\mathbf{r}$. The question is to represent this vector-field $\mathbf{f}$ as a combination of actual vectors which are defined in each point of the moving fluid. In reality, such a vector is the velocity $\mathbf{v}$ of the moving liquid/gas. To expand an arbitrary vector $\mathbf{f}$ we need to use three different non-complanar vectors defined in each spatial point of the moving liquid/gas. Formally, we can write the following expansion for the vector-field $\mathbf{f}$ in the moving liquid

$$\mathbf{f} = a\cdot\mathbf{v} + b\cdot\nabla\times\mathbf{v} + c\cdot(\mathbf{v}\times(\nabla\times\mathbf{v})) = a\cdot\mathbf{v} + b\cdot\vec{\omega} + c\cdot(\mathbf{v}\times\vec{\omega}) \tag{9}$$

where the vectors $\mathbf{v}$ and $\mathbf{v}\times\vec{\omega} = (\mathbf{v}\times(\nabla\times\mathbf{v}))$ are the two polar vectors, while the vorticity vector $\vec{\omega} = \nabla\times\mathbf{v}$ is an axial vector, or pseudo-vector. However, the representation of $\mathbf{f}$ in the form of Eq.(9) cannot be used in actual applications, since there is no basic property in an arbitrary moving fluid which is represented by a non-zero pseudo-scalar b defined in each local point of the moving fluid. An alternative form of such a representation is

$$\mathbf{f} = a\cdot\mathbf{v} + b\cdot\nabla\times\vec{\omega} + c\cdot\mathbf{v}\times\vec{\omega} \tag{10}$$

where now only three true (or polar) vectors are used in the right-hand side. Note that the expression in the right-hand side contains spatial derivatives of the second order, while analogous expression in the right-hand side of Eq.(9) includes only spatial derivatives of the first order. Formally, the presence of additional equations, e.g., the Navier-Stocks equation, allows one to reduce the order of spatial derivatives in analytical formulas such as Eq.(10).

Now, we want to discuss a general situation which can be found in any moving liquid/gas which occupy a finite volume V. In this case we can use the following formula from [3] which represents an arbitrary vector $\mathbf{a}$ in the 'two-gradient' form

$$\mathbf{a} = \nabla\phi + \chi\cdot\nabla\psi = \nabla\phi + \mathbf{a}_2 = \mathbf{a}_1 + \mathbf{a}_2 \tag{11}$$

where ϕ, ψ and χ are the three scalar functions, while ∇ is the gradient operator defined above. If we apply this formula to the velocity of the moving fluid and/or gas, then we can write

$$\mathbf{v} = \nabla\phi + \chi\cdot\nabla\psi = \nabla\phi + \mathbf{v}_r = \mathbf{v}_p + \mathbf{v}_r \tag{12}$$

where the notation $\mathbf{v}_p$ stands for the velocity which equals to the gradient of some potential $\phi(\mathbf{r}$. In practical applications it is often called the 'potential' velocity, or velocity of the potential flow (see below). It follows from Eq.(11) that $\nabla\times\mathbf{v}_p = 0$ and $\nabla\cdot\mathbf{v}_p = \Delta\phi = s(\mathbf{r})$, where $s(\mathbf{r})$ is the density-distribution function of the internal sources of the liquid/gas located inside of the moving liquid and/or gas. Here and below the notation Δ means the Laplace operator, i.e. $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Briefly, we can say that the gradient velocity $\mathbf{v}_p$ describes the potential flow in the moving liquid/gas. The notation $\mathbf{v}_r$ in Eq.(12) designates the rotational component of the liquid velocity $\mathbf{v}$, which has non-zero rotor/curl. It follows from the definition of the $\mathbf{v}_r$ vector, Eq.(12), that

$$\begin{aligned} curl\mathbf{v}_r = \nabla \times \mathbf{v}_r = \nabla\chi \times \nabla\psi \quad and \\ div\mathbf{v}_r = \nabla \cdot \mathbf{v}_r = \nabla\chi \cdot \nabla\psi + \chi\Delta\psi \end{aligned} \qquad (13)$$

Note also that as follows from Eqs.(12) and (13) we always have $\nabla\times\mathbf{v} = \nabla\times\mathbf{v}_r = \nabla\chi\times\nabla\psi = \vec{\omega}$. This means that the potential component $\mathbf{v}_p$ of the liquid velocity $\mathbf{v}$ does not contribute to the vorticity $\vec{\omega}$ of the moving liquid/gas. In the general case, it would be nice to have an additional equality $\nabla \cdot \mathbf{v}_r = 0$, but such an equality is obeyed, if (and only if) the two following conditions are true: (1) $\Delta\psi = 0$, i.e. if ψ is a harmonic function, and (2) $\nabla\chi \cdot \nabla\psi = 0$, or $\nabla\chi \perp \nabla\psi$. Such two scalar fields form a pair of conjugate fields.

Let us assume for now that it is possible to choose the two scalar functions ψ and χ in such a way that the equality $\nabla \cdot \mathbf{v}_r = 0$ is obeyed in each spatial point. This means that all possible motions which exist in hydrodynamics of liquids and/or gases can be described in terms of the two-velocity approach, i.e. in any spatial point we need to know the vector $\mathbf{v}_p$, which is associated with the potential flow of the liquid, and vector $\mathbf{v}_r$, which describes rotational flows in the moving liquid and/or gas. It is clear that such a two-velocity picture governs a large number of phenomena in the moving liquid and/or gas. Formally, the two velocity representation means that in an arbitrary spatial point of the moving liquid we always have $\mathbf{v} = \mathbf{v}_p + \mathbf{v}_r$, where $\nabla \times \mathbf{v}_p = 0$ and $\nabla \cdot \mathbf{v}_r = 0$. It follows from here that the first component of the velocity $\mathbf{v}_g$ is represented as a gradient of some scalar function $\Phi(\mathbf{r})$, while the second component $\mathbf{v}_r$ is the rotor (or curl) of some axial vector $\mathbf{F}(\mathbf{r})$, i.e. $\mathbf{v}_r = \nabla \times \mathbf{F}$.

It was mentioned above that for the potential component of the velocity $\mathbf{v}_p$ the following equality is always obeyed: $\nabla \cdot \mathbf{v}_p = s(\mathbf{r})$. This leads to the Poisson equation for the scalar potential function $\Phi(\mathbf{r})$ mentioned above, i.e. $\Delta\Phi = s$, which has a well known solution

$$\Phi(\mathbf{r}) = -\frac{1}{4\pi}\int\int\int \frac{s(\mathbf{r}')dV'}{\mid \mathbf{r}-\mathbf{r}' \mid} \quad , \qquad (14)$$

where $dV' = dx'dy'dz'$. It is straightforward to calculate the gradient of the potential $\Phi(\mathbf{r})$ function defined by Eq.(14). Now, consider the analogous equation which determines the pseudo-vector $\mathbf{F}$ uniformly related with the vorticity $\vec{\omega}$. From the definition of $\vec{\omega}$ and relation between $\mathbf{F}$ and $\mathbf{v}$ given above ($\mathbf{v} = \nabla \times \mathbf{F}$) one finds

$$\nabla \times \nabla \times \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \Delta\mathbf{F} = \vec{\omega} \qquad (15)$$

where $\vec{\omega}$ is the axial vector of vorticity. Let us assume that the pseudo-vector $\vec{\omega}$ is known. In this case Eq.(15) is written as the three Poisson equations $\Delta\mathbf{F} = -\vec{\omega}$, or $\Delta F_x = -\omega_x, \Delta F_y = -\omega_y, \Delta F_z = -\omega_z$,i.e. we have one equation per each component of the vorticity vector. The solution of these three Poisson vector-equations can also be written in the following vector-form

$$\mathbf{F}(\mathbf{r}) = \frac{1}{4\pi}\int\int\int \frac{\vec{\omega}(\mathbf{r}')dV'}{\mid \mathbf{r}-\mathbf{r}' \mid} \quad , \qquad (16)$$

where all notations under the integral are exactly the same as in Eq.(14). The formulas, Eqs.(14) - (16), allow one to solve the Navier-Stocks equations in the infinite space by using the given s and $\vec{\omega}$ functions. The scalar function s and pseudo-vector function $\vec{\omega}$ are assumed to be continuous functions defined in the whole space. The first-order derivatives of the $\Phi(\mathbf{r})$ and $\mathbf{F}(\mathbf{r})$, Eqs.(14) - (16) are also continuous functions of the spatial coordinates.

In 99 % of all applications to hydrodynamics the volume occupied by the moving liquid/gas is always finite and very often some amount of liquid/gas constantly moves through the boundaries of such a volume. Therefore, we need to develop an approach to describe motions of the liquid/gas through the boundaries in and out of the finite volume V occupied by the same liquid/gas. Below we shall assume that the boundary surface S of the finite volume is 'sufficiently smooth'. The motion of liquid through boundaries of the finite volume V can be described by an additional component of the liquid velocity. In other words, we introduce the following three-component representation of the liquid velocity $\mathbf{v}$:

$$\mathbf{v} = \mathbf{v}_p + \mathbf{v}_r + \mathbf{v}_b \qquad (17)$$

where the first two vectors $\mathbf{v}_p$ and $\mathbf{v}_r$ are exactly the same as defined above, while the third vector has zero divergence and zero rotor (or curl), i.e. $\nabla \cdot \mathbf{v}_b = 0$ and $\nabla \times \mathbf{v}_b = 0$. In reality, it is easy to find that the three-term representation of the velocity vector $\mathbf{v}$, Eq.(17), is not very practical, since it does not contain any 'free' parameter which can be used to sew together three different velocities in 'intermediate areas' of the moving liquid/gas. In general, it is better to introduce a slightly different expression for the velocity $\mathbf{v}$ which is based on Eq.(17)

$$\mathbf{v} = C_p\mathbf{v}_p + C_r\mathbf{v}_r + C_b\mathbf{v}_b \qquad (18)$$

where C_p, C_r and C_b are unknown linear coefficients which are some slow-varying functions of all three velocities $\mathbf{v}_p, \mathbf{v}_r, \mathbf{v}_b$ and local coordinates $\mathbf{r}$. Substitution of Eq.(18) into the Navier-Stocks equation produces a set of coupled differential equations for these functions (C_p, C_r and C_b). By solving these equations we can determine the exact solutions of the Navier-Stocks equations. In the lowest-order approximation these coefficients can be considered as numerical constants. This allows one to represent the unknown flow of the moving liquid/gas by using the known expressions for the $\mathbf{v}_p, \mathbf{v}_r, \mathbf{v}_b$ velocities.

The velocity $\mathbf{v}_b$ has been introduced in Eq.(17) to describe all possible flows of the liquid and/or gas which arise from the outer sources of this liquid, or, in other words, from sources located outside of the finite volume occupied by the liquid/gas. It follows from here that $\mathbf{v}_b = \nabla\Psi$, i.e. $\mathbf{v}_b$ is the gradient of some scalar field $\Psi(\mathbf{r})$ which is called the second velocity potential, or surface potential. Substitution of this equation into the condition $\nabla \cdot \mathbf{v}_b = 0$ leads to the Laplace equation for the second velocity potential $\Psi(\mathbf{r})$, i.e. $\Delta\Psi(\mathbf{r}) = 0$. In other words, the function $\Psi(\mathbf{r})$ is a harmonic function in the finite volume V occupied by the liquid/gas. The gradient of this function at the outer surface S of the finite volume V must be equal to the flow of liquid/gas which moves into and/or out of this finite volume. This can be written in the form

$$\frac{\partial \Psi}{\partial n} = (\mathbf{n} \cdot \nabla)\Psi = H(M) \tag{19}$$

where $\mathbf{n}$ is the unit normal vector (or normal, for short) to the surface S at the point M. The derivative $\frac{\partial \Psi}{\partial n}$ is the gradient of the Ψ function along this normal. The scalar function $H(M)$ in Eq.(19) is a function which describes the flow of a liquid/gas at the point M of the boundary surface S.

Numerous advantages of the three-velocity representations of the total velocity $\mathbf{v}$ can be illustrated by a substitution of Eq.(19) into Eq.(3). After such a substitution it is easy to find that many terms in these equations equal zero identically. Finally, we can say that the three-component representation of the local velocity Eq.(17) allows one to solve the general Navier-Stocks equation in those case when we know the 'source function' $s(\mathbf{r})$ and vorticity pseudo-vector $\vec{\omega}(\mathbf{r})$ in each spatial point of the volume occupied by the liquid/gas. In addition to these values, we also need to know the function $H(M)$ which describes the flow of liquid/gas at each point of the boundary surface S. The knowledge of these values allows one to solve the general Navier-Stocks equations everywhere in the volume V surrounded by the surface S. It can be shown that such solutions are unique and uniform functions of the spatial coordinates. In this form the problem of solving general Navier-Stocks equations essentially coincides with the Neumann problem known from the Mathematical Physics (see, e.g., [5]).

4 Applications to incompressible fluids

Let us consider a few simplified cases when our basic equations, i.e. Eq.(6) are Eq.(7), can be written in a relatively simple and short form. For incompressible fluids, where $\nabla \cdot \mathbf{v} = 0$, we have the following system of coupled differential equations

$$\frac{\partial \vec{\omega}}{\partial t} = (\vec{\omega} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\vec{\omega} + \nu \Delta \vec{\omega} \quad , \tag{20}$$

$$\frac{\partial D}{\partial t} = -\frac{1}{2}\Delta v^2 - \mathbf{v} \cdot (\nabla \times \vec{\omega}) + \omega^2 - \frac{1}{\rho}\Delta p + \Big(\mu + \frac{4}{3}\nu\Big)\Delta D \quad , \tag{21}$$

In the case of statitonary processes one finds:

$$\nu \Delta \vec{\omega} = (\mathbf{v} \cdot \nabla)\vec{\omega} - (\vec{\omega} \cdot \nabla)\mathbf{v} \quad , \tag{22}$$

$$\Big(\mu + \frac{4}{3}\nu\Big)\Delta D = \frac{1}{2}\Delta v^2 + \mathbf{v} \cdot (\nabla \times \vec{\omega}) + \frac{1}{\rho}\Delta p - \omega^2 \quad , \tag{23}$$

These systems of the coupled differential equations are completely equivalent to the general Navier-Stocks equation, Eq.(4) and such an equation in the stationary case.

The explicit form of the equations Eq.(20) and Eq.(21) allows one to investigate some general properties of the moving, incompressible fluids. As follows from Eq.(20) and Eq.(21) and from analogous equations mentioned in Sectiond II and III above all possible motions in an actual fluid can be represented as a combination of the potential flow, rotational flow (or flow with non-zero vorticity) and outer flow through the finite volume occupied by the same liquid. In general, the potential flow corresponds to the laminar flow, while the flow with non-zero vorticity can be considered as a turbulent flow. In actual liquid/gas we always have a mixture of laminar and turbulent flow. Furthermore, the velocity representation in the form of Eq.(18) is not unique, i.e. if we have an identity

$$\mathbf{v} = C_p \mathbf{v}_p + C_r \mathbf{v}_r + C_b \mathbf{v}_b = C'_p \mathbf{v}_p + C'_r \mathbf{v}_r + C'_b \mathbf{v}_b \tag{24}$$

then, in the general case, it is impossible to show that $C_p = C'_p, C_r = C'_r$ and/or $C_b = C'_b$. This substantially complicates analysis of the general flow of a liquid/gas. In many cases it is also leads to various instabilities in actual liquids/gases.

5 Conclusion

The new 'rotor-divergence' approach to the Navier-Stocks equation is developed. This approach is based on the fact that the Navier-Stocks equation is linear upon the vorticity of the moving liquid/gas ($\vec{\omega} = rot\mathbf{v}$) and scalar D ($D = div\mathbf{v}$). It is used to transform the Navier-Stocks equation to a differnet form which is simpler than the incident equation. Our approach is based on the derivation of the governing equations for the divergence and rotor (or curl) of the liquid/gas velocity $\mathbf{v}$. The knowledge of the divergence and rotor (or curl) of the velocity $\mathbf{v}$ of the moving liquid/gas allows one to determine the velocity $\mathbf{v}$ of the liquid/gas, i.e. to find the solutions of the Navier-Stocks equation. The arising equations for the divergence $D = \nabla \cdot \mathbf{v}$ and rotor/curl $\vec{\omega} = \nabla \times \mathbf{v}$ are significantly simpler than the incident Navier-Stocks equation. Formally, by using Eq.(6) and Eq.(7) we can determine the velocity vector $\mathbf{v}$. In general, this approach is an alternative method developed to determine analytical and numerical solutions of the Navier-Stocks equation. In reality, it is better to apply the following approach based on the process of consecutive iteractions. First, we chose the velocity vector $v_0(\mathbf{r}, t)$ and calculate its rotor (or curl) $\vec{\omega}_0$ and divergence D_0. By substituting these values of $\vec{\omega}_0$ and D_0 into Eq.(6) and Eq.(7) we obtain equations. Solution of these equations gives us the new velocity $\mathbf{v}_1$. By using this velocity $\mathbf{v}_1$ we determine its rotor $\vec{\omega}_1$ and divergence D_1, respectively. Again, by using these values in Eq.(6) and Eq.(7) we derive the new equation. Solution of these equations gives us the new velocity $\mathbf{v}_2$. Then, we determine the rotor $\vec{\omega}_2$ and divergence D_2 of this velocity. At the third stage of this process we solve Eq.(6) and Eq.(7) and obtain the velocity $\mathbf{v}_3$ and its rotor $\vec{\omega}_3$ and divergence D_3. It can be shown that after a number of steps in this procedure we obtain a velocity vector $\mathbf{v}_n$ which is very close to the actual velocity $\mathbf{v}$ in each spatial point of the moving fluid. Furthermore, its rotor $\vec{\omega}$ and divergence D coincide with the actual vorticity and density distribution of the liquid sources. In other words, the process converges to the actual solution of the Navier-Stocks equation.

References

[1] L.D. Landau and E.M. Lifshitz, *Theory of Elasticity* (Pergamon Press, London, UK, 1970).

[2] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics* (Pergamon Press, Oxford, UK, 1966).

[3] N.E. Kochin, *Vector Calculus and Foundations of Tensor Calculus* (Academy of Science Publ., 8th ed., Moscow, 1961) [in Russian].

[4] I.M. Gel'fand, R.A. Minlos and Z.Ya. Shapiro, *Representations of the Rotation and Lorentz Groups and Their Applications* (MacMilan Publ., New York, 1963).

[5] A.N. Tikhonov and A.A. Samarskii, *Equations of Mathematical Physics* (Dover Publ. Inc., New York, 1996).