On Irresolute Multifunctions and Games

A.A. El-Atik

Department of Mathematics, Faculty of Science,
Tanta University, Tanta, Egypt
Email: aelatik55@yahoo.com

Department of mathematics, Faculty of Science and Human sciences in El-Dawadmi, 11911, Box 1040 Shaqra University, KSA

Abstract

A topological game is a game in which positions are the points of a topological space, the set of possible moves from any such points varies continuously with the point. The play can start from any point of the space and at each point it is specified which player has the initiate, and ends when the set of positions from which the player with the initiate can choose is empty. The payoff to each player depends on the set of positions met in the play. The aim of this paper is to introduce and study a new topological game for irresolute multifunctions.

Keywords–Topological games, upper an lower irresolute multifunctions, the tactic, a winning tactic.

1 Introduction

The notion of topological games with perfect information has introduced and studied by Berge [2]. Many authors used it to solve some topological problems (e.g. [12] and [13]). For further details, see [2, 6, 9, 3]. A. R. Pears [10] has defined and studied the topological games for continuous multifunctions. A subset A of a topological space X is said to be semi-open [8] if there exists an open set U of X such that $U \subset A \subset cl(U)$, where $cl(U)$ denotes the closure of U. Ewert and Lipski [7] introduced the concept of irresolute multifunctions. V. Papa and T. Noiri [11] obtained further characterizations of irresolute multifunctions. For $A \subset X$, the interior of A will be denoted by int(A). Also,$cl_\tau(A)(int_\tau(A))$ denotes the closure (interior) of A with respect to in order to avoid confusion when there exist more than one topology on X. Throughout this paper, X and Y always mean topological spaces. A multifunction of X into Y is a function $\Gamma : X \longrightarrow P(Y)$, where $P(Y)$ the set of all subsets of Y. For a multifunction $\Gamma : X \longrightarrow Y$, we shall denote the upper and lower inverse of a subset B of Y by $\Gamma^+(B)$ and $\Gamma^-(B)$ respectively that is, $\Gamma^+(B) = \{x \in X : \Gamma(x) \subset B\}$ and $\Gamma^-(B) = \{x \in X : \Gamma(x) \cap B \neq \phi\}$.

The family of semi-open sets of X is denoted by $SO(X)$ and $SO(X, x)$ represents the family of semi-open sets of X containing a point x of X. The complement of a semi-open set is called semi-closed [4].

Definition 1.1. *[7]. A multifunction $\Gamma : X \longrightarrow Y$ is said to be:*

(a) upper irresolute (resp. lower irresolute) at a point x of X if for each $V \in SO(Y)$ such that $\Gamma(x) \subset V$ (resp.$\Gamma(x) \cap V \neq \phi$, there exists $U \in SO(X, x)$ such that $\Gamma(u) \subset V$ (resp.$\Gamma(u) \cap V \neq \phi$) for every $u \in U$.

(b) upper irresolute (resp. lower irresolute) if it is upper irresolute (resp. lower irresolute) at each point of X.

Definition 1.2. *[5] A subset A of a space $(X, \tau)$is said to be semi-compact if every cover of A by semi-open sets of (X, τ) has a finite subcover.*

2 Topological games

In this article, consider each X_i is a topological space for each $i = 1, 2, ..., n$ and X is a topological sum of $\{X_1, X_2, ..., X_n\}$.

Definition 2.1. *A game on X for the players $(1), (2), ..., (n)$ is defined as follows :*

(i) *A partition $\{N^+, N^-\}$ of the set N of players.*

(ii) *A partition $\{X_1, X_2, ..., X_n\}$ of X.*

(iii) *An irresolute multifunction Γ of X onto itself such that $\Gamma(X_i) \cap X_i = \phi$ for $i = 1, 2, 3, ..., n$.*

(iv) *n-bounded real valued functions $f_1, f_2, f_3, ..., f_n$ on X.*

The points of X are the positions of the game, play can start from any position. If $x \in X_i$ is the move of player (i) at the position x. A play with x_0 as initial position proceeds as follows: If $x \in X_i$, player (i) chooses a position x_1 in the set $\Gamma(x_0)$: If $x_1 \in X_j$, player (j) chooses a position $x_2 \in \Gamma(x_1)$, and so on. If in the course of the play a position x is reached where $\Gamma(x) = \phi$, then the play terminates at x. Thus a play is a sequence $< x_0, \Gamma(x_0), x_1, \Gamma(x_1), ... >$ such that $x_0 \in \Gamma(x_0)$, $x_1 \in \Gamma(x_1)$ and so on.

Definition 2.2. *For a finite sequence* $< x_0, \Gamma(x_0), x_1, \Gamma(x_1), ..., x_k, \Gamma(x_k) >$ *with* $k+1$ *elements of a play, the length of the play is said to be* k*, note that the last element* x_k *must satisfy* $\Gamma(x_k) = \phi$.

If the length of each play of the game is finite, the game is called locally finite. If S is the set of positions in a play, the payoff to player (i) for the play is $\sup\{f_i(x) : x \in S\}$ or $\inf\{f_i(x) : x \in S\}$ according as $(i) \in N^+$ or $(i) \in N^-$. The aim of each player is to obtain as large payoff as possible.

Definition 2.3. *Player* (i) *is said to guarantee* γ *(where* γ *is a real number) from the initial position* x *if he can ensure that, whatever the other players do, his payoff for all plays starting at* x *is greater than or equal to* γ*. If he can ensure a payoff is greater than* γ *for a play begins at* x*, he is called strictly guarantee* γ *from* x.

Lemma 2.4. *[1] For a topological space* (X, τ)*, if* $G \in \tau$ *and* $A \subset X$*, then* $G \cap cl(A) \subset cl(G \cap A)$.

Proposition 2.5. *If* X *is a topological sum of the family* $X_i : i \in I$ *and* $A \in SO(X)$*, then* $A \cap X_i \in SO(X_i)$ *for each* $i \in I$*. If* A *is open in* X*, the inverse conclusion will be hold.*

Proof. Let $A \in SO(X)$, then $A \cap X_i \subset cl(intA) \cap X_i$. Since each X_i is open in X, by Lemma 2.4, we get $A \cap X_i \subset cl(intA \cap X_i) = cl(int(A \cap X_i))$. Then $A \cap X_i \subset cl(int(A \cap X_i)) \cap X_i = cl_{X_i} int(A \cap X_i) \cap X_i$, each X_i is a subspace of X, then $int(A \cap X_i) \cap X_i$ is open in X_i, for each $i \in I$. Therefore $A \cap X_i \subset cl_{X_i}(int_{X_i}(int(A \cap X_i) \cap X_i) = cl_{X_i}(int_{X_i}(int(A \cap X_i))) \subset cl_{X_i}(int_{X_i}(A \cap X_i))$. Hence $A \cap X_i \in SO(X_i)$, for each $i \in I$. Conversely, suppose that A is open in X and $A \in SO(X_i)$, then $A \cap X_i \subset cl_{X_i} int_{X_i}(A \cap X_i) \subset cl_{X_i}(A \cap X_i) = X_i \cap cl(A \cap X_i) \subset cl(A) \cap X_i$, implies $A \subset cl(A)$. For the openness of A, we get $A \subset cl(intA)$ and $A \in SO(X)$. ☐

3 Preservation theorems

In this section, suppose that Γ is an irresolute multifunction of X into itself. There are two types of games depend on the real valued function f_i.

Definition 3.1. *For a topological space* X *, a real valued function* $f : X \longrightarrow R$ *is upper (lower) s-continuous if for every* $x \in X$ *and every real number* r *satisfying* $f(x) < r \quad (f(x) > r)$*, there exists a semi-open nbd* $U \subset X$ *of* x *such that* $f(x^{'}) < r \quad (f(x^{'}) > r)$ *for every* $x^{'} \in U$.

Definition 3.2. *The game is said to be*

(1) *lower topological for player* (i) *if in addition the real valued function* f_i *is lower s-continuous.*

(2) *upper topological for player* (i) *if* f_i *is upper s-continuous.*

Theorem 3.3. *If a game is lower topological for* $(1) \in N^+$*, then the set of positions from which* (1) *can strictly guarantee a gain* γ *is semi-open in* X.

Proof. Let A_γ be the set of initial positions from which (1) can strictly guarantee γ. Then $(X_1 \cap \Gamma^-(A_\gamma) \cup (\bigcup_{j=1}^{n}(X_j \cap \Gamma^+(A_\gamma))) \subset A_\Gamma$ .Let us note that $\Gamma^+(A_\gamma) = \{x \in X : \Gamma(x) \subset A_\gamma\}$ and $\Gamma^-(A_\gamma) = \{x \in X : \Gamma(x) \cap A_\gamma \neq \phi\}$. We construct by transfinite induction a semi-open set $A(\alpha)$ such that $A(\alpha) \subset A_\gamma$ for each ordinal α as follows:

Let $A(0) = \{x \in X : f_1(x) > \gamma\}$. Since the game is lower topological, then f_1 is lower s-continuous. Thus $A(0)$ is semi-open and $A(0) \subset A_\gamma$. Suppose that we have defined a semi-open set $A(\beta) \subset A_\gamma$ for all ordinal $\beta < \alpha$. If α is a limit ordinal, let $A(\alpha) = \bigcup_{\beta<\alpha} A(\beta)$, then $A(\alpha)$ is semi-open and $A(\alpha) \subset A_\gamma$. If α is not a limit ordinal, this means $\alpha = \alpha^{'} + 1$ say, let $A(\alpha) = A(\alpha^{'}) \cup (X_1 \cap \Gamma^-(A(\alpha^{'}))) \cup \bigcup_{n=2}^{n}(X_j \cap \Gamma^+(A(\alpha')))$ by hypothesis,$A(\alpha^{'})$ is semi-open and by proposition 2.5,$X_1 \cap \Gamma^-(A(\alpha^{'}))$, $X_j \cap \Gamma^+(A(\alpha^{'}))$ for each $j = 2, 3, ..., n$ are semi-open. Since X is the topological sum of $\{X_1, X_2, ..., X_n\}$ and Γ is an irresolute multifunction, thus $A(\alpha)$is semi-open and $A(\alpha) \subset A_\gamma$. For $A(\alpha^{'}) \subset A_\gamma$ and $(X_1 \cap \Gamma^-(A(\alpha^{'}))) \cup \bigcup_{j=2}^{n}(X_j \cap \Gamma^+(A(\alpha^{'}))) \subset (X_1 \cap \Gamma^-(A_\gamma)) \cup \bigcup_{j=2}^{n}(X_j \cap \Gamma^+(A_\gamma)) \subset A_\gamma$. Therefore for each ordinal α we have a semi-open set $A(\alpha)$ and $A(\alpha) \subset A_\gamma$. The transfinite sequence $\{A(\alpha)\}$ is increasing and so must became ultimately constant. This means that, we have $A(\alpha_0) = A(\alpha_0 + 1) = ...$ for some α_0. Let $A^{'} = X - A(\alpha_0)$. If $x \in A^{'} \cap X_1$, then $\Gamma(x) \subset A^{'}$ whilst if $x \in A^{'} \cap X_j$, where $j \neq 1$, then $\Gamma(x) \cap A^{'} \neq \phi$. Therefore if a play begins from a point in $A^{'}$, whatever (1) does, players (2), (3),..., (n) can ensure that a position in $A(\alpha_0)$ is never reached. But $A(\alpha_0) \supset A(0) = \{x : f_1(x) > \gamma\}$. Thus if $x \in A^{'}$, implies $x \notin A(\alpha_0)$, then $x \notin A_\gamma$ and so $A_\gamma \subset A(\alpha_0)$. But $A(\alpha_0) \subset A_\gamma$ by construction and so we have $A_\gamma = A(\alpha_0)$. Hence $A_\gamma \in SO(X)$. ☐

The complement of the conditions in theorem 3.3 does not satisfy that the set of positions from which (1) can guarantee a gain γ is semi-closed.

Remark 3.4. *Since the complement of each semi-open set is semi-closed, and* Γ *is irresolute,* $\Gamma^+(X)$ *is semi-open, then* $X_0 = X - \Gamma^+(X)$ *is semi-closed.*

The following theorem gives additional conditions.

Theorem 3.5. *Suppose that a game is locally finite and upper topological for* $(1) \in N^+$ *and that* $X_0 = \{x : \Gamma(X) = \phi\}$ *is a semi-open set. Then the set of positions from which* (1) *can gurantee a gain is semi-closed.*

Proof. We define a semi-open set $X(\alpha)$ for each ordinal α. Let $X(0) = X_0 = \{x : \Gamma(x) = \phi\}$. Then $X(0)$ is semi-open. Now suppose we have constructed semi-open sets $X(\beta)$ for all ordinals $\beta < \alpha$. If α is a limit ordinal, let $X(\alpha) = \bigcup_{\beta<\alpha} X(\beta)$ which is semi-open. If α has an immediate predecessor $\alpha^{'}$ i.e. $\alpha = \alpha^{'} + 1$ say, let $X(\alpha) = X(\alpha^{'}) \cup \Gamma^{+}(X(\alpha^{'}))$, since Γ is an irresolute multifunction, then $X(\alpha)$ is semi-open. Thus for each ordinal α we have by transfinite induction semi-open set $X(\alpha)$. We note that if $\beta < \alpha$, $X(\beta) < X(\alpha)$. Now, we define H_γ to be the set of positions from which (1) can guarantee γ. Then $(X_1 \cap \Gamma^{-}(H_\gamma)) \cup \bigcup_{j02}^{n} (X_j \cap \Gamma^{+}(H_\gamma)) \subset H_\gamma$. We define a set $H(\alpha)$ for each α such that

(i) $H(\alpha) \subset H_\gamma$;

(ii) if $\beta < \alpha$, $H(\beta) \subset H(\alpha)$;

(iii) if $\beta < \alpha$, $H(\alpha) \cap X(\beta) = H(\beta) \cap X(\beta)$ and (iv)$H(\alpha) \cap X(\alpha)$ is semi-closed in $X(\alpha$.

Claim (1) Let $H(0) = \{x : f_1(x) \geq \gamma\}$, since f_1 is upper s-continuous, then $H(0)$ is semi-closed in X; also $H(0) \subset H_\gamma$; and so $H(0) \cap X(0)$ is semi-closed in $X(0)$. Suppose that the set $H(\beta)$ satisfying (i)-(iv) have been constructed for all ordinals $\beta < \alpha$.

Claim (2) If α is a limit ordinal, let $H(\alpha) = \bigcup_{\beta<\alpha} H(\beta)$. Since each $H(\beta) \subset H_\gamma$, then $H(\alpha) \subset H_\gamma$. Also if $\beta < \alpha$, then $H(\beta) \subset H(\alpha)$ and if $\beta^{'} < \alpha$, $H(\alpha) \cap X(\beta^{'}) = (\bigcup_{\beta<\alpha} H(\beta)) \cap X(\beta^{'}) = \bigcup_{\beta<\alpha}(H(\beta)) \cap X(\beta^{'})$ .If $\beta < \beta^{'}$, then $H(\beta) \cap X(\beta^{'}) \subset H(\beta^{'}) \cap X(\beta^{'})$, and if $\beta^{'} \leq \beta < \alpha$, $H(\beta) \cap X(\beta^{'}) = H(\beta^{'}) \cap X(\beta^{'})$. Hence $H(\beta) \cap X(\beta^{'}) = H(\beta^{'}) \cap X(\beta^{'})$ and (iii) is satisfied. If $x \in X(\alpha)$ and $x \notin H(\alpha)$, then $x \in X(\beta)$ for some $\beta < \alpha$ and $x \notin H(\beta)$. Now $H(\beta) \cap X(\beta)$ is semi-closed in $X(\beta)$ and so there is a semi-open nbd A of x in $X(\beta)$ such that $A \cap H(\beta) = \phi$. Now, since $X(\beta)$ and A are semi-open sets in X and $A \subset X(\beta) \subset X(\alpha)$, then A is semi-open in $X(\alpha)$. By (iii), $X(\beta) \cap H(\alpha) = X(\beta) \cap H(\beta)$ and so A is a semi-open nbd of x in $X(\alpha)$ such that $A \cap H(\alpha) = \phi$. Thus (iv) is satisfied.

Claim (3) If α has an immediate predecessor $\alpha^{'}$ i.e. $\alpha = \alpha^{'} + 1$ say, let $H(\alpha) = H(\alpha^{'}) \cup (X_1 \cap \Gamma^{-}(H(\alpha^{'}))) \cup \bigcup_{j=2}^{n} (X_j \cap \Gamma^{+}(H(\alpha^{'})))$. Since $H(\alpha^{'}) \subset H_\gamma$ and $X_1 \cap \Gamma^{-}(H(\alpha^{'})) \cup \bigcup_{j=2}^{n} (X_j \cap \Gamma^{+}(H(\alpha^{'}))) \subset (X_1 \cap \Gamma^{-}(H_\gamma)) \cup \bigcup_{j=2}^{n} (X_j \cap \Gamma^{+}(H_\gamma)) \subset H_\gamma$, then (i) is satisfied, (ii) is clearly satisfied. Suppose $\beta^{'} < \alpha$. If $x \in X(\beta^{'})$ and $\Gamma(x) \neq \phi$, then $\Gamma(x) \subset X(\beta)$ for some $\beta < \beta^{'}$. Thus if $x \in X(\beta^{'}) \cap (X_1 \cap \Gamma^{-}(H(\alpha^{'})))$, $\Gamma(x) \cap \{X(\beta) \cap H(\alpha^{'})\} \neq \phi$ for $\beta < \beta^{'} \leq \alpha^{'}$. Thus $x \in X(\beta^{'}) \cap (X_1 \cap \Gamma^{-}(H(\beta))) \subset X(\beta^{'}) \cap H(\beta + 1) \subset X(\beta^{'} \cap H(\beta^{'})$. Similarly if $j \neq 1$,$X(\beta^{'}) \cap (X_j \cap \Gamma^{+}(H(\alpha^{'}))) \subset X(\beta^{'}) \cap H(\beta^{'})$. Also,$X(\beta^{'}) \cap H(\alpha^{'}) = X(\beta^{'}) \cap H(\beta^{'})$. Thus $X(\beta^{'}) \cap X(\alpha) = X(\beta^{'}) \cap [H(\alpha^{'}) \cup (X_1 \cup \Gamma^{-}(H(\alpha^{'}))) \cup \bigcup_{j=2}^{n} X_j \cap \Gamma^{+}(H(\alpha^{'}))] = X(\beta^{'}) \cap H(\beta^{'})$ and so (iii) is satisfied. Finally, we prove (iv), suppose that $x \in X(\alpha)$ and $x \notin H(\alpha)$. If $x \in X(\alpha^{'})$, then $x \notin H(\alpha^{'})$ and since $H(\alpha^{'}) \cap X(\alpha^{'})$ is semi-closed in $X(\alpha^{'})$, there is a semi-open nbd A of x in $X(\alpha^{'})$ such that $A \cap H(\alpha^{'}) = \phi$. Since $X(\alpha^{'})$ is semi-open in X and $A \subset X(\alpha^{'}) \subset X(\alpha)$, then A is a semi-open nbd of x in $X(\alpha)$, and since $X(\alpha^{'}) \cap H(\alpha) = X(\alpha^{'}) H(\alpha^{'})$, by (iii), then $A \cap H(\alpha) = \phi$. If $x \in (X(\alpha) - X(\alpha^{'})) \cap X_1$, then $\Gamma(x) \subset (X(\alpha^{'}) - H(\alpha^{'}))$. $X(\alpha^{'}) - H(\alpha^{'})$ is semi-open in $X(\alpha^{'})$ and so is semi-open in X. $X_1 \cap \Gamma^{+}(X(\alpha^{'}) - H(\alpha^{'}))$ is semi-open nbd of x such that $[X_1 \cap \Gamma^{+}(X(\alpha^{'}) - H(\alpha^{'}))] \cap H(\alpha) = \phi$. If $x \in (X(\alpha) - X(\alpha^{'})) \cap X_j$, then $\Gamma(x) \cap (X(\alpha) - H(\alpha^{'})) \neq \phi$ and $X_j \cap \Gamma^{-}(X(\alpha^{'}) - H(\alpha^{'}))$ is semi-open nbd of x such that $[X_j \cap \Gamma^{-}(X(\alpha^{'}) - H(\alpha^{'}))] \cap H(\alpha) = \phi$. In either cases, if $x \in X(\alpha)$ and $x \notin H(\alpha)$, there is a semi-open nbd of x in $X(\alpha)$ not meeting $H(\alpha)$. Therefore $H(\alpha) \cap X(\alpha)$ is semi-closed in $X(\alpha)$ and (iv) is satisfied. Thus, by transfinite induction we can construct $H(\alpha)$ for each ordinal α such that (i) - (iv) are satisfied. By Berge [2], since the game is locally finite, then $X = X(\alpha_0)$ for some ordinal α_0. Thus $H(\alpha_0$ is semi-closed set and if $\alpha > \alpha_0$, $H(\alpha) = H(\alpha) \cap H(\alpha_0) = H(\alpha_0)$. Let $H^{'} = X - H(\alpha_0)$. If $x \in H^{'} \cap X_1$, then $\Gamma(x) \subset H^{'}$, and if $x \in H^{'} \cap X_j$ where $j \neq 1$, then $\Gamma(X) \cap H^{'} \neq \phi$. Thus if a play begins from a position in $H^{'}$, whatever (1) does, players (2), (3), ... ,(n) can ensure that a position in $H(\alpha_0)$ is never reached. But $H(\alpha_0) \supset H(0) = \{x : f_1(x) \geq \gamma\}$ and so $H_\gamma \subset H(\alpha_0)$. But $H(\alpha_0) \subset H_\gamma$ by construction and so we have $H(\alpha_0) = H_\gamma$.

Thus H_γ is semi-closed. This complete the proof. □

The hypothesis of Theorem 3.5 can not be weakened. Example 3.6 shows that if X_0 is not semi-open, then the conclusion of Theorem 3.5 is false even there is a bound to the length of a play of the game.

Example 3.6. *Consider two players ONE and TWO played on the space consisting of the topological sum of* X_1 *and* X_2 *of the segment* $(-1, m]$ *of the real line. Let* $(x; i)$ *be the point* $x \in X_i$ *and suppose that*

$$\Gamma(x; i) = \begin{cases} (x-1, j) & i \neq j \ : \ x > 0 \\ \phi & : \ x \leq 0 \end{cases}$$

Suppose that $(1) \in N^{+}$ *and*

$$f_1(x) = \begin{cases} 1 & : \ x \in X_2 \ \text{ and } \ x \leq 0 \\ 0 & : \ \text{otherwise} \end{cases}$$

f_1 *is upper s-continuous, and so the game is upper topological for* $(1) \in N^{+}$*. The set of initial positions from which* (1) *can guarantee unit gain is* $\{(x; 1) : 0 < x \leq 1, 2 < x \leq 3, ...\} \cup \{(x; 2) : 1 < x \leq 2, 3 < x \leq 4, ...\}$ *which is not semi-closed.*

The following example shows that the conclusion of Theorem 3.5 may be false if the game is not locally finite.

Example 3.7. *Suppose that X is the topological sum of X_1 and X_2 where X_1 is the real line and X_2 is the topological sum of Y and Z of subsets of the real line. Let $(x;1)$, $(x;2)$; $(x;0)$ denote the point x in X_1, Y, Z respectively. Let*

$$\begin{aligned}\Gamma(x;1) &= \{(x;0)\} \cup \{(y;2) :\mid x-y \mid\leq 3 \mid x \mid\} \\ \Gamma(x;2) &= \{(y;1) :\mid x-y \mid\leq 1/2 \mid x \mid\} \\ \Gamma(x;0) &= \phi, \quad \text{then} \quad X_0 = Z\end{aligned}$$

Suppose that $(1) \in N^+$ and that $f_1(x;0) = 1$ if $\mid x \mid$ whilst $f_1 = 0$ otherwise. This is upper topological game for $(1) \in N^+$ and $X - 0$ is semi-open but the game is not locally finite. The set of positions in X_1 from which (1) can guarantee unit gain is $\bigcup_{n=0}^{\infty} \{(x;1) :\mid x \mid\geq 1/2^n\} = \{(x;1) :\mid x \mid> 0\}$ which is not semi-closed. But X is the topological sum of X_1 and X_2 and so the set of initial positions from which (1) can guarantee unit gain is not semi-closed.

Corollary 3.8. *If a game is upper topological for $(1) \in N^-$, the set of positions from which (1) can guarantee γ is semi-closed.*

Proof. Let A_γ denotes the set of initial positions from which (1) can not guarantee γ. Similar to the proof of Theorem 3.3, we can construct a semi-open set A such that $\{x : f_1(x) < \gamma\} \subset A \subset A_\gamma$. Set $A^c = X - A$, then A^c is semi-closed in X.If $x \in A^c \cap X_1$, then $\Gamma(x) \cap A^c \neq \phi$, which if $x \in A^c \cap X_j$ such that $j \neq 1$, then $\Gamma(x) \subset A^c$. Therefore if a play begins at a position in A^c, (1) can ensure that a position in H is never reached. Thus if H_γ is the set of initial positions from which (1) can guarantee γ,$H_\gamma \supset A^c$. But $A_\gamma \supset A$ and $H_\gamma \cap A_\gamma = \phi$ and so $H_\gamma = A^c$. Thus H_γ is semi-closed. □

Corollary 3.9. *Suppose that a game is locallty finite and lower topological for $(1) \in N^-$ and that $X_0 = \{x : \Gamma(x) = \phi\}$ is semi-open set. Then the set of positions from which (1) can strictly guarantee a gain γ is semi-open set.*

Proof. Let K_γ denotes the set of initial positions from which (1) can not strictly gurantee γ. By the modification of the argument used in proving Theorem 3.5, we can show that K_γ is semi-closed. But if A_γ is the set of initial positions from which (1) can strictly gurantee γ, $A_\gamma = X - K_\gamma$ and so A_γ is semi-open. □

4 The tactic for topological games

Definition 4.1. *Let $X_0 = \{x : \Gamma(x) = \phi\}$, a tactic for player (i) is a function $\sigma : X_i - X_0 \longrightarrow X$ such that $\sigma(x) \in \Gamma(x)$ for all $x \in X_i - X_0$. A tactic for each player and an initial position determine a play of the game.*

Definition 4.2. *A tactic σ for player (1) is said to guarantee him γ from an initial position x if whenever play begins at x and (1) uses a tactic σ, he obtains a payoff greater than or equal to γ whatever tactics the other players are employed.*

Definition 4.3. *If $x \in X$, let $\psi(x) = \sup\{\gamma : x \in H_\gamma\}$, where H_γ is the set of positions from which (1) can gurantee γ. A tactic for player (1) is called optimal if it gurantees that $\psi(x)$ from the initial position x for all $x \in X$.*

Now, let $\sum$ denotes the set of tactics for player (1). Since each tactic for (1) is a function from $X_1 - X_0$ to X. If $X^{X_1 - X_0}$ is the set of functions from $X_1 - X_0$ to X, then $\sum \subset X^{X_1 - X_0}$. We suppose that $\sum$ have the relativized product topology.

Theorem 4.4. *Suppose we have either*

(a) *a locally finite game upper topological for $(1) \in N^+$ such that X_0 is a semi-open set in X, or*

(b) *an upper topological game for $(1) \in N^-$. If $\Gamma(x)$ is semi-compact for all $x \in X_1 - X_0$,then the set of optimal tactics for (1) is non empty and semi-closed in $\sum$*

Proof. Since each $\Gamma(x)$ is semi-compact for all $x \in X_1 - X_0$ then $\sum = \prod_{x \in X_1 - X_0} \Gamma(x)$ is semi-compact. Let S_γ denotes the set of tactics with which (1) can gurantee γ from any initial position in H_γ, where H_γ is the set of positions from which (1) can gurantee γ. Clearly, $S_\gamma \neq \phi$ if $H_\gamma \neq \phi$. Now, we prove that S_γ is semi-closed in $\sum$. Suppose $\sigma \in \sum$, $\sigma \notin S_\gamma$. Then for some $x \in X_1 \cap H_\gamma$, $\sigma(x) \notin H_\gamma$. By condition (a) and Theorem 3.5 or condition (b) and Corollary 3.8, we get H_γ is semi-closed, so that there is a semi-open nbd N of $\sigma(x)$ in X such that $N \cap H_\gamma = \phi$. If $M(\sigma) = \{\delta : \delta \in \sum\}$ and $\delta(x) \in N$, $M(\sigma)$ is a semi-open nbd of σ and $M(\sigma) \cap S_\gamma = \phi$. Thus S_γ is semi-closed in $\sum$. Let $x_0 \in X$, consider the family $\{S_{\gamma:\gamma<\psi(x_0)}\}$. Suppose $\gamma_1 < \gamma_2 < .... < \gamma_n < \psi(x_0)$. Then $x_0 \in H_{\gamma_i}$ for each i and $H_{\gamma_1} \supset H_{\gamma_2} \supset .... \supset H_{\gamma_n}$. Suppose $H_{\gamma_k} \cap (X_1 - X_0) \neq \phi$ and that $k \leq n$ the largst integer for which this is true. Then for $k < j \leq n$, $\gamma(H_{\gamma_j}) \cap (X_1 - X_0) = \phi$ and $S_{\gamma_j} = \sum$. Let $\sigma_i \in S_{\gamma_i}$ for $1 \leq i \leq k$ and consider the tactic $\sigma \in \sum$, where for $x \in X_1 - X_0$,

$$\sigma(x) = \begin{cases} \sigma_k(x) & : \quad x \in H_{\gamma_k} \\ \sigma_i(x) & : \quad x \in H_{\gamma_i} - H_{\gamma_{i+1}}, i = 1, 2, ...k-1 \\ \sigma_1(x) & : \quad x \in H_{\gamma_2} \end{cases}$$

Then $\sigma \in S_{\gamma-1} \cap S_{\gamma_2} \cap ... \cap S_{\gamma_n}$. Thus for each $x_0 \in X$, $\{S_\gamma : \gamma < \psi(x_0)\}$ is a family of nonempty semi-closed sets with the finite intersection property. Let $S(x) = \bigcap_{\gamma<\psi(x_0)} S_\gamma$. So $S(x)$ is nonempty semi-closed. Now, consider the family $\{S(x) : x \in X\}$. Suppose $x_1, x_2, ..., x_n \in X$. If $\psi(x_m) = \max_{1\leq i\leq n} \psi(x_i)$, $S(x_m) \subset S(x_i)$, Thus $\{S(x) : x \in X\}$ is the family of nonempty semi-closed sets with the finite intersection property. Let $S = \bigcap_{x\in X} S(x)$. Thus S is nonempty and semi-closed in $\sum$. But S is precisely the set of optimal tactics for (1). For if $\sigma \in S(x)$ for all $\gamma < \psi(x)$ and so guarantees for (1) that $\psi(x)$ if the play begins from x. Thus if $\sigma \in \bigcap_{x \in X} S(x)$, σ is an optimal tactic. Conversely, if σ is an optimal tactic for (1), σ guarantees (1) if the play begins from x and so $\sigma \in \bigcap_{\gamma < \psi(x)} S_\gamma$. This is true for all $x \in X$ and so $\sigma \in S = \bigcap_{x \in X} \bigcap_{\gamma < \psi(x)} S_\gamma$. □

References

[1] Andrijevic, D: On SPO-equivalent topologies, Supp. Rend. Circ. Math. Palermo 24, 2 (1990), 251-265.

[2] Berge, C.: Topological games with perfect information, Contribution to the theory of games III (1957), 165-178.

[3] Cao,J.; Moors, W.; and Reilly,I. : Topological properties defines by games and their applications, Topology and its applications, 123(2002),47-55.

[4] Crossley, S. G. and Hildebrand, S. K., Semi-closure, Texas, J. Sci. 22 (1971), 99-112.

[5] Dorsett, C.: Semi-convergence and compactness, Indian J. Math. Mech. 19 (1981), 11-17.

[6] Garcia-Ferreira,S. and Gonzalez-Silva, R.A.: Topological games defined by ultrafilters, Topology and its applications, 137(2004),159-166.

[7] Ewert, J. and Lipski, T.: Quasi-continuous multivalued mapping. Math. Slovaca 33 (1983), 69-74.

[8] Levine, N.: Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly 70 (1963), 36-41.

[9] Martin, K.: Topological games in domian theory, Topology and its applications, 129(2003),177-186.

[10] Pears, A. R.: On topological games, Proc. Cambridge Philos. Soc. 61 (1965), 165-171.

[11] Popa, V. and Noiri T.: Some properties of irresolute multifunctions, Math. Bech. 43 (1991), 11-17.

[12] Telgársky, R.: Spaces defined by topological games, Fund. Math. 88 (1975), 193-223.

[13] Yajima, Y.: Topological games and applications, Topics in general topology (1989), 523-562.