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Convex Cone Generated by Chern Numbers of Complete Intersection Surfaces of General Type

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Abstract—For a given positive integer n, we determine all linear Chern number inequalities satisfied by any complete intersection surface of general type of order n. Similar results are obtained in the general case. As a corollary, we improve the inequalities satisfied by the Chern-invariants of a surface of general type.

Keywords—Convex cone; general type; complete intersection surfaces.

I. INTRODUCTION

This paper is motivated by the results established in [1] in the case of smooth complete intersection threefolds with ample canonical bundle. Let c_1^2, χ be the Chern-invariants of a surface of general type, Bogomolov Miyauka-Yau inequalities [2] say that

$$c_1^2, \chi > 0, \qquad c_1^2 \geq 2\chi - 6, \qquad c_1^2 \leq 9\chi.$$

As these inequalities are, one may naturally asks if there exist more sharpened inequalities. This is possible because ULF Persson [3] listed those complete intersections whose Chern-numbers are fairly small, gave the geography of them and established the following result.

Theorem 1.1: If X is a complete intersection surface of general type, then $c_1^2(X) \leq 8\chi(\mathcal{O}_X)$

However, this inequality can be slightly sharpened, although the coefficient 8 is the best possible. Our plan is, first, to determine for each natural number n, the convex hull in $\mathbb{R}^2$ generated by Chern- invariants (c_1^2, χ) of complete intersection surfaces of general type of order n. Secondly, we deduce the whole convex hull of all complete intersection surfaces (CIS) of general type. A CIS X in P^{n+2} is defined by n hypersurfaces of degrees $d_1, \cdots, d_n$ with $d_i \geq 2$ for each $1 \leq i \leq n$. The Chern-invariants of X are uniquely determined by the tuple $(d_1, \cdots, d_n)$ and we use the notation

$$P(n; d_1, \cdots, d_n) = (c_1^2, \chi)$$

for Chern-invariants of X. We know that X is of general type if and only if $\sum_{i=1}^{n} > n+3$.

Set

$$P_n = \left\{ P(n; d_1, \cdots, d_n) \mid d_i \geq 2, \sum_{i=1}^{n} d_i > n+3 \right\},$$

$$Q_1 = (6,11), Q_2 = \left(11, \frac{151}{24}\right), Q_3 = \left(14, \frac{131}{18}\right)$$

and for $n \geq 4$

$$Q_n = \left((n^2 - 5n + 9)2^n + 1; \frac{(n^2 - 5n + 8)2^{n-3} + 3n + 1}{24n}\right).$$

Let C_n be the convex hull of P_n. We obtain the following theorem.

Theorem 1.2: C_n is an unbounded domain with infinitely many faces defined as follow.

1. C_1 is given by all the points in P_1, which form its upper bound, and the half-line $[P(5)Q_1)$ defined by

   $$y = \frac{1}{6}x + \frac{25}{6},$$

   which is the lower bound.

2. C_2 is delimited by segments of lines defined by points $P(2,k)$ with $k \geq 4$, $P(3,3)$ and the half-line $[P(3,3)Q_2)$ defined by

   $$48y - 7x - 225 = 0.$$

3. C_3 is given by points $P(2,2,k)$ with $k \geq 3$, which form its upper bound, and the half-line $[P(2,2,3)Q_3)$ its lower bound defined by

   $$36y - 5x - 192 = 0.$$

4. For $n \geq 4$, C_n is defined as follows:

a) The upper bound is given by:

i. $P(2,2,2,k), \quad k \geq 2$ if $n = 4$,

ii. $P(2,2,2,2,k), \quad k \geq 2$ if $n = 5$,

iii. the line defined by

$$6y - x + (n^2 - 9n + 12)2^{n-2} = 0,$$

if $n \geq 6$.

b) The lower bound is the half-line $[P(2,2,\cdots,2)Q_n)$ defined by

$$24ny - (3n+1)x - (2n^2 + 3n - 9)2^n = 0.$$

We have now set the stage. Complete intersection surfaces of general type of a fixed degree n, live in a very cut out region of the universe.

With these results at hands, one may naturally ask how looks like the shape of the union of all P_n.

Denote P by:

$$\bigcup_{n=1}^{\infty} P_n = \left\{P(n; d_1, \cdots, d_n) \mid n \geq 1, d_i \geq 2, \sum_{i=1}^{n} d_i > n+3\right\}.$$

One way to look at this question is to search a deduction from the previous particular cases. If it true for its lower bound to be deduced, the deduction of the upper bounds requires more effort as shows the proof of the following theorem.

Theorem 1.3: The convex hull of P is generated by the following points

$$Q_\infty = (13,6);$$

$$P(1;d) = (d^3 - 8d^2 + 16d; \frac{d^3 - 6d^2 + 11d}{6}),$$

where $d \geq 5$.

For two distinct points A and B in $\mathbb{R}^2$, denote the line segment connecting A and B by S_{AB}, and the line passing through A and B by L_{AB}. Denote the slope of L_{AB} by $\alpha(AB)$.

II. Proof of Theorem 1.2

Suppose $n \in \mathbb{R}$ is a positive integer, and $d_1, \ldots, d_n \in \mathbb{R}$ We define:

$$s_1 = d_1 + d_2 + \cdots + d_n,$$
$$s_{1'} = d_1 + d_2 + \cdots + d_n - n - 3,$$
$$s_2 = d_1^2 + d_2^2 + \cdots + d_n^2,$$
$$s_{2'} = d_1^2 + d_2^2 + \cdots + d_n^2 - n - 3,$$
$$p = d_1 d_2 \cdots d_n,$$
$$p_{in} = \prod_{k \neq i,n} d_k,$$
$$A_k^n = P(n; 2, \cdots, 2, k).$$

(1) Let $\in \mathbb{R}$; $d \geq 5$. The slope of $L_{P(d)P(d+1)}$ is given as follows

$$\alpha\big(P(d)P(d+1)\big) = \frac{3d^2 - 9d + 6}{6(3d^2 - 13d + 9)}.$$

Then the family of slopes $\big(\alpha\big(P(d)P(d+1)\big)\big)_{d\geq 5}$ is a sequence with positive terms.

Moreover

$$\alpha\big(P(d+1)P(d+2)\big) - \alpha\big(P(d)P(d+1)\big)$$
$$= \frac{-2d^2 + d + 1}{(3d^2 - 7d - 1)(3d^2 - 13d - 9)} \leq 0$$

and

$$\lim_{d\to+\infty} \alpha\big(P(d)P(d+1)\big)$$
$$= \lim_{d\to+\infty} \frac{3d^2 - 9d + 6}{6(3d^2 - 13d + 9)} = \frac{1}{6},$$

then $\big(\alpha\big(P(d)P(d+1)\big)\big)_{d\geq 5}$ is decreasing and tending to $\frac{1}{6}$. This concludes that all the points $P(d)$, for $d \geq 5$, are above the line of slope $\frac{1}{6}$ passing through $P(5) = (5,5)$, i.e the line whose equation is

$$y = \frac{1}{6}x + \frac{25}{6}.$$

Lemma 2.1 Let $n \geq 2$ and $d_1, d_2 \ldots, d_n \in \mathbb{R}^n$ satisfying the following: $2 \leq d_1 \leq d_2 \leq \cdots \leq d_n$ and $d_1 + d_2 + \cdots + d_n \geq n + 4$.

There exist $k \geq 2$ and a finite number of points $R_1, R_2, \cdots, R_l$ in P_n such that

$$c_1^2(P(2,\cdots,2,k)) < c_1^2(R_1) < c_1^2(R_2) < \cdots < c_1^2(R_l).$$

Moreover

$$\alpha\big(R_l P(d_1, d_2, \cdots, d_n)\big) \leq \frac{1}{6}$$

and

$$\alpha(R_iR_{i+1}) \leq \frac{1}{6}$$

for $i \in \{1, 2, \cdots, l-1\}$.

Proof. Define $A = \{j \in \{1, 2, \cdots, n-1\} \mid d_j \geq 3\}$. Case $A = \varnothing$ is obvious.

Suppose $A \neq \varnothing$. Let $i = minA$. We know that If X is a complete intersection surface of general type of order n, then we have

$$c_1^2(X) = ps_{1'}^2$$

and

$$\chi(X) = \frac{p}{24}\left(s_{2'} + 3s_{1'}^2\right).$$

Consider $T_i = P(d_1, d_2, \cdots, d_{i-1}, d_i - 1, d_{i+1}, \cdots, d_{n-1}, d_n + 1)$ Set (x_1, y_1) and (x_2, y_2) the coordinates of $P(d_1, d_2, \cdots, d_n)$ and T_i respectively. We obtain

$$y_2 - y_1 = \frac{p}{24}\left(s_{2'} + 3s_{1'}^2\right)$$

$$-\frac{p_{in}}{24}(d_id_n + d_i - d_n - 1)(s_{2'} + 3s_{1'}^2 - 2d_i + 2d_n + 2)$$

$$= \frac{p_{in}}{24}\left[-2d_id_n(d_n - d_i + 1) + (d_n - d_i + 1)(s_{2'} + 3s_{1'}^2 - 2d_i + 2d_n + 2)\right]$$

$$= \frac{p_{in}}{24}(d_n - d_i + 1)\left(s_{2'} + 3s_{1'}^2 - 2d_i + 2d_n + 2 - 2d_id_n\right).$$

$$x_2 - x_1 = ps_{1'}^2 - p_{in}(d_id_n + d_i - d_n - 1)s_{1'}^2$$

$$= p_{in}s_{1'}^2(d_n - d_i + 1).$$

We get the following slope $\alpha\left(T_iP(d_1, d_2, \cdots, d_n)\right)$

$$= \frac{s_{2'} + 3s_{1'}^2 - 2d_i + 2d_n + 2 - 2d_id_n}{24s_{1'}^2}.$$

Now, let us prove the inequality. We use induction to do that.

$$\alpha\left(T_iP(d_1, d_2, \cdots, d_n)\right) \leq \frac{1}{6} \text{ iff}$$

$$s_{2'} + 2d_n - 2d_i + 2 \leq s_{1'}^2 + 2d_id_n. \qquad (1)$$

For $d_1 = \cdots = d_{i-1} = 2$, $d_i = d_{i+1} = \cdots = d_n = 3,$ (1) is equivalent to

$$i^2 + (-4n + 9)i + 4n^2 - 16n + 18 \geq 0.$$

The latter inequality is true because the discriminant of its left hand side polynomial in i is $-8n + 9 < 0$.

Now, suppose (1) is true for $2 \leq d_1 \leq d_2 \leq \cdots \leq d_n$ such that $d_1 = \cdots = d_{i-1} = 2$ and that the following holds $3 \leq d_i \leq d_{i+1} \leq \cdots \leq d_n$.

Let $l = min\{l' \in \{i, i+1, \cdots, n\} \mid 3 \leq d_i \leq \cdots \leq d_{l'-1} \leq d_{l'} + 1 \leq d_{l'+1} \leq \cdots \leq d_n\}$. Let us prove that (1) is also verified for the following series of inequalities $2 \leq d_1 \leq d_2 \leq \cdots \leq d_{l-1} \leq d_l + 1 \leq d_{l+1} \leq \cdots \leq d_n$.

If $l = i$ then we have

(1) true

$$\Leftrightarrow s_{2'} + 1 + 2d_n \leq s_{1'}^2 + 2s_{1'} + 1 + 2d_id_n + 2d_n$$

$$\Leftarrow d_l - 1 \leq s_{1'} + d_n; \text{because } d_i \leq d_n$$

$$\Leftarrow -1 \leq s_{1'}$$

The last inequality is true by definition of $s_{1'}$.

If $l = n$ then the inequality (1) is true

$$\Leftrightarrow s_{2'} + 4d_n - 2d_i + 5 \leq s_{1'}^2 + 2s_{1'} + 1 + 2d_id_n + 2d_i$$

$$\Leftarrow d_n + 1 \leq s_{1'} + d_i$$

$$\Leftarrow 1 \leq s_{1'} - d_n + d_i$$

The last inequality is true because $s_1 - d_n + d_i \geq 2(n-2) + 6 = 2n + 2$. If $i \neq l \neq n$ then $n \geq 3$ and we have

$$(1) \quad \Leftrightarrow s_{2'} + 2d_l + 1 + 2d_n - 2d_i + 2 \leq s_{1'}^2 + 2s_{1'} + 1 + 2d_id_n$$

$$\Leftarrow d_l \leq d_1 = s_1 - d_1 - n - 3$$

$$\Leftarrow \sum_{k \neq l} d_k \geq n + 3$$

The last inequality is true because $\sum_{k \neq l} d_k \geq 2(n-1) + 2 = 2n.$

To prove $c_1^2(P_1) < c_1^2(P_2) < \cdots < c_1^2(P_l)$, note that $\alpha\left(P(d_1, d_2, \cdots, d_{i-1}, d_i - 1, d_{i+1}, \cdots, d_{n-1}, d_n + 1)P(d_1, d_2, \cdots, d_n)\right) > 0$

Lemma 2.2 Consider an integer $\in \mathbb{R}$, $n \geq 2$ and $P(d,d,\cdots,d)$ be a point in P_n. Then the sequence of slopes

$$\left(\alpha(P(d,d,\cdots,d)P(d+1,d+1,\cdots,d+1)\right)_d$$

converges to the number

$$\frac{3n+1}{24n}.$$

Proof. From the fact that if X is a complete intersection surface of general type of order n, then $c_1^2(X) = ps_{1'}^2$ and $\chi(X) = \frac{p}{24}\left(s_{2'} + 3s_{1'}^2\right)$, we get $P(d,d,\cdots,d) = \left(d^n(nd-n-3)^2; \frac{d^n}{24}\left(nd^2 - n - 3 + 3(nd-n-3)^2\right)\right)$. And clearly

$$\lim_{d \mapsto +\infty} \left(\alpha(P(d,d,\cdots,d)P(d+1,d+1,\cdots,d+1)\right)_d = \frac{3n+1}{24n}$$

(2) It is easy to see that all points in Q_2 are above the segment line $\left[P(2,4)P(3,3)\right]$. Let $P(d_1,d_2)$ a point in Q_2. Then

$$P(d_1,d_2) = \left((d_1+d_2-5)^2 d_1 d_2; \frac{d_1 d_2}{24}\left(d_1^2 + d_2^2 - 5 + 3(d_1+d_2-5)^2\right)\right).$$

Now $P(d_1,d_2)$ is above $\left[P(3,3)P_2\right)$ if and only if

$$d_1 d_2 \left((d_1-d_2)^2 + 10(d_1+d_2) - 35\right) \geq 225.$$

The left hand side of the latter inequality attains its minimum for $(d_1,d_2) = (2,4)$ and the minimum is 225. By Lemma 2.2 we conclude that $\left[P(3,3)P_2\right)$ is the lower bound. For the upper bounds, from computations we have

$$\alpha\left(P(2,k)P(2,k+1)\right) = \frac{k^2 - 2k + 1}{2(3k^2 - 9k + 4)}$$

and

$$\alpha\left(P(2,k+1)P(2,k+2)\right) - \alpha\left(P(2,k)P(2,k+1)\right) = \frac{-3k^2 - k + 2}{2(3k^2 - 9k + 4)(3k^2 - 3k - 2)}.$$

The sequence $\left(\alpha\left(P(2,k)P(2,k+1)\right)\right)_{k\geq 4}$ is then decreasing. It is easy to see that

$$\lim_{k \mapsto +\infty} \alpha\left(P(2,k)P(2,k+1)\right) = \frac{1}{6}.$$

Lemma 2.1 concludes that points $P(2,k)$ form the upper bound of the convex hull C_2.

(3) Let $P(d_1,d_2,d_3)$ be a point in Q_3. Then

$$P(d_1,d_2,d_3) = \left((d_1+d_2+d_3-6)^2 d_1 d_2 d_3; \frac{d_1 d_2 d_3}{24}\left(d_1^2 + d_2^2 + d_3^2 - 6 + 3(d_1+d_2+d_3-6)^2\right)\right).$$

The point $P(d_1,d_2,d_3)$ is above $\left[A_3^3 P_3\right)$ if and only if

$$d_1 d_2 d_3 \left(\frac{3}{2}(d_1^2 + d_2^2 + d_3^2) - 9 - \frac{1}{2}(d_1+d_2+d_3-6)^2\right) \geq 192 \quad (2)$$

The inequality (2) is true for $(d_1,d_2,d_3) = (2,2,3)$. If $2 \leq d_1 \leq d_2 \leq d_3$ and $d_1 + d_2 + d_3 \geq 8$ then

(2) is true

$$\Leftarrow \frac{3}{2}(d_1^2 + d_2^2 + d_3^2) - 9 - \frac{1}{2}(d_1+d_2+d_3-6)^2 \geq 16$$

$$\Leftarrow 3(d_1^2 + d_2^2 + d_3^2) + 12(d_1+d_2+d_3) \geq (d_1+d_2+d_3)^2 + 86$$

The latter inequality is true because $3(d_1^2 + d_2^2 + d_3^2) \geq (d_1+d_2+d_3)^2$ and also $12(d_1+d_2+d_3) \geq 12 \cdot 8 = 96$. By Lemma 2.2 we conclude that $\left[A_3^3 P_3\right)$ is the lower bound. For the upper bound, we have

$$\alpha\left(A_k^3 A_{k+1}^3\right) = \frac{2k^2 - 2k + 1}{4(3k^2 - 5k + 1)}.$$

Moreover

$$\alpha\left(A_{k+1}^3 A_{k+2}^3)\right) - \alpha\left(A_k^3 A_{k+1}^3)\right) = \frac{-4k^2 - 6k + 2}{4(3k^2 - 5k + 1)(3k^2 + k - 1)}.$$

Then the sequence $\left(\alpha\left(A_k^3 A_{k+1}^3\right)\right)_{k\geq 3}$ is decreasing. We have

$$\lim_{k\mapsto+\infty}\alpha\left(A_k^3A_{k+1}^3\right)=\frac{1}{6}.$$

By lemma 2.1 we conclude that points $P(2,2,k)$ give the upper bound.

(4) (a)

We have

$$A_k^n=\left(\left(k^3+2(n-5)k^2+(n^2-10n+25)k\right)2^{n-1};\ \frac{2^{n-4}k}{3}\left(4k^3+6(n-5)k^2+(3n^2-27n+68)k\right)\right).$$

Computations give

$$\alpha\left(A_k^nA_{k+1}^n\right)=\frac{12k^2+12(n-4)k+3(n^2-7n+14)}{24(3k^2+(4n-17)k+n^2-8n+16)}.$$

(i) If $n=4$ then

$$\alpha\left(A_k^4A_{k+1}^4\right)=\frac{2k^2+1}{4(3k^2-k)}.$$

We get

$$\alpha\left(A_{k+1}^4A_{k+2}^4\right)-\alpha\left(A_k^4A_{k+1}^4\right)=\frac{-2k^2-8k-2}{4(3k^2+5k+2)(3k^2-k)}.$$

Then $\left(\alpha\left(A_k^4A_{k+1}^4\right)\right)_k$ is decreasing and tending to $\frac{1}{6}$. Lemma 2.1 concludes.

(ii) If $n=5$ then

$$\alpha\left(A_k^5A_{k+1}^5\right)=\frac{k^2+k+1}{2(3k^2+3k+1)}.$$

We get $\alpha\left(A_{k+1}^5A_{k+2}^5\right)-\alpha\left(A_k^5A_{k+1}^5\right)=\frac{-4(k+1)}{2(3k^2+9k+7)(3k^2+3k+1)}.$ Then $(\alpha(A_k^5A_{k+1}^5))_k$ is decreasing and tending to $\frac{1}{6}$. By Lemma 2.1, we get the result.

(iii) For $n\geq 6$, we prove that $\alpha\left(A_k^nA_{k+1}^n\right)\leq\frac{1}{6}$. In fact

$$\alpha\left(A_k^nA_{k+1}^n\right)\leq\frac{1}{6}$$

$$\Leftrightarrow\frac{12k^2+12(n-4)k+3(n^2-7n+14)}{24(3k^2+(4n-17)k+n^2-8n+16)}\leq\frac{1}{6}$$

$$\Leftrightarrow 4(n-5)k+n^2-11n+22\geq 0.$$

We check that the latter inequality is true. Moreover, we have the following limit

$$\lim_{k\mapsto+\infty}\alpha(A_k^nA_{k+1}^n)=\frac{1}{6}.$$

Lemma 2.1 permits to say that all points in Q_n are below the line passing through A_2^n whose slope is $\frac{1}{6}$ i.e the line whose equation is

$$6y-x+(n^2-9n+12)2^{n-2}=0.$$

(b) Let $P(d_1,d_2,\cdots,d_n)$ a point in Q_n.

$P(d_1,d_2,\cdots,d_n)$ is above the half line $\left[A_2^nP_n\right)$

$$\Leftrightarrow np\left(s_{2'}+3s_{1'}^2\right)-(3n+1)s_{1'}^2p-(2n^2+3n-9)2^n\geq 0$$

$$\Leftrightarrow p\left(ns_{2'}-s_{1'}^2\right)\geq(2n^2+3n-9)2^n$$

$$\Leftrightarrow ns_2-n(n+3)\geq s_1^2-2(n+3)s_1+3n^2+9n$$

$$\Leftrightarrow 2(n+3)s_1\geq 4n^2+12n$$

$$\Leftrightarrow s_1\geq 2n$$

The latter inequality is true. By Lemma 2.2 we conclude that $\left[A_2^nP_n\right)$ is the lower bound.

3 Proof of Theorem 1.3

From Theorem 1.2, we see that for a fixed integer n, the lower bound of C_n is a line with slope

$$\frac{3n+1}{24n}.$$

Besides, all points in P have their $x-$ coordinate and $y-$ coordinate greater than those of $P(5)=(5,5)$. As

$$\lim_{n\to\infty}\frac{3n+1}{24n}=\frac{1}{8},$$

the lower bound of P is given by the line of slope $\frac{1}{8}$, passing through $P(5)$. That concludes that $L_{P(5)Q_\infty}$ defines the lower bound. Let us prove now the upper bound.

Lemma 3.1. Let $\epsilon\,\mathbb{R}$ $n\geq 2$, $b_1,b_2,\ldots,b_n\,\epsilon\,\mathbb{R}$ such that for all $A\subseteq\{1,2,\cdots,n\}$ with $|A|=n-1,\sum_{i\in A}b_i\geq n+3$. If

$$b_1^2+b_2^2+\cdots+b_n^2-n-3\le(b_1+b_2+\cdots+b_n-n-3)^2$$

then

$$d_1^2+d_2^2+\cdots+d_n^2-n-3\le(d_1+d_2+\cdots+d_n-n-3)^2$$

for all $i\in\{1,2,\cdots,n\}$ and $n\in\mathbb{R}$ such that $d_i\ge b_i$.

Proof. Let $d_1, d_2 \dots, d_n\in\mathbb{R}$ such that $d_i\ge b_i$ for all $i\in\{1,2,\cdots,n\}$ and set $k_i=d_i-b_i$. We have

$$b_1^2+b_2^2+\cdots+b_{i-1}^2+(b_i+k_i)^2+b_{i+1}^2+\cdots+b_n^2-n-3$$
$$=b_1^2+b_2^2+\cdots+b_n^2-n-3+2k_ib_i+k_i^2$$
$$\le(b_1+b_2+\cdots+b_n-n-3)^2+2k_ib_i+k_i^2$$
$$\le(b_1+b_2+\cdots+b_n-n-3)^2+2k_i(b_1+b_2+\cdots$$
$$+b_n-n-3)+k_i^2$$
$$\le(b_1+b_2+\cdots+b_{i-1}+d_i+b_{i+1}+\cdots+b_n-n-3)^2.$$

Applying the same reasoning consecutively n times with distinct i we obtain

$$d_1^2+d_2^2+\cdots+d_n^2-n-3\le(d_1+d_2+\cdots+d_n-n-3)^2.$$

We will now prove that for all $n\in\mathbb{R}$; $n\ge2$, $d\in\mathbb{R}; d\ge5$, for all $d_1, d_2 \dots, d_n\in\mathbb{R}$ such that $2\le d_1\le d_2\le d_3\le\cdots\le d_n$ and $\sum_{i=1}^{n}d_i>n+3$, $P(n;d_1,d_2,\cdots,d_n)\in P$ is below $L_{Q_dQ_{d+1}}$.

A point $P(n;d_1,\cdots,d_n)$ is below the line $L_{Q_dQ_{d+1}}$ if and only if

$$\frac{p}{24}\left[\left(3d^2-13d+9\right)s_{2'}-3\left(d^2+d-1\right)s_{1'}^2\right]-\left(2d^4-6d^3+13d^2-75d+54\right)\le0.$$

We will use Lemma 3.1 to prove the cases $n\ge6$ and Lemma 2.1 to check the case $n=2,3,4,5$. Now, let us set $f_n(d;d_1,\cdots,d_n)$ to be the following function

$$\frac{p}{24}\left[\left(3d^2-13d+9\right)s_{2'}-3\left(d^2+d-1\right)s_{1'}^2\right]$$
$$-\left(2d^4-6d^3+13d^2-75d+54\right).$$

We need to prove that $f_n(d;d_1,\cdots,d_n)$ is negative.

Case $n\ge8$**:** We first notice that for $d\ge5$, we have

$$0\le3d^2-13d+9\le3(d^2+d-1)$$

and

$$2d^4-6d^3+13d^2-75d+54\ge0.$$

Moreover

$$\underbrace{2^2+2^2+\cdots+2^2}_{\text{n times}}-n-3\le(\underbrace{2+2+\cdots+2}_{\text{n times}}-n-3)^2.$$

The conditions of Lemma 3.1 are satisfied for $b_1=b_2=\cdots=b_n=2.$ Thus $s_{2'}\le s_{1'}^2$. Then $f_n(d;d_1,d_2,\cdots,d_n)\le0.$

Case $\mathbf{n}=7$**:** We get the following inequality

$$2^2+2^2+2^2+2^2+2^2+2^2+3^2-7-3=23\le25=(2+2+2+2+2+2+3-7-3)^2$$

and the conditions of Lemma 3.1 are satisfied for

$$b_1=b_2=b_3=b_4=b_5=b_6=2, b_7=3.$$

Then

$$d_1^2+d_2^2+d_3^2+d_4^2+d_5^2+d_6^2+d_7^2-7-3$$
$$\le(d_1+d_2+d_3+d_4+d_5+d_6+d_7-7-3)^2$$ for all $d_1,d_2,\cdots,d_6\ge2, d_7\ge3.$ It is easy to check that $f_7(d;2,\dots,2)=-2d^4+6d^3+179d^2-8949d+6666\le0.$

Case $\mathbf{n}=\mathbf{6}$**:** We have

$$2^2+2^2+2^2+2^2+3^2+3^2-6-3=(2+2+2+2+3+3-6-3)^2$$

$$2^2+2^2+2^2+2^2+3^2+4^2-6-3\le(2+2+2+2+3+4-6-3)^2$$

$$2^2+2^2+2^2+2^2+2^2+5^2-6-3\le(2+2+2+2+2+5-6-3)^2$$

By Lemma 3.1, we come to the conclusion that $d_1^2+d_2^2+d_3^2+d_4^2+d_5^2+d_6^2-6-3\le(d_1+d_2+d_3+d_4+d_5+d_6-6-3)^2$

holds for all $(d_1,d_2,d_3,d_4,d_5,d_6)$ such that

$d_1,d_2,d_3,d_4\ge2$ and $d_5\ge3$

and $d_6\ge3$ or $d_5\ge2$ and $d_6\ge5.$ It is easy to verify that $f_6(d;d_1,d_2,d_3,d_4,d_5,d_6)\le0$ when

$$(d_1,d_2,d_3,d_4,d_5,d_6)\in\{(2,2,2,2,2,2),(2,2,2,2,2,3),(2,2,2,2,2,4)\}.$$

Case $n=5$**:** By theorem 1.2(2), we only need to check $f_5(d;d_1,d_2,d_3,d_4,d_5)\le0$ for

$(d\_1,\ d\_2,\ d\_3,\ d\_4,\ d\_5) = (2,\ 2,\ 2,\ 2,\ k)$ with $k \geq 2$ and $d \geq 5$. It is easy to see that

$$f_5(d;2,2,2,2,k) = -2d^4 + 6d^3 + (96k-13)d^2 + (-64k^3 - 416k + 75)d + 48k^3 + 288k - 54 \leq 0.$$

Founding the same argumentation and seeing that

$$f_4(d;2,2,2,k) = -2d^4 + 6d^3 + (12k^2 + 24k - 13)d^2 + (-32k^3 + 12k^2 - 136k + 75)d + 24k^3 - 12k^2 + 96k - 54 \leq 0,$$

$$f_3(d;2,2,k) = -2d^4 + 6d^3 + (12k^2 - 6k - 13)d^2 + (-16k^3 + 12k^2 - 38k + 75)d + 12k^3 - 12k^2 + 30k - 54 \leq 0$$

And

$$f_2(d;2,k) = -2d^4 + 6d^3 + (9k^2 - 15k - 13)d^2 + (-8k^3 + 9k^2 - 7k + 75)d + 6k^3 - 9k^2 + 9k - 54 \leq 0,$$

we also conclude for the cases $n = 4$, $n = 3$ and $n = 2$.

Corollary 3.1 If X is a complete intersection surface of general type, then

$$c_1^2(X) \geq 5, \quad \chi(X) \geq 5,$$

$$c_1^2(X) \leq 8\chi(X) - 35, \quad \frac{19}{6}\chi(X) - \frac{65}{6} \leq c_1^2(X).$$

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