

# On the 150th Anniversary of Maxwell Equations

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## Abstract

In this lecture we discuss some interesting developments in the modern theory of electromagnetic field(s). In particular, by using the methods developed in Dirac's constraint dynamics we derive the Schrödinger equation for the free electromagnetic field. The electromagnetic field that arises only contains combinations of transverse photons and does not include any scalar and/or longitudinal photons. This approach is also used to determine and investigate the actual symmetry of the free electromagnetic field. Then we discuss the Majorana representation of the Maxwell equations, the symmetric form of Maxwell equations and an original approach to electrodynamics called the "scalar electrodynamics".

## 1 Introduction

In 2012 we have celebrated an important anniversary for the whole modern physics and for the theory of fields. Indeed, one hundred and fifty years ago James Clerk Maxwell presented his famous equations [1] in a closed and relatively simple form. These equations described results of various electric and magnetic experiments known at that time (see, e.g., [2] and references therein). Moreover, it followed directly from those equations that all electric and magnetic phenomena can be explained with the use of the united 'electro-magnetic field'. In the general case, the corresponding vectors ( $\mathbf{E}$ ,  $\mathbf{H}$ ) include six components each of which is a function of the spatial (Cartesian) coordinates  $\mathbf{r}$  and time  $t$ . The propagation of the free electromagnetic fields in four-dimensional space-time continuum was discovered later by Hertz (theoretically) and has become a most crucial part of today's human communications. Step-by-step, Maxwell equations started to be applied to very large spectrum of phenomena many of which were not originally considered as being electric, or magnetic. These equations successfully survived two great events in physics of the last century: (a) the Galilean-Lorentz crisis around 1900 - 1905, and (b) the appearance of Quantum Mechanics around 1925 - 1927. Presently the whole Maxwell theory of electro-magnetic phenomena is considered to be a solid and absolute construction in modern physics. The age of this theory is also outstanding and it is in many dozens times larger than an average 'time-life' of 99.99 % of the 'fundamental' theories being developed today.

In this brief notice I want to show that our understanding of the Maxwell equations is still a very interesting and evolving area of theoretical physics. In particular, by following Dirac we will see how Maxwell equations survived another crucial event in the physics of XX century - a transition to another mechanics which was created by Dirac in 1951 for Hamiltonian systems with constraints and now it is known as the 'constraint dynamics'. We also discuss the dynamical symmetry of the free electromagnetic field(s), the Majorana form of the Maxwell equations and the co-existence of our 'electric' and alternative 'magnetic' worlds. These two worlds can be located at the same spatial place, but cannot interact with each other directly. The only possible communication between our 'real' and alternative 'ghost' worlds comes in the lowest-order approximation in the fine structure constant  $\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$  due to radiation. Finally, we discuss so-called scalar electrodynamics - a new approach to electrodynamics which is based on the using only four scalar functions.

In general, a detailed description of the time-evolution of various physical systems and fields is a fundamental problem which arises in many areas of physics. Explicit derivation of the equations which govern the time-evolution of physical systems and fields is the most interesting part of physics. In Quantum Electrodynamics (QED) the time-evolution of the electromagnetic field (or, EM-field, for short) is governed by the Schrödinger (or Heisenberg) equations for each of the field components. For the free EM-field(s) such an equation is known since the end of 1920's [3] - [5]. Later, analogous equations were obtained for arbitrary electromagnetic fields which interact with electrons and positrons [6], [7]. By solving such equations people answered a significant number of questions which arose in Quantum Electrodynamics. However, the main disadvantage of these (Schrödinger) equations was a presence of indefinite numbers of the scalar and longitudinal photons. In general, the presence of arbitrary numbers of scalar and longitudinal photons transforms all QED-calculations into extremely painful process, which in many cases does not lead to a consistent answer. To avoid complications related to the presence of large (even infinite) numbers of the scalar and longitudinal photons scientists working in this area developed quite a number of clever tricks and procedures. The most recent and widely accepted approach is based on exact compensation of the scalar photons by an equal number of longitudinal photons. All such procedures, however, are not based on

an internal logic of the original QED-theory of the EM-field(s).

In 1950's Dirac developed his famous mechanics [8], [9] of the constrained dynamical systems with Hamiltonians. By applying this mechanics to the free electromagnetic field one finds that it produces the EM-field which is represented as a linear combination of only transverse photons and does not include any scalar and/or longitudinal photons. In other words, such a EM-field can directly be used in QED calculations to determine the probabilities of different processes. Briefly, we can say that Dirac constraint dynamics (or Dirac mechanics) allows one to produce electromagnetic field which is physical, i.e. it does not include any of the ghost components. Therefore, we can investigate the properties of such a field and assume that this field coincides with the actual (or physical) EM-field. In particular, we can determine the actual symmetry of the equations of the free electromagnetic field.

## 2 Hamiltonian

Following Dirac [9] we begin our analysis from the Lagrangian  $L$  of the free electromagnetic field written in the Heaviside-Lorentz units (see, e.g., [10])

$$L = -\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} dx dy dz = -\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} d^3x \quad , \quad (1)$$

where the integration is over three-dimensional space and  $F_{\mu\nu}$  and  $F^{\mu\nu}$  are the covariant and contravariant components of the  $F$ -tensor which is uniformly related with the corresponding derivatives of the field potential  $A_\mu$  (or  $A^\nu$ ) by the relations

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \quad , \quad (2)$$

where the suffix with the comma before it means differentiation according to the following general scheme  $T_{,\mu} = \frac{dT}{dx^\mu}$ , where  $T$  is an arbitrary quantity (or tensor) and  $x = (x^0, x^1, x^2, x^3)$  is the point in the four-dimensional space-time. Note that the suffix '0' with the comma before it designates the temporal derivative (or time derivative), while analogous notations with suffixes 1, 2 and 3 mean the corresponding spatial derivatives.

Following Dirac [9] we need to construct the Hamiltonian of the free electromagnetic field by using the Lagrangian  $L$  from Eq.(1). First, by varying the corresponding velocities, i.e. temporal components of the tensor  $F$ , we introduce the momenta  $B^\mu$

$$\delta L = -\frac{1}{2} \int F^{\mu\nu} \delta F_{\mu\nu} d^3x = \int F^{\mu 0} \delta A_{\mu,0} d^3x = \int B^\mu \delta A_{\mu,0} d^3x \quad , \quad (3)$$

As follows from Eq.(3) the momenta  $B^\mu$  are defined by the equalities  $B^\mu = F^{\mu 0} = -F^{0\mu}$ , which follow from Eq.(3), and antisymmetry of the  $F$ -tensor, i.e. from  $F^{\mu\nu} = -F^{\nu\mu}$  (see, e.g., [6], [9] and [10]). From this definition of the momenta one finds that  $B^0$  equals zero identically, since  $B^0 = F^{00} = -F^{00} = -B^0$ . This is the primary

constraint which is designated in Dirac's constrained dynamics as  $B^0 \approx 0$ . In Quantum Electrodynamics this can be written in the more informative form  $B^0 \Psi = 0$  (or  $B^0 | \Psi \rangle = 0$ ), where  $\Psi$  (also  $| \Psi \rangle$ ) is the wave function of the free electromagnetic field. Briefly, this means that for all states of the free electromagnetic field which are of interest for our purposes below we have  $B^0 \Psi = 0$ , or  $B^0 | \Psi \rangle = 0$ .

Now, we can construct the Hamiltonian of the free electromagnetic field, or EM-field, for short. It should be mentioned here that any Hamiltonian determines a symplectic structure with the dimension  $2n + 1$ , where  $2n$  is the number of dynamical variables, i.e.  $n$  coordinates and  $n$  momenta conjugate to these coordinates. For the free electromagnetic field in three-dimensional space we have four generalized coordinates  $A_\mu = (A_0, A_1, A_2, A_3)$  of the field, or four-vector  $(\phi, \mathbf{A})$  of the field potentials in the traditional  $EM$ -notations. The momenta  $B^\mu$  conjugate to these coordinates also form 4-vector  $(B^0, B^1, B^2, B^3)$ . The Poisson brackets between these dynamical variables must be equal to the delta-function, i.e.

$$[B^\mu(x), A_\nu(x')] = -[A_\nu(x), B^\mu(x')] = -g_\nu^\mu \delta^3(x - x') \quad (4)$$

All other Poisson brackets between these dynamical variables, i.e. the  $[B^\mu(x), B^\nu(x')]$  and  $[A_\mu(x), A_\nu(x')]$  brackets, equal zero identically.

By using the Lagrangian, Eq.(1), and explicit formulas for the momenta  $B^\mu = F^{\mu 0} = -F^{0\mu}$  we can obtain the explicit expression for the Hamiltonian  $H$ . The first step here is to write the Hamiltonian in terms of the velocities ( $A_{\mu,0}$  and  $F_{r0}$ ):

$$H = \int B^\mu A_{\mu,0} d^3x - L = \int (F^{r0} A_{r,0} + \frac{1}{4} F^{rs} F_{rs} + \frac{1}{2} F^{r0} F_{r0}) d^3x \quad , \quad (5)$$

where the indexes  $r$  and  $s$  stand for the spatial indexes, i.e.  $r = 1, 2, 3$ , and  $s = 1, 2, 3$ . For the first term in the second equation we can write  $A_{r,0} = F_{0r} - A_{0,r} = -F_{r0} - A_{0,r}$  (this follows from the definition of  $F_{\mu\nu}$ , Eq.(2)). This allows one to transform the Hamiltonian, Eq.(5), to the form

$$H = \int (\frac{1}{4} F^{rs} F_{rs} - \frac{1}{2} F^{r0} F_{r0} + F^{r0} A_{0,r}) d^3x = \int (\frac{1}{4} F^{rs} F_{rs} + \frac{1}{2} B^r B^r - (B^r)_{,r} A_0) d^3x \quad , \quad (6)$$

where we introduce the momenta  $B^r$  and integrated the last term  $(F^{r0} A_{0,r})$  by parts. This is the explicit formula for the Hamiltonian  $H$  of the free electromagnetic field. Let us investigate this Hamiltonian  $H$ , Eq.(6). First, it is easy to see that the Poisson bracket of the momentum  $B^0$  and the Hamiltonian  $H$  (i.e.  $[B^0, H]$ ) equals  $(B^r)_{,r} \delta^3(x - x')$ . In Dirac's constrained dynamics the Poisson brackets between the primary constraints and Hamiltonian determine the secondary constraints. In other words, the secondary constraint for the free electromagnetic field equals  $(B^r)_{,r}$ , i.e. to the sum of spatial derivatives of the corresponding components of the momenta  $B^r$ . In three-dimensional notations this value equals to  $div \mathbf{B}$ .

By determining the Poisson bracket between the secondary constraint  $(B^r)_r$  and the Hamiltonian, Eq.(6), one finds that it equals zero identically. This means that Dirac's procedure is closed, since no (non-zero) tertiary constraints have been found. The final expression for the total Hamiltonian  $H_T$  of the electromagnetic field is

$$H_T = H + \int v(x_1, x_2, x_3) B^0 d^3x = \int \left( \frac{1}{4} F^{rs} F_{rs} + \frac{1}{2} B^r B^r - A_0 (B^r)_r + v B^0 \right) d^3x \quad (7)$$

where  $v = v(x_1, x_2, x_3)$  is an arbitrary coefficient defined in each point of three-dimensional space. This Hamiltonian is a 'classical' expression. Our next goal is to quantize and obtain the quantum Hamiltonian operator which corresponds to the classical Hamiltonian, Eq.(7). This problem is considered in the next Section.

### 3 Quantization

The total Hamiltonian  $H_T$ , Eq.(7), derived above allows one to perform the quantization of the free electromagnetic field and derive the Schrödinger equation which describes time-evolution of the EM-field. The general process of quantization for various classical systems with Hamiltonians is described in detail in various textbooks (see, e.g., [11], [12] and [13]). Briefly, such a process of quantization can be represented as a following two-step procedure. The first step is the replacement of the classical fields by the corresponding quantum operators. The classical Poisson bracket, Eq.(4), is replaced by the quantum Poisson bracket in which all classical momenta are replaced by the differential operators. The quantum Poisson bracket for two operators of the electromagnetic field must include the reduced Plank constant  $\hbar = \frac{h}{2\pi}$  and, may be, the speed of light in vacuum  $c$ . The presence of the speed of light in the expressions for Poisson brackets depends upon the explicit form of the field operators and the units used. For the operators  $B^\mu(x)$  and  $A_\nu(x')$  defined above the transformation from the classical to the quantum Poisson bracket is written in the form

$$[B^\mu(x), A_\nu(x')]_C = -g_\nu^\mu \delta^3(x - x') \rightarrow [B^\mu(x), A_\nu(x')]_Q = \hbar (-g_\nu^\mu) \delta^3(x - x') \quad (8)$$

where  $B^\mu(x)$  and  $A_\nu(x')$  are the operators and  $B^\mu(x)$  is the differential operator in the  $A_\nu(x)$ -representation (or coordinate representation). Other notations in Eq.(8) have the same meaning as in Eq.(4). The second step of the quantization process is the explicit introduction of the wave function  $\Psi$  which depends upon time  $t$  and all coordinates of the dynamical system, i.e. upon the  $A_\mu = (A_0, A_1, A_2, A_3)$  components of the electromagnetic field, i.e.  $\Psi = \Psi(A_0, A_1, A_2, A_3)$ . The Hamiltonian and other 'observable' quantities must now be considered as operators which act (or operate) on such wave functions. At this point we have to introduce the system of traditional notations for different components of the electromagnetic field and their derivatives. The four-vector potential of the electromagnetic field is represented as the unique combination

of its scalar component  $A_0$ , which is usually designated as  $\phi$ , and three remaining components, which form a three-dimensional vector  $\mathbf{A} = (A_1, A_2, A_3) = (A_x, A_y, A_z)$  (see, e.g., [6] and references therein). The wave function of the electromagnetic field  $\Psi$  is a function of the scalar  $\phi$  and vector  $\mathbf{A}$ , i.e.  $\Psi = \Psi(\phi, \mathbf{A})$ .

As follows from the definition of momenta of the free electromagnetic field ( $B^\mu = F^{\mu 0} = -F^{0\mu}$ ) such momenta essentially coincide with the corresponding components of the electric field  $\mathbf{E}$ , i.e.  $B^\mu = -F^{0\mu} = E^\mu = -E_\mu$ . On the other hand, as follows from Eq.(8) the same momenta can be considered as differential operators in the  $A_\nu(x)$ -representation, or coordinate representation. In other words, we can also choose the following definition of the momenta  $B^\mu(x) = -\frac{\partial}{\partial A_\nu(x)}$ , or  $B^\mu(x) = -\hbar \frac{\partial}{\partial A_\nu(x)}$  in the case of quantum Poisson brackets. For general Hamiltonian systems such a twofold representation of momenta are acceptable, since transition from one to another does not change the fundamental Poisson brackets and, therefore, does not lead to any noticeable contradiction with the reality and/or with the first principles of the Hamiltonian approach. Now, we can write for the primary constraint

$$-\hbar \frac{\partial}{\partial \phi} | \Psi(\phi, \mathbf{A}) \rangle = 0 \quad (9)$$

This means that the wave function  $| \Psi \rangle$  of the free electromagnetic field cannot depend upon the scalar component (or  $\phi$ -component), i.e.  $| \Psi \rangle = \Psi(A_1, A_2, A_3) = \Psi(A_x, A_y, A_z)$ , where  $A_x, A_y$  and  $A_z$  are the three Cartesian coordinates of the vector  $\mathbf{A}$ .

An arbitrary three-dimensional vector  $\mathbf{A} = (A_x, A_y, A_z)$  can always be represented (see, e.g., [14]) as a linear combination of its longitudinal  $A_{||}$  and two transverse  $A_{\perp}^{(1)}, A_{\perp}^{(2)}$  components, i.e.  $\mathbf{A} = (A_x, A_y, A_z) = (A_{||}, A_{\perp}^{(1)}, A_{\perp}^{(2)})$ . By using the standard methods of vector analysis (see, e.g., [14]) it can be shown that the condition  $div \mathbf{A} = 0$  in each spatial point is equivalent to the equality  $A_{||} = 0$  which must be obeyed in each spatial point. Now, the secondary constraint is written in the form

$$-\hbar \frac{\partial}{\partial A_{||}} | \Psi(\mathbf{A}) \rangle = 0 \quad (10)$$

which leads to the conclusion that the vector  $| \Psi(\mathbf{A}) \rangle$  depends upon the two transverse components  $(A_{\perp}^{(1)}, A_{\perp}^{(2)})$  only. In other words, for the free electromagnetic field only those states (or wave functions) are acceptable for which  $| \Psi \rangle = | \Psi(A_{\perp}^{(1)}, A_{\perp}^{(2)}) \rangle$ . Formally, for the free electromagnetic field one can use only such spatial vectors which have only two components (at arbitrary time  $t$ ). Moreover, since  $\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$ , then the vector of electric field  $\mathbf{E}$  also has the two spatial components only. To simplify our notation below, we shall assume that electromagnetic wave always propagates into  $z$ -direction (in each spatial point) and it has two non-zero components ( $x$ - and  $y$ -components). This means that  $| \Psi \rangle = | \Psi(A_x, A_y) \rangle$  and  $A_z = A_{||} = 0$ . This important result will be used below.

The knowledge of the Hamiltonian  $H$  written in the canonical variables of 'momenta'  $\mathbf{E}$  and 'coordinates'  $\mathbf{A}$

of the electromagnetic field allows one to obtain all equation(s) of the time-evolution of the free electromagnetic field. In reality, there is an additional problem here related with the fact that the Hamiltonian contains only special combinations of spatial derivatives of the coordinates, i.e.  $\text{curl} \mathbf{A}$ , rather than coordinates  $\mathbf{A} = (A_x, A_y, A_z)$  themselves. This problem is solved by considering the spatial Fourier transform of the ‘coordinates’, or components of the vector  $\mathbf{A}$ . To simplify analysis even further we represent the original Fourier transform in a ‘discrete’ form, i.e. as an infinite sum, e.g.,

$$\mathbf{A} = \sum_{\mathbf{k}\alpha} \left( c_{\mathbf{k}\alpha} \mathbf{A}_{\mathbf{k}\alpha} + c_{\mathbf{k}\alpha}^* \mathbf{A}_{\mathbf{k}\alpha}^* \right) = \sum_{\mathbf{k}\alpha} \sqrt{\frac{c^2}{2\omega}} \left[ \mathbf{e}^{(\alpha)} \exp(i\mathbf{k} \cdot \mathbf{r}) c_{\mathbf{k}\alpha} + \mathbf{e}^{(\alpha)*} \exp(-i\mathbf{k} \cdot \mathbf{r}) c_{\mathbf{k}\alpha}^* \right] \quad (11)$$

where  $\mathbf{A}_{\mathbf{k}\alpha} = \mathbf{e}^{(\alpha)} \exp(i\mathbf{k} \cdot \mathbf{r}) \sqrt{\frac{c^2}{2\omega}}$  are the normalized plane waves (in the Heaviside-Lorentz units),  $\omega = c |\mathbf{k}|$  and  $\mathbf{e}^{(\alpha)} \cdot \mathbf{e}^{(\beta)*} = \delta_{\alpha\beta}$ . Analogous plane-wave expansions for the electric  $\mathbf{E}$  and magnetic  $\mathbf{H}$  fields are

$$\mathbf{E} = \sum_{\mathbf{k}\alpha} \left( c_{\mathbf{k}\alpha} \mathbf{E}_{\mathbf{k}\alpha} + c_{\mathbf{k}\alpha}^* \mathbf{E}_{\mathbf{k}\alpha}^* \right), \quad \mathbf{H} = \sum_{\mathbf{k}\alpha} \left( d_{\mathbf{k}\alpha} \mathbf{H}_{\mathbf{k}\alpha} + d_{\mathbf{k}\alpha}^* \mathbf{H}_{\mathbf{k}\alpha}^* \right) \quad (12)$$

where  $\mathbf{E}_{\mathbf{k}\alpha} = i\omega \mathbf{A}_{\mathbf{k}\alpha}$  and  $\mathbf{H}_{\mathbf{k}\alpha} = i\omega (\mathbf{n} \times \mathbf{A}_{\mathbf{k}\alpha})$ , The amplitudes  $c_{\mathbf{k}\alpha}, d_{\mathbf{k}\alpha}$  and their complex conjugate  $c_{\mathbf{k}\alpha}^*, d_{\mathbf{k}\alpha}^*$  in such expansions are now considered as a canonical (Hamiltonian) variables. Sometimes it is more convenient to introduce the new canonical variables which are the linear combinations of the  $c_{\mathbf{k}\alpha}$  and  $c_{\mathbf{k}\alpha}^*$  amplitudes and  $d_{\mathbf{k}\alpha}$  and  $d_{\mathbf{k}\alpha}^*$ . The only non-trivial Poisson bracket is  $[c_{\mathbf{k}\alpha}, c_{\mathbf{k}\alpha}^*] = 1$  and  $[d_{\mathbf{k}\alpha}, d_{\mathbf{k}\alpha}^*] = 1$  (for classical amplitudes), or  $[c_{\mathbf{k}\alpha}, c_{\mathbf{k}\alpha}^\dagger] = \hbar$  and  $[d_{\mathbf{k}\alpha}, d_{\mathbf{k}\alpha}^\dagger] = \hbar$  (in the case of quantum amplitudes when  $c_{\mathbf{k}\alpha}^* \rightarrow c_{\mathbf{k}\alpha}^\dagger$  and analogously for  $d$ -amplitudes). All other Poisson brackets equal zero identically. Note that the both Hamiltonian and Poisson brackets are the quadratic expressions in the Fourier amplitudes of the free electromagnetic field. Therefore, it is possible to re-define these amplitudes in the quantum case (by multiplying them by a factor  $\frac{1}{\sqrt{\hbar}}$ ). After such a re-definition the Poisson brackets between quantum and classical amplitudes look identically, but the normalized plane waves take an additional factor  $\sqrt{\hbar}$ , i.e. we must write now:  $\mathbf{A}_{\mathbf{k}\alpha} = \mathbf{e}^{(\alpha)} \exp(i\mathbf{k} \cdot \mathbf{r}) \sqrt{\frac{\hbar c^2}{2\omega}}$ . Such a representation has a number of advantages in applications, since in this case the operators  $c_{\mathbf{k}\alpha}^\dagger$  and  $c_{\mathbf{k}\alpha}$  are dimensionless, i.e. they act on the number of photons (with the two possible polarizations) in the field (or photon) wave function. In respect with this, the whole procedure of quantization of the amplitudes of the Fourier expansion is called the second quantization.

Finally, the Hamiltonian of the free electromagnetic field (in this case  $c_{\mathbf{k}\alpha} = d_{\mathbf{k}\alpha}$ ) is reduced to the infinite

sum of Hamiltonians of independent harmonic oscillators

$$H = \sum_{\mathbf{k}\alpha} \frac{1}{2} \hbar \omega \left( c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}\alpha} + c_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha}^\dagger \right) \quad (13)$$

where for each spatial vector  $\mathbf{k}$  one finds two independent harmonic oscillators (for  $\alpha = +1$  and  $\beta = -1$ , or  $\alpha = 1$  and  $\beta = 2$ ). Note that the operators  $c_{\mathbf{k}\alpha}^\dagger$  and  $c_{\mathbf{k}\alpha}$  in the last equation are dimensionless, i.e. they act on the total number of photons only. All such transformations are described in [15] and here we do not want to repeat them. Note only that the Hamiltonian approach for the free electromagnetic field leads to the well known Planck formula for the thermal energy distribution of electromagnetic radiation. This indicates the correctness of our Hamiltonian  $H$ , Eq.(13), derived above. This Hamiltonian is used to solve a large number of actual problems, e.g., we apply this Hamiltonian below to determine the dynamical symmetry of the free electromagnetic field.

## 4 On the dynamical symmetry of the free electromagnetic field

The Hamiltonian of the free electromagnetic field, Eq.(13), is reduced to the form

$$H = \sum_{\mathbf{k}\alpha} \hbar \omega \left( c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}\alpha} + \frac{1}{2} \right) = \sum_{\mathbf{k}} \hbar \omega \left( a_1^\dagger(\mathbf{k}) a_1(\mathbf{k}) + a_2^\dagger(\mathbf{k}) a_2(\mathbf{k}) + 1 \right) \quad (14)$$

where  $a_1(\mathbf{k}) = c_{\mathbf{k}\alpha} = a_1$  and  $a_2(\mathbf{k}) = c_{\mathbf{k}\beta} = a_2$ . For any given  $\mathbf{k}$  we can write

$$H_{\mathbf{k}} = \hbar \omega \left( a_1^\dagger a_1 + a_2^\dagger a_2 + 1 \right) \quad (15)$$

An interesting question is to determine the symmetry of the corresponding Schrödinger equation with the Hamiltonian, Eq.(15). To answer this question let us construct the four following operators:  $A_j^i = a_i^\dagger a_j$  which commute with the Hamiltonian, Eq.(15). The operator  $A = \sum_i a_i^\dagger a_i = \frac{1}{\hbar \omega} H_{\mathbf{k}} - 1$  also commute with  $H_{\mathbf{k}}$ . The commutation relations between  $A_j^i$  operators are:

$$[A_j^i, A_l^k] = \delta_j^k A_l^i - \delta_l^i A_j^k \quad (16)$$

These commutation relations coincide with the well known relations between four generators of the  $U(2)$ -group (the group of unitary  $2 \times 2$  matrixes). Now, we introduce three  $B_j^i$  operators defined by the relations  $B_j^i = A_j^i - \frac{1}{2} \delta_j^i A$ . Note that for these operators the condition  $B_1^1 + B_2^2 = 0$  is always obeyed. The three operators  $B_j^i$  are the generators of the  $SU(2)$ -group, i.e. the group of unitary  $2 \times 2$  matrixes and determinants of these matrixes equal unity). Thus, the group of dynamical symmetry of the free electromagnetic field is the three-parameter  $SU(2)$ -group. The physical representations of this group which are only of interest in applications are  $D(p, q) = D(n, 0)$ , where  $p \geq q$  are non-negative integer and  $n = p + q$ . Note that the total number of parameters in this  $SU(2)$ -group coincides with the total number of Stokes parameters.

## 5 Maxwell equations and wave propagation

In all Sections above we ignored an obvious (or internal) relation between Maxwell equations and wave propagation. The nature of this relation directly follows from the Hamiltonian approach mentioned in Section II. Indeed, we can write the following expression for the Hamiltonian  $H$  which acts on the wave function of the free electromagnetic field

$$H | \Psi \rangle = \frac{1}{2} \int \left[ \mathbf{E}^2 + (\text{curl} \mathbf{A})^2 \right] d^3x | \Psi \rangle = \frac{1}{2} \int \left[ \frac{1}{c^2} \left( \frac{\partial \mathbf{A}}{\partial t} \right)^2 + (\text{curl} \mathbf{A})^2 \right] d^3x | \Psi \rangle \quad (17)$$

where it is assumed that  $\phi = 0$  and  $\text{div} \mathbf{A} = 0$  (or  $\text{div} \mathbf{E} = 0$ ) in each spatial point. These two conditions must be written in the form of the following equations for the wave function  $(\Psi)$   $\phi | \Psi \rangle = 0$  and  $\text{div} \mathbf{A} | \Psi \rangle = 0$ . From here one finds

$$\begin{aligned} \langle \Psi | (\delta H) | \Psi \rangle &= \int (\delta \mathbf{A}) \left[ -\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \text{curl}(\text{curl} \mathbf{A}) \right] d^3x | \Psi \rangle \\ &= \langle \Psi | \int (\delta \mathbf{A}) \left[ -\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} + \Delta \mathbf{A} \right] d^3x | \Psi \rangle = 0 \end{aligned} \quad (18)$$

This is the energy functional of the free electromagnetic field  $H(\mathbf{A})$ .

The condition that the variational derivative  $\frac{\delta H}{\delta \mathbf{A}}$  equals zero in each spatial point leads to the following equation (wave equation) for each component of the vector-potential  $\mathbf{A}$ :

$$\begin{aligned} \left( \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta \mathbf{A} \right) | \Psi \rangle &= 0 \quad , \\ \text{or} \quad \left( \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \frac{\partial^2 \mathbf{A}}{\partial x^2} - \frac{\partial^2 \mathbf{A}}{\partial y^2} - \frac{\partial^2 \mathbf{A}}{\partial z^2} \right) | \Psi \rangle &= 0 \end{aligned} \quad (19)$$

This is the Schrödinger equation of the free electromagnetic field. Briefly, we can say that the minimum of the functional  $H(\mathbf{A})$  (also called the energy functional of the free electromagnetic field) uniformly leads to the wave equations for the vector  $\mathbf{A}$  and, therefore, for the vectors of the electric  $\mathbf{E}$  and magnetic  $\mathbf{H}$  fields, respectively. The equation, Eq.(19), can be written in the form of one of the Maxwell equations:

$$\text{curl} \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \quad (20)$$

Another Maxwell equation follows from the definition of momentum of the free electromagnetic field  $\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$ . By calculating  $\text{curl}$  of both sides of this equation one finds

$$\text{curl} \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} \quad (21)$$

where all these equations must be considered on the wave functions  $| \Psi \rangle$  for which the condition  $(\text{div} \mathbf{E}) | \Psi \rangle = 0$  is obeyed in each spatial point. This secondary constraint coincides with another Maxwell equation. The last

(fourth) Maxwell equation directly follows from the definition  $\mathbf{H} = \text{curl} \mathbf{A}$ . It should be mentioned that in modern literature on constraint dynamics the role of constraints is often considered as relatively minor. For electrodynamics of the free electromagnetic field it is not true and the constraint  $(\text{div} \mathbf{E}) | \Psi \rangle = 0$  allows one to determine many important features of the propagating electromagnetic field. This question is discussed in the Appendix. Below, we briefly consider some special form of Maxwell equations known as Majorana representation.

## 6 Majorana representation

Let us discuss another form of Maxwell equations which has a number of advantages in some applications. In this form (obtained first by Majorana) the system of Maxwell equations is represented in the form of Dirac equation(s) for massless particle. To simplify our discussion below we consider the case of the free electromagnetic field. Let us introduce the two new vectors  $\mathbf{F} = \mathbf{E} + i\mathbf{H}$  and  $\mathbf{G} = \mathbf{E} - i\mathbf{H}$ , where  $i$  is the imaginary unit, and the gradient vector  $\mathbf{p} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \nabla$ . In these notations Maxwell equations of the free electromagnetic field(s) are written in the following form

$$\frac{1}{c} \frac{\partial \mathbf{F}}{\partial t} = (\mathbf{s} \cdot \mathbf{p}) \mathbf{F} \quad , \quad \mathbf{p} \cdot \mathbf{F} = 0 \quad (22)$$

$$\frac{1}{c} \frac{\partial \mathbf{G}}{\partial t} = (\mathbf{s} \cdot \mathbf{p}) \mathbf{G} \quad , \quad \mathbf{p} \cdot \mathbf{G} = 0 \quad (23)$$

where the vector-matrix  $\mathbf{s} = (s_x, s_y, s_z) = (s_1, s_2, s_3)$  is the vector with the three following components  $(s_i)_{kl} = -ie_{ikl}$ , where  $e_{ikl}$  is the absolute antisymmetric tensor. Note that the four Maxwell equations are now reduced to the two groups of two equations in each and one group contains only vector  $\mathbf{F}$ , while another group contains only vector  $\mathbf{G}$ . Moreover, each of the equations with the time derivative is similar to the corresponding Dirac equations for the spinor wave function. The total equation of the free electron field is a bi-spinor function, while the total wave function of the free-electromagnetic field is a bi-vector function. It is interesting to note that the  $3 \times 3$  matrixes  $s_x, s_y, s_z$  in Eqs.(22) - (23) play the same role for photons as the Pauli matrixes  $\frac{1}{2}\sigma_x, \frac{1}{2}\sigma_y, \frac{1}{2}\sigma_z$  play for electrons. Therefore, they can be considered as spin matrixes. The commutation rules for the matrixes from these two groups of generators of transformations are similar:  $s_i s_k - s_k s_i = ie_{ikl} s_l$  and  $(\frac{1}{2}\sigma_i)(\frac{1}{2}\sigma_k) - (\frac{1}{2}\sigma_k)(\frac{1}{2}\sigma_i) = ie_{ikl} \frac{1}{2}\sigma_l$ . For electrons the vector with the components  $(\frac{1}{2}\sigma_x, \frac{1}{2}\sigma_y, \frac{1}{2}\sigma_z)$  is the spin vector. Therefore, for the vector  $s_x, s_y, s_z$  is the spin vector of a photon. The Casimir operator of the second order  $C_2$  for this algebra equals 2, i.e.  $C_2 = s(s+1) = 2$  and we can say that the spin  $s$  of a single photon equals unity.

The main advantage of such a form of Maxwell equations is very simple and transparent formulas which describe behavior of the bi-vectors  $(\mathbf{F}, \mathbf{G})$  under Lorentz transformations, i.e. under rotations and velocity shifts.

The explicit formulas take the form

$$\mathbf{F} \rightarrow \left[ 1 + \frac{i}{4\pi} \mathbf{s} \cdot \delta(\vec{\theta}) - \frac{1}{c} \mathbf{s} \cdot \delta \mathbf{v} \right] \mathbf{F} \quad (24)$$

$$\mathbf{G} \rightarrow \left[ 1 + \frac{i}{4\pi} \mathbf{s} \cdot \delta(\vec{\theta}) - \frac{1}{c} \mathbf{s} \cdot \delta \mathbf{v} \right] \mathbf{G} \quad (25)$$

These formulas for the Lorentz transformations of bi-vectors of the free electromagnetic field are very similar to the formulas for the Lorentz transformations derived for the electron wave functions which is a bi-spinor  $(\xi, \eta)$

$$\xi \rightarrow \left[ 1 + \frac{i}{8\pi} \vec{\sigma} \cdot \delta(\vec{\theta}) - \frac{1}{2c} \vec{\sigma} \cdot \delta \mathbf{v} \right] \xi \quad (26)$$

$$\eta \rightarrow \left[ 1 + \frac{i}{8\pi} \vec{\sigma} \cdot \delta(\vec{\theta}) - \frac{1}{2c} \vec{\sigma} \cdot \delta \mathbf{v} \right] \eta \quad (27)$$

where  $\frac{1}{2}\vec{\sigma} = \frac{1}{2}(\sigma_x, \sigma_y, \sigma_z)$  is the electron spin vector. In this case the Casimir operator  $C_2$  equals  $\frac{3}{4}$  and the electron spin equals  $\frac{1}{2}$ . Here we do not want to discuss other properties of the Majorana representation [17] of Maxwell equations. Note only that this representation is very useful in application to some electromagnetic problems. In the next Section, we briefly discuss the so-called ‘symmetric form’ of Maxwell equations.

## 7 On the symmetric form of Maxwell equations

The ideas discussed in this Sections were originally stimulated by Dirac’s research on magnetic monopole [18]. It is a controversial matter which recently attracted a very substantial attention. Originally, my plan was to extend and publish this Section as a separate manuscript, but these days it is really hard to publish a manuscript, if its subject contradicts foundations of something (e.g., classical electrodynamics) known to everybody. On the other hand, our conclusions agree with a number of facts known from everyday life. Finally, I decided to write the text in the form which allows readers to make personal decisions about this subject.

Let us consider the general Maxwell equations for the classical EM-field (see, e.g., [6])

$$\begin{aligned} \text{curl} \mathbf{H} &= \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{j}_e, & \text{curl} \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} \\ \text{div} \mathbf{E} &= 4\pi \rho_e, & \text{div} \mathbf{H} &= 0 \end{aligned} \quad (28)$$

where  $\rho_e$  and  $\mathbf{j}_e$  are the electric charge density distribution and the current of electric charges. Note that in Eq.(28) the  $\rho_e$  is the true scalar and  $\mathbf{j}_e$  is a true vector. The equations, Eqs.(28) are written in the so-called non-symmetric form, since they contain, e.g., the electric current  $\mathbf{j}_e$ , but no analogous magnetic current  $\mathbf{j}_m$ . Their ‘manifestly symmetric’ form of these equations is

$$\begin{aligned} \text{curl} \mathbf{H} &= \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{j}_e, & \text{curl} \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} + \frac{4\pi}{c} \mathbf{j}_m \\ \text{div} \mathbf{E} &= 4\pi \rho_e, & \text{div} \mathbf{H} &= 4\pi \rho_m \end{aligned} \quad (29)$$

where  $\rho_m$  is the magnetic charge density distribution (pseudo-sclar) and  $\mathbf{j}_m$  (pseudo-vector) is the current of

magnetic charges. It is clear that the four quantities ( $\rho_m$  and three components of  $\mathbf{j}_m$ ) form the four-vector (or four-pseudo-vector) which is properly transformed under the Lorentz transformation. In our ‘real’ world we have no free magnetic charges. Not even a single stream (or current) of magnetic charges was ever observed. On the other hand, it can be another world where free magnetic charges and currents of such charges do exist, but the presence of free electric charges is impossible. It can be shown that events in these two worlds proceed absolutely independent, i.e. these events cannot affect each other, since the cross-sections between events which proceed in these two worlds are always equal zero. In the lowest order approximation upon  $\alpha$  (the fine structure constant) interaction and/or communication between these two worlds can be produced by radiation, i.e. by regular electromagnetic waves.

First, let us find Maxwell equations which govern all electromagnetic phenomena in that ‘alternative’ (or magnetic) world. Assuming the absolute separation of the two worlds we can write from Eqs.(29)

$$\begin{aligned} \text{curl} \mathbf{H} &= \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}, & \text{curl} \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} + \frac{4\pi}{c} \mathbf{j}_m \\ \text{div} \mathbf{E} &= 0, & \text{div} \mathbf{H} &= 4\pi \rho_m \end{aligned} \quad (30)$$

In other words, the electric field vector is now solenoidal (i.e.  $\text{curl} \mathbf{E} = 0$ ), while the magnetic field has sources. Another interesting observation follows from Eqs.(30). If pseudo-scalar  $\rho_m$  and pseudo-vector  $\mathbf{j}_m$  equal zero identically, then Eqs.(30) coincide with Maxwell equations known for the free electromagnetic field. The same equations are correct in our ‘real’ space. This means that we can register EM-waves which are coming from that ‘alternative’ world. It works in the opposite way too: they can observe EM-waves which have been emitted in our space. Briefly, this means that two our worlds are complement to each other. Furthermore, these two worlds can be considered as the two separated parts of one United super-world. It is impotant that our (or real) world and ghost world can be located at the same place (a local piece of ‘our’ three-dimensional space) and the ‘door’ between these two worlds is the reflection in some ‘actual’ mirror. Here by an ‘actual’ mirror I mean a mirror which: (1) reflects all objects as a regular mirror, and (2) transforms all scalar, vectors and tensors into pseudo-scalars, pseudo-vectors and pseudo-tensors, respectively. An opposite transformation of ‘pseudo-’values into the ‘actual’ values also takes place during such a reflection. The second point in this definition is crucial, since currently the word ‘reflection’ in physics is overloaded with different meanings.

As mentioned above the electric and magnetic worlds which are complement to each other. In general, these two worlds can be considered as the two independent components of one united super-space of events, or Super-World. This is of great interest for a large number of applications. For instance, for theology this means that the life and death are the two complementary forms of one super-life (or super-existence) and transformation between these two form is the reflection in the ‘actual’ mirror as defined above. Remarkably, that all facts known from the ‘old’ re-

ligous points of view, e.g., ‘spiritual transitions from life to death and vice versa, i.e. from death to life’, ‘spirit risen from the death’, etc are supported by Maxwell equations in their symmetric form and can formally be described by these equations. The question about observation and registration of radiation which comes from the ‘magnetic’ world (or ‘ghost’ world) is very interesting. However, here one finds two questions which must be answered before such observations will be possible. First, right now we know almost nothing about frequencies and amplitudes of radiation which arrives into our world from its ‘magnetic’ counterpart. Very likely, it has relatively low frequencies and small amplitudes. Second, we do not know the exact moment when any pulse of radiation will be emitted in the magnetic world. Therefore, it is hard to predict the moment of registration and the corresponding frequencies. Probably, a few ‘gifted’ people can see such radiation and respond to it, but any systematical, experimental study of radiation arriving from the magnetic world into our electric world is an extremely complex process. Despite a large number of unanswered questions at this moment we can only predict that old-fashioned Maxwell equations (without any modification) are the appropriate, accurate and sufficient ‘tool’ to provide communications between our real world and the ‘ghost’ world.

## 8 On scalar electrodynamics

As is well known the equations of electromagnetic field(s), or Maxwell equations, contain one polar vector  $\mathbf{E}$  and one axial vector  $\mathbf{H}$ . All components of these two vectors are unknown functions of the spatial coordinates  $\mathbf{r}$  and time  $t$ . From here one can conclude that an arbitrary electromagnetic field always has six independent (unknown) components which are the scalar components of these two vectors. However, this conclusion is not correct, since by using formulas known for the Lorentz transformations between two different inertial systems we can reduce the total number of independent components to four. It is clear that a possibility of such a reduction is closely related to the well known fact that there are two independent field invariants  $\mathbf{E}^2 - \mathbf{H}^2$  (scalar, which up to the constant is the Lagrangian (or Lagrangian density) of the electromagnetic field) and  $\mathbf{E} \cdot \mathbf{H}$  (pseudoscalar). This allows us to introduce the four-vector of field potentials  $(\Phi, \mathbf{A})$  and re-write all Maxwell equations in terms of  $\phi$  and  $\mathbf{A}$ . In the general form these equations are (in regular units):

$$\frac{4\pi}{c} \rho \mathbf{v} = \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} + \text{grad} \left( \text{div} \mathbf{A} + \frac{1}{c} \frac{\partial \Phi}{\partial t} \right) \quad (31)$$

$$4\pi \rho = -\nabla^2 \Phi - \frac{1}{c} \frac{\partial}{\partial t} (\text{div} \mathbf{A}) \quad (32)$$

where the second equation can be re-written to the form

$$4\pi \rho = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi - \frac{1}{c} \frac{\partial}{\partial t} \left( \text{div} \mathbf{A} + \frac{1}{c} \frac{\partial \Phi}{\partial t} \right) \quad (33)$$

The vector  $\mathbf{A}$  is defined by the differential equation  $\text{curl} \mathbf{A} = \mathbf{H}$ . It follows from here that the vector  $\mathbf{A}$  is defined up to a gradient of some scalar function, i.e. our

equations must be invariant during the transformation:  $\mathbf{A}' \rightarrow \mathbf{A} + \nabla \Psi$ . The choice of this function ( $\Psi$ ) can be used to simplify the both equations, Eqs.(31) and (33). This is very well known gauge invariance (or gauge freedom) of the Maxwell equations. It is well described in numerous books on classical electrodynamics (see, e.g., [6] and [10]). A freedom to choose different gauges is often used to solve actual problems in electrodynamics. Here we do not want to discuss it. Instead let us consider a slightly different approach which can be very effective for many complex problems in electrodynamics. This approach is called the ‘scalar electrodynamics’.

By analyzing equations Eqs.(31) - (33) one finds that to solve these equations we need to determine the four scalar functions, e.g., the scalar potential  $\Phi$  and three components of the vector-potential  $\mathbf{A} = (A_x, A_y, A_z)$ , or their linear combinations. This approach is absolutely equivalent to the use of one four-vector  $(\Phi, \mathbf{A})$ , but the use of non-covariant notations instead of one four-vector does not lead to any simplification in the general case. However, there is another approach which is based on the following theorem from Vector Calculus [14]. An arbitrary vector  $\mathbf{a}$  is uniformly represented in the form

$$\mathbf{a} = \phi \cdot \text{grad} \psi + \text{grad} \chi = \phi \nabla \psi + \nabla \chi \quad (34)$$

where  $\phi, \psi, \chi$  are the three scalar functions which depend upon three spatial coordinates  $\mathbf{r}$  and time  $t$ . The proof of this theorem is relatively simple (see, e.g., [14]) and it leads to the following identity:  $\text{curl} \mathbf{a} = \text{grad} \phi \times \text{grad} \psi = \nabla \phi \times \nabla \psi$ . The expression for the  $\text{div} \mathbf{a} = \nabla \mathbf{a}$  is slightly more complex:  $\text{div} \mathbf{a} = \text{grad} \phi \cdot \text{grad} \psi + \phi \Delta \psi + \Delta \chi = \nabla \phi \cdot \nabla \psi + \phi \Delta \psi + \Delta \chi$ . If we can choose the functions  $\psi$  and  $\chi$  as the solutions of the Laplace equations, i.e.  $\Delta \psi = 0$  and  $\Delta \chi = 0$  (i.e. these two functions are the harmonic functions), then from the last equation one finds  $\text{div} \mathbf{a} = \text{grad} \phi \cdot \text{grad} \psi = \nabla \phi \cdot \nabla \psi$ . In classical electrodynamics we can always represent the vector-potential  $\mathbf{A}$  in the form of Eq.(34). Then the solution of the incident problem is reduced to the derivation of the corresponding equations for the three scalars  $\phi, \psi, \chi$  in Eq.(34) and scalar-potential  $\Phi$  from the four-vector  $(\Phi, \mathbf{A})$ . An obvious advantage of this method follows from the fact that we can choose three functions  $\phi, \psi, \chi$  step-by-step and by using the known boundary and initial conditions. For many problems it provides crucial simplifications of arising equations and allows one to find the explicit solutions. However, in this approach Maxwell equations become a system of the non-linear equations. For theoretical development of the classical/quantum electrodynamics this approach (based on Eq.(34)) has never been used.

## 9 Conclusion

Thus, we have applied the methods of constraint dynamics developed by Dirac [8], [9] to derive the Hamiltonian of the free electromagnetic field. This Hamiltonian and arising primary and secondary constraints are used to derive the corresponding Schrödinger equation for the free electromagnetic field. One of the advantages of this method is the

absence of any scalar and/or longitudinal photons. Both scalar and longitudinal photons arise in standard QED, since without them this theory cannot be considered as being a closed, relativistic procedure. However, the presence of such photons in the expression for the electric and magnetic fields makes all QED calculations extremely difficult. Furthermore, initially in QED the physical (or internal) reasons for the appearance of scalar and longitudinal photons was not clear. Fermi proposed to exclude all such 'non-physical' photons by using re-definition of the field wave functions [16]. At the same time a number of other ideas and recipes were proposed which lead to complete exclusion of the scalar and/or longitudinal photons from QED calculations. Only after development of the constrained dynamics by Dirac it became clear that the original Fermi's idea is essentially correct. Based on Dirac's methods we have developed a new approach to perform QED calculations which are correct at each step. This approach will be described elsewhere.

We also briefly discuss questions related with the dynamical symmetry of the free electromagnetic field, Majorana form of the Maxwell equations, communications between our (electric) and ghost (or magnetic) worlds and scalar approach to electrodynamics which is based on the use of four scalar functions only. It should be mentioned that Maxwell theory of electromagnetic equations is one of the oldest (150 years old!) physical theories which is still in active use in many areas of modern science and technology. Nevertheless we are still far away from that moment when we can say that we know everything about Maxwell equations and predictions which follow from these equations at different experimental conditions. This indicates clearly that Maxwell theory of electromagnetic field(s) is healthy and it is still a subject of intense theoretical and experimental development.

## Appendix.

Here we discuss the role of the secondary constraint  $div\mathbf{E} = 0$  in Dirac's electrodynamics. Recently, in many books and textbooks it became a tradition to consider all primary and secondary constraints for the free electromagnetic field as some secondary conditions which play a non-significant role (in contrast with the Hamiltonian equations) for the field itself. From my point such a view is absolutely wrong and may lead to serious mistakes, if it applies to other fields. Even for the free electromagnetic field the constraint  $div\mathbf{E} = 0$  allows one to predict many important details of its propagations. Let us discuss this problem here. First, note that the constraint  $div\mathbf{E} | \Psi\rangle = 0$  exactly coincides with one of the field equations (or Maxwell equations). There is no easy way to derive this equation by using the Hamiltonian of the free electromagnetic field. This condition means that no new (non-zero) electric charge can be created during any possible time-evolution of the free electromagnetic field in our three-dimensional space. In addition to this, the condition  $div\mathbf{E} | \Psi\rangle = 0$  substantially determines the actual shape and time-evolution of the free electromagnetic field. Indeed, let us consider the formula for the divergence of the

vector  $\mathbf{E}$  in spherical coordinates  $(r, \theta, \phi)$  is

$$div\mathbf{E} = \frac{1}{r^2} \frac{\partial(r^2 E_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\sin \theta E_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial(\sin \theta E_\phi)}{\partial \phi} = 0 \quad (35)$$

where  $E_r, E_\theta$  and  $E_\phi$  are the spherical components of the  $\mathbf{E}$  vector (the vector of the electric field intensity). Let us discuss possible choices of the spherical components of the vector  $\mathbf{E} = (E_r, E_\theta, E_\phi)$  which will automatically lead to the identity  $div\mathbf{E} = 0$ . To obey the condition, Eq.(35), the radial component  $E_r$  of the vector  $\mathbf{E}$  must be a very special function of  $r$ . The dependence of the  $E_r$  component upon  $r$  is general and it is crucially important for the whole electro-dynamics in three-dimensional space. From the condition  $\frac{\partial(r^2 E_r)}{\partial r} = 0$  one finds that at large  $r$  the electric field intensity  $\mathbf{E}$  decreases as  $r^{-2}$ , i.e.  $E_r \simeq \frac{C}{r^2}$ , where  $C$  is some numerical constant. It can be shown that the same conclusion is true for the magnetic field intensity  $\mathbf{H}$ . Such a radial dependence at large  $r$  is the known general property of the free electromagnetic field which propagates in three-dimensional space. Analogous conclusion about angular dependence of the  $E_\theta$  and  $E_\phi$  components (e.g.  $E_\theta = \frac{F(r, \phi)}{\sin \theta}$  and  $E_\phi = G(r, \theta)$ , where  $F$  and  $G$  are the arbitrary (regular) functions of two arguments) cannot be considered as universal and general.

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