Abstract—Hyper-Wiener index, edge-vertex Szeged index and vertex-edge Szeged index are important topological indices in theoretical chemical. In this paper, we first determine the minimum Hyper-Wiener index of graph with connectivity or edge-connectivity. Then, the edge-vertex Szeged index and vertex-edge Szeged index of fan molecular graph, wheel molecular graph, gear fan molecular graph, gear wheel molecular graph, and their r-corona molecular graphs are presented.

Keywords—Chemical graph theory, organic molecules, Hyper-Wiener index, connectivity, edge-connectivity, edge-vertex Szeged index, vertex-edge Szeged index

I. INTRODUCTION

Hyper-Wiener index, edge-vertex Szeged index and vertex-edge Szeged index are introduced to reflect certain structural features of organic molecules. Several papers contributed to determine the distance-based index of special molecular graphs (See Yan et al., [1] and [2], Gao and Shi [3], Gao and Gao [4] and [5], and Xi and Gao [6] for more detail). Let $P_n$ and $C_n$ be path and cycle with $n$ vertices. The molecular graph $F_n=(V) \cup P_n$ is called a fan molecular graph and the molecular graph $W_n=(V) \cup C_n$ is called a wheel molecular graph. Molecular graph $l(G)$ is called r-crown molecular graph of $G$ which splicing $r$ hang edges for every vertex in $G$. By adding one vertex in every two adjacent vertices of the fan path $P_n$ of fan molecular graph $F_n$, the resulting molecular graph is a subdivision molecular graph called gear fan molecular graph, denote as $\tilde{F}_n$. By adding one vertex in every two adjacent vertices of the wheel cycle $C_n$ of wheel molecular graph $W_n$ the resulting molecular graph is a subdivision molecular graph, called gear wheel molecular graph, denoted as $\tilde{W}_n$.

The Hyper-Wiener index $WW$ is one of the recently distance-based graph invariants. That $WW$ clearly encodes the compactness of a structure and the $WW$ of $G$ is defined as:

$$WW(G) = \frac{1}{2} \left( \sum_{\{u,v\} \subseteq V(G)} d(u,v)^2 + \sum_{\{u,v\} \subseteq V(G)} d(u,v) \right).$$

Let $e=uv$ be an edge of the molecular graph $G$. The number of vertices of $G$ whose distance to the vertex $u$ is smaller than the distance to the vertex $v$ is denoted by $n_u(e)$. Analogously, $n_v(e)$ is the number of vertices of $G$ whose distance to the vertex $v$ is smaller than the distance to the vertex $u$. The number of edges of $G$ whose distance to the vertex $u$ is smaller than the distance to the vertex $v$ is denoted by $m_{uv}(e)$. Analogously, $m_{vu}(e)$ is the number of edges of $G$ whose distance to the vertex $v$ is smaller than the distance to the vertex $u$.

Some conclusions on edge-vertex Szeged index and vertex-edge Szeged index can refer to [7].

In this paper, we first determine the minimum Hyper-Wiener index of graph with connectivity or edge-connectivity, then present the edge-vertex Szeged index and vertex-edge Szeged index of $I_r(F_n)$, $I_r(W_n)$, $I_r(\tilde{F}_n)$ and $I_r(\tilde{W}_n)$.

II. MAIN RESULTS AND PROOFS

Let $G$ be a connected graph on $n$ vertices. It is clear that the Hyper-Wiener index is minimal if and only if $G=K_n$, in which case, $WW(G) = \frac{1}{2} n(n-1)$. In what follows, we investigate when a graph with a given vertex or edge-connectivity has minimum Hyper-Wiener index.

**Theorem 1.** Let $G$ be a $k$-connected, $n$-vertex graph, $1 \leq k \leq n-2$. Then

$$WW(G) \geq \frac{n(n-1)}{2} + 2(n-k-1).$$
Equality holds if and only if $G = K_k \cup (K_1 \cup K_{n-k+1})$.

**Proof.** Let $G_{\text{min}}$ be the graph that among all graphs on $n$ vertices and connectivity $k$ has minimum Hyper-Wiener index. Since the connectivity of $G_{\text{min}}$ is $k$, there is a vertex-cut $X \subseteq V(G_{\text{min}})$, such that $|X| = k$.

Denote the components of $G_{\text{min}} \setminus X$ by $G_1, G_2, \ldots, G_ω$. Then each of the sub-graphs $G_1, G_2, \ldots, G_ω$ must be complete. Otherwise, if one of them would not be complete, then by adding an edge between two nonadjacent vertices in this sub-graph we would arrive at a graph with the same number of vertices and same connectivity, but smaller Hyper-Wiener index, a contradiction.

It must be $ω = 2$. Otherwise, by adding an edge between a vertex from one component and a vertex from another component $G_1, G_2, \ldots, G_ω$, if $ω > 2$, then the resulting graph would still have connectivity $k$, but its Hyper-Wiener index would decrease, a contradiction. Hence, $G_{\text{min}} \setminus X$ has two components $G_1$ and $G_2$. By a similar argument, we conclude that any vertex in $G_1$ and $G_2$ is adjacent to any vertex in $X$.

Denote the number of vertices of $G_1$ by $n_1$ and that of $G_2$ by $n_2$. Then $n_1 + n_2 + k = n$ and by direct calculation we get

$$WW(G_{\text{min}}) = \frac{1}{2} \left( \frac{1}{2} n_1(n_1 - 1) + \frac{1}{2} n_2(n_2 - 1) + \frac{1}{2} k(k - 1) + \frac{1}{2} n_1(n_1 + n_2) + 4n_1n_2 + \frac{1}{2} n_2(n_2 + n_1) + 2n_1n_2 \right) + k \cdot a_1a_2$$

which for fixed $n$ and $k$ is minimum for $n_1 = 1$ or $n_2 = 1$.

This in turn means that $G_{\text{min}} = K_k \cup (K_1 \cup K_{n-k+1})$.

Direct calculation yields

$$WW(G_{\text{min}}) = \frac{n(n-1)}{2} + 2(n-k-1)$$

which completes the proof.

The edge-connectivity version for Theorem 1 is also valid. Here the case $k = n-1$ needs not be considered, since the only $(n-1)$-edge connected graph is $K_1$.

**Theorem 2.** Let $G$ be a $k$-edge connected, $n$-vertex graph, $1 \leq k \leq n-2$. Then

$$WW(G) \geq \frac{n(n-1)}{2} + 2(n-k-1).$$

Equality holds if and only if $G = K_k \cup (K_1 \cup K_{n-k+1})$.

**Proof.** Let now $G_{\text{min}}$ denote the graph that among all graphs with $n$ vertices and edge-connectivity $k$ has minimum Hyper-Wiener index. Let $X$ be an edge-cut of $G_{\text{min}}$ with $|X| = k$. Then $G_{\text{min}} \setminus X$ has two components, $G_1$ and $G_2$. Both $G_1$ and $G_2$ must be complete graphs. Let $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$, $n_1 + n_2 = n$.

Denote the set of the end-vertices of the edges of $X$ in $G_1$ by $S$, and that in $G_2$ by $T$. Let $|V(G_1 - S)| = a_1$ and $|V(G_2 - S)| = a_2$.

There are $\frac{n}{2} E(G_{\text{min}}) \cdot a_1a_2$ pairs of vertices at distance 1, and $a_1a_2$ pairs of vertices at distance of 3. All other vertex pairs, namely

$$\binom{n}{2} - E(G_{\text{min}}) \cdot a_1a_2$$

are at distance 2. Consequently,

$$WW(G_{\text{min}}) = \frac{1}{2} \left( \frac{1}{2} n_1(n_1 - 1) + \frac{1}{2} n_2(n_2 - 1) + \frac{1}{2} k(k - 1) + \frac{1}{2} n_1(n_1 + n_2) + 4n_1n_2 + \frac{1}{2} n_2(n_2 + n_1) + 2n_1n_2 \right) + k \cdot a_1a_2$$

$$+ \binom{n}{2} - E(G_{\text{min}}) \cdot a_1a_2$$

$$= \frac{1}{2} \left( \frac{1}{2} n_1(n_1 - 1) + \frac{1}{2} n_2(n_2 - 1) + \frac{1}{2} k(k - 1) + \frac{1}{2} n_1(n_1 + n_2) + 4n_1n_2 + \frac{1}{2} n_2(n_2 + n_1) + 2n_1n_2 \right) + k \cdot a_1a_2$$

$$= \frac{n(n-1)}{2} + 2(n-k-1)$$

which for fixed $n$ and $k$ is minimum for $n_1 = 1$, $a_1 = 0$ or $n_2 = 1$, $a_2 = 0$. This, as before, implies $G_{\text{min}} = K_k \cup (K_1 \cup K_{n-k+1})$.

Hence,

$$WW(G_{\text{min}}) = \frac{n(n-1)}{2} + 2(n-k-1).$$

**Theorem 3.** $S_{\text{min}}(I_r(F_n)) = r^2(2n^2 + 4n - 6)$.
Proof. Let \( P_n = v_1 v_2 \ldots v_n \) and the \( r \) hanging vertices of \( v_i \) be \( v_1^i, v_2^i, \ldots, v_r^i \) \((1 \leq i \leq n)\). Let \( v \) be a vertex in \( F_n \) beside \( P_n \) and the \( r \) hanging vertices of \( v \) be \( v_1^r, v_2^r, \ldots, v_r^r \). Using the definition of edge-vertex Szeged index, we have

\[
S_{cv}(I_r(F_n)) = \\
\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{r} \left[ (m_{i,j}(v_{i,j})n_i(v_{i,j}) + m_i(v_{i,j})n_j(v_{i,j})) + \sum_{j=1}^{r} (m_{i,j}(v_{i,j})n_i(v_{i,j}) + m_i(v_{i,j})n_j(v_{i,j})) \right] \\
+ \frac{1}{2} \sum_{j=1}^{r} \left[ (m_{i,j}(v_{i,j})n_i(v_{i,j}) + m_i(v_{i,j})n_j(v_{i,j})) \right]
\]

In view of the definition of edge-vertex Szeged index, we infer

\[
S_{cv}(I_r(F_n)) = \\
\frac{1}{2} \sum_{i=1}^{n} \sum_{r=1}^{n} \left( m_i(v_i)n_i(v_i) + m_i(v_i)n_i(v_i) \right) \\
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{r} \left( m_{i,j}(v_{i,j})n_i(v_{i,j}) + m_i(v_{i,j})n_j(v_{i,j}) \right)
\]

\[
 = \frac{1}{2} \left[ (1 \times (r + n(r + 1)) + (2n + r + nr - 2) \times 1) \right]
\]

\[
+ \frac{1}{2} \left[ (2r + 3) \times (2(r + 1) + (2r + 3) \times 2(r + 1)) \right]
\]

\[
+ \frac{n}{2} \left[ (2r + 3) \times (2(r + 1) + (2r + 3) \times 2(r + 1)) \right]
\]

\[
= \frac{1}{2} \left[ (2n^2 + 4n - 6) + r \left( \frac{9}{2} n^2 + 4n - 20 \right) + (2n^2 + \frac{3}{2} n - 12) \right]
\]

\[
= 2n^2 + \frac{3}{2} n - 12
\]

Corollary 1. \( S_{cv}(F_n) = 2n^2 + \frac{3}{2} n - 12 \).

Theorem 4. \( S_{cv}(I_r(F_n)) = r^2 (n^2 + 8n - 3) \)

\[
+ r \left( \frac{3}{2} n^2 + 17n - \frac{27}{2} \right) + (10n - 9)
\]

Proof. Let \( C_n = v_1 v_2 \ldots v_n \) and \( v_{i,1}^{i,1}, v_{i,2}^{i,2}, \ldots, v_{i,r}^{i,r} \) be the \( r \) hanging vertices of \( v_i \) \((1 \leq i \leq n)\). Let \( v \) be a vertex in \( F_n \) beside \( P_n \) and the \( r \) hanging vertices of \( v \) be \( v_1^r, v_2^r, \ldots, v_r^r \). Using the definition of edge-vertex Szeged index, we yield

\[
S_{cv}(I_r(F_n)) = \\
\frac{1}{2} \sum_{i=1}^{n} \sum_{r=1}^{n} \left( m_i(v_i)n_i(v_i) + m_i(v_i)n_i(v_i) \right)
\]

\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{r} \left( m_{i,j}(v_{i,j})n_i(v_{i,j}) + m_i(v_{i,j})n_j(v_{i,j}) \right)
\]

\[
+ \frac{1}{2} \sum_{j=1}^{r} \left( m_{i,j}(v_{i,j})n_i(v_{i,j}) + m_i(v_{i,j})n_j(v_{i,j}) \right)
\]

\[
= \frac{1}{2} \left[ (1 \times (r + n(r + 1)) + (2n + r + nr - 2) \times 1) \right]
\]

\[
+ \frac{1}{2} \left[ (2r + 3) \times (2(r + 1) + (2r + 3) \times 2(r + 1)) \right]
\]

\[
+ \frac{n}{2} \left[ (2r + 3) \times (2(r + 1) + (2r + 3) \times 2(r + 1)) \right]
\]

\[
= \frac{1}{2} \left[ (2n^2 + 4n - 6) + r \left( \frac{9}{2} n^2 + 4n - 20 \right) + (2n^2 + \frac{3}{2} n - 12) \right]
\]

\[
= 2n^2 + \frac{3}{2} n - 12
\]

Corollary 1. \( S_{cv}(F_n) = 2n^2 + \frac{3}{2} n - 12 \).

Theorem 4. \( S_{cv}(I_r(F_n)) = r^2 (n^2 + 8n - 3) \)

\[
+ r \left( \frac{3}{2} n^2 + 17n - \frac{27}{2} \right) + (10n - 9)
\]

Proof. Let \( C_n = v_1 v_2 \ldots v_n \) and \( v_{i,1}^{i,1}, v_{i,2}^{i,2}, \ldots, v_{i,r}^{i,r} \) be the \( r \) hanging vertices of \( v_i \) \((1 \leq i \leq n)\). Let \( v \) be a vertex in \( F_n \) beside \( P_n \) and the \( r \) hanging vertices of \( v \) be \( v_1^r, v_2^r, \ldots, v_r^r \). Using the definition of edge-vertex Szeged index, we yield

\[
S_{cv}(I_r(F_n)) = \\
\frac{1}{2} \sum_{i=1}^{n} \sum_{r=1}^{n} \left( m_i(v_i)n_i(v_i) + m_i(v_i)n_i(v_i) \right)
\]

\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{r} \left( m_{i,j}(v_{i,j})n_i(v_{i,j}) + m_i(v_{i,j})n_j(v_{i,j}) \right)
\]

\[
+ \frac{1}{2} \sum_{j=1}^{r} \left( m_{i,j}(v_{i,j})n_i(v_{i,j}) + m_i(v_{i,j})n_j(v_{i,j}) \right)
\]

\[
= \frac{1}{2} \left[ (1 \times (r + n(r + 1)) + (2n + r + nr - 2) \times 1) \right]
\]

\[
+ \frac{1}{2} \left[ (2r + 3) \times (2(r + 1) + (2r + 3) \times 2(r + 1)) \right]
\]

\[
+ \frac{n}{2} \left[ (2r + 3) \times (2(r + 1) + (2r + 3) \times 2(r + 1)) \right]
\]

\[
= \frac{1}{2} \left[ (2n^2 + 4n - 6) + r \left( \frac{9}{2} n^2 + 4n - 20 \right) + (2n^2 + \frac{3}{2} n - 12) \right]
\]

\[
= 2n^2 + \frac{3}{2} n - 12
\]
Theorem 6. \( \text{Sz}_v(I_{1}((\tilde{W}_n})) = r^2(22n^2 - 16n) + r\left(\frac{85}{2}n^2 - \frac{79}{2}n + 2\right) + \left(\frac{39}{2}n^2 - \frac{47}{2}n - 1\right) \).

Proof. Let \( C_i = v_1v_2\ldots v_{n_i} \) and \( v \) be a vertex in \( W_n \) beside \( C_i \), and \( v_{j,i+1} \) be the adding vertex between \( v_i \) and \( v_{i+1} \). Let \( v^1, v^2, \ldots, v^r \) be the \( r \) hanging vertices of \( v_i \) and \( v^1, v^2, \ldots, v^r \) be the \( r \) hanging vertices of \( v_{j+1} \) (\( 1 \leq j \leq n \)). Let \( v_{n+1} = v_1 \). In view of the definition of edge-vertex Szeged index, we deduce

\[
\text{Sz}_v(I_{1}((\tilde{W}_n})) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{r} (m_{i,j}(v^j_v) n_i(v^j_v) + m_{i,j}(v^j_v) n_i(v^j_v)) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{r} (m_{i,j}(v^j_{j+1}) n_i(v^j_{j+1}) + m_{i,j}(v^j_{j+1}) n_i(v^j_{j+1})) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{r} (m_{i,j}(v^j_{j+1}) n_i(v^j_{j+1}) + m_{i,j}(v^j_{j+1}) n_i(v^j_{j+1}))
\]

\[
= r \left(\frac{85}{2}n^2 - \frac{79}{2}n + 2\right) + \left(\frac{39}{2}n^2 - \frac{47}{2}n - 1\right)
\]

Corollary 3. \( \text{Sz}_v(\tilde{F}_n) = \frac{39}{2}n^2 - \frac{101}{2}n + 30 \).
\[
\frac{n}{2} \left[ (2nr - 2r + 3n - 5) \times 3(r + 1) + (3r + 2) \times (2n - 2)(r + 1) \right] + \\
\frac{n}{2} \left[ (2nr - 2r + 3n - 5) \times 3(r + 1) + (3r + 2) \times (2n - 2)(r + 1) \right] + \\
\frac{nr}{2} \left[ 1 \times ((2n + 1)(r + 1) - 1) + (3n + 2nr + r - 1) \times 1 \right] + \\
r^2 \left( 2n^2 - 16n \right) + r \left( \frac{85}{2} n^2 - \frac{79}{2} n + 2 \right) + \left( \frac{39}{2} n^2 - \frac{47}{2} n - 1 \right) \\
= S_{sz}(\tilde{W}_n) = \frac{39}{2} n^2 - \frac{47}{2} n - 1.
\]

**Corollary 4.**

\[
S_{sz}(I, (F_n)) = \frac{39}{2} n^2 - \frac{47}{2} n - 1.
\]

**Theorem 7.**

\[
S_{sz}(I, (F_n)) = r^3 \left( - \frac{1}{2} n^3 + \frac{3}{2} n^2 + \frac{3}{2} n + \frac{1}{2} \right) + \\
r^2 \left( 2n^2 + 7n - 10 \right) + \left( \frac{5}{2} n^3 - \frac{13}{2} n^2 + \frac{13}{2} n - \frac{55}{2} \right) + \left( n^2 - \frac{9}{2} n^2 + 15n - 19 \right)
\]

**Proof.** Using the definition of vertex-edge Szeged index, we have

\[
S_{sz}(I, (F_n)) = \frac{1}{2} \sum_{i,j=1}^{v} (m_i (v_i v_j) m_j (v_j v_i) + m_i (v_i v_j) n_j (v_j v_i)) + \\
\frac{1}{2} \sum_{i,j=1}^{v} (m_i (v_i v_j) n_j (v_j v_i) + m_i (v_i v_j) n_j (v_j v_i)) + \\
\frac{1}{2} \sum_{i,j=1}^{v} (m_i (v_i v_j) n_j (v_j v_i) + m_i (v_i v_j) n_j (v_j v_i)) + \\
\frac{1}{2} \sum_{i,j=1}^{v} (m_i (v_i v_j) n_j (v_j v_i) + m_i (v_i v_j) n_j (v_j v_i))
\]

\[
r^2 \left[ 1 \times 1 + (2n + r + nr - 2) \times (r + n(r + 1)) \right] + \\
\frac{1}{2} \left[ 2(r + 2) \times (r + 1) + (2n + nr - 2r - 4) \times (n - 1)(r + 1) \right] + \\
(n - 2) \left[ (r + 2) \times (r + 1) + (2n + nr - 2r - 5) \times (n - 2)(r + 1) \right] + \\
\frac{1}{2} \left[ (2r + 3) \times 2(r + 1) + (r + 1) \times (r + 1) \right] + \\
\frac{1}{2} \left[ (2r + 3) \times 2(r + 1) + (2r + 2) \times 2(r + 1) \right] + \\
(n - 5) \left[ (2r + 3) \times 2(r + 1) + (2r + 3) \times 2(r + 1) \right] + \\
\frac{nr}{2} \left[ 1 \times 1 + (2n + r + nr - 2) \times (r + n(r + 1)) \right] + \\
r^3 \left( \frac{1}{2} n^3 + \frac{3}{2} n^2 + \frac{3}{2} n + \frac{1}{2} \right) + r^2 \left( 2n^2 + 7n - 10 \right) + \\
\frac{5}{2} n^3 - \frac{13}{2} n^2 + \frac{13}{2} n - \frac{55}{2} + \left( n^2 - \frac{9}{2} n^2 + 15n - 19 \right)
\]

**Corollary 5.**

\[
S_{sz}(F_n) = n^3 - \frac{9}{2} n^2 + 15n - 19.
\]

**Theorem 8.**

\[
S_{sz}(I, (W_n)) = r^3 \left( - \frac{1}{2} n^3 + \frac{3}{2} n^2 + \frac{3}{2} n + \frac{1}{2} \right) + \\
r^2 \left( 3n^2 + \frac{1}{2} n^2 + \frac{13}{2} n - \frac{55}{2} \right) + r \left( \frac{5}{2} n^3 - 6n^2 + \frac{35}{2} n \right) + \left( n^2 - \frac{9}{2} n^2 + 12n \right)
\]

**Proof.** In view of the definition of vertex-edge Szeged index, we infer

\[
S_{sz}(I, (W_n)) = \frac{1}{2} \sum_{i,j=1}^{v} (m_i (v_i v_j) n_j (v_j v_i) + m_i (v_i v_j) n_j (v_j v_i)) + \\
\frac{1}{2} \sum_{i,j=1}^{v} (m_i (v_i v_j) n_j (v_j v_i) + m_i (v_i v_j) n_j (v_j v_i)) + \\
\frac{1}{2} \sum_{i,j=1}^{v} (m_i (v_i v_j) n_j (v_j v_i) + m_i (v_i v_j) n_j (v_j v_i)) + \\
\frac{1}{2} \sum_{i,j=1}^{v} (m_i (v_i v_j) n_j (v_j v_i) + m_i (v_i v_j) n_j (v_j v_i))
\]

\[
r^2 \left[ 1 \times 1 + (2n + r + nr - 2) \times (r + n(r + 1)) \right] + \\
\frac{1}{2} \left[ 2(r + 2) \times (r + 1) + (2n + nr - 2r - 4) \times (n - 1)(r + 1) \right] + \\
(n - 2) \left[ (r + 2) \times (r + 1) + (2n + nr - 2r - 5) \times (n - 2)(r + 1) \right] + \\
\frac{1}{2} \left[ (2r + 3) \times 2(r + 1) + (r + 1) \times (r + 1) \right] + \\
\frac{1}{2} \left[ (2r + 3) \times 2(r + 1) + (2r + 2) \times 2(r + 1) \right] + \\
(n - 5) \left[ (2r + 3) \times 2(r + 1) + (2r + 3) \times 2(r + 1) \right] + \\
\frac{nr}{2} \left[ 1 \times 1 + (2n + r + nr - 2) \times (r + n(r + 1)) \right] + \\
r^3 \left( \frac{1}{2} n^3 + \frac{3}{2} n^2 + \frac{3}{2} n + \frac{1}{2} \right) + r^2 \left( 2n^2 + 7n - 10 \right) + \\
\frac{5}{2} n^3 - \frac{13}{2} n^2 + \frac{13}{2} n - \frac{55}{2} + \left( n^2 - \frac{9}{2} n^2 + 15n - 19 \right)
\]

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117
$$+ r \left( \frac{5}{2} n^3 - 6n^2 + \frac{35}{2} n \right) + \left( n^3 - \frac{9}{2} n^2 + 12n \right)$$

**Corollary 6.**
$$S_{\text{v-e}}(W_n) = n^3 - \frac{9}{2} n^2 + 12n$$

**Theorem 9.**
$$S_{\text{v-e}}(I, (\tilde{F}_n)) = r^3 (4n^3) + r^2 (16n^3 - 30n^2 + 43n - 28)$$
$$+ r(2n^3 - \frac{143}{2} n^2 + 117n - \frac{139}{2}) + (9n^3 - \frac{81}{2} n^2 + \frac{141}{2} n - 42)$$

**Proof.** By virtue of the definition of vertex-edge Szeged index, we yield
$$S_{\text{v-e}}(I, (\tilde{F}_n)) =$$
$$\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{r} (m_{i,j}(v_{i,j}')) n_{i,j}(v_{i,j}') + m_i(v_{i,j})n_i(v_{i,j})$$
$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{r} (m_{i,j}(v_{i,j}'')) n_{i,j}(v_{i,j}'') + m_i(v_{i,j})n_i(v_{i,j})$$
$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{r} (m_{i,j}(v_{i,j+1})) n_{i,j+1}(v_{i,j+1}) + m_i(v_{i,j+1})n_i(v_{i,j+1})$$
$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{r} (m_{i,j}(v_{i,j+1}'')) n_{i,j+1}(v_{i,j+1}'') + m_i(v_{i,j+1})n_i(v_{i,j+1})$$
$$r^2 \left[ 1 \times 1 + (3n + 2nr - 3) \times (r + (r + 1)(2n - 1)) \right]$$
$$+ \frac{1}{2} (2(2r + 1) \times (2r + 1) + (2nr + 3n - 2r - 5) \times (2n - 2)(r + 1))$$
$$(n - 2)(3r + 2) \times (r + 1) + (2nr + 3n - 3r - 7) \times (2n - 3)(r + 1))$$
$$n^2 r \left[ 1 \times 1 + (3n + 2nr - 3) \times (2n(r + 1) - 1) \right]$$
$$+ \frac{n - 1}{2} \left[ (2nr - 3r + 3n - 7) \times (2n - 3)(r + 1) + (3r + 2) \times (r + 1) \right]$$
$$n - 1 \left[ (2nr - 3r + 3n - 7) \times (2n - 3)(r + 1) + (3r + 2) \times (r + 1) \right]$$

$$= \frac{n - 1}{2} \left[ (2nr - 3r + 3n - 7) \times (2n - 3)(r + 1) + (3r + 2) \times (r + 1) \right]$$

$$= \frac{n - 1}{2} \left[ (3n + 2nr - 3) \times (2n(r + 1) - 1) + 1 \times 1 \right]$$

$$= r^3 (4n^3) + r^2 (16n^3 - 30n^2 + 43n - 28)$$
$$+ r(2n^3 - \frac{143}{2} n^2 + 117n - \frac{139}{2}) + (9n^3 - \frac{81}{2} n^2 + \frac{141}{2} n - 42)$$

**Corollary 7.**
$$S_{\text{v-e}}(W_n) = 9n^3 - \frac{81}{2} n^2 + \frac{141}{2} n - 42$$

**Theorem 10.**
$$S_{\text{v-e}}(I, (\tilde{W}_n)) = r^3 (n^3 + \frac{5}{2} n^2 + 2n + \frac{1}{2})$$
$$+ r^2 \left( \frac{19}{2} n^3 - \frac{17}{2} n^2 - 19n - \frac{1}{2} \right) + (r(18n^3 - 34n^2 + \frac{85}{2} n))$$
$$+ (9n^3 - 24n^2 + 24n)$$

**Proof.** In view of the definition of vertex-edge Szeged index, we deduce
$$S_{\text{v-e}}(I, (\tilde{W}_n)) =$$
$$\frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{n} (m_{i,j}(v_{i,j}')) n_{i,j}(v_{i,j}') + m_i(v_{i,j})n_i(v_{i,j})$$
$$+ \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{n} (m_{i,j}(v_{i,j}'')) n_{i,j}(v_{i,j}'') + m_i(v_{i,j})n_i(v_{i,j})$$
$$+ \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{n} (m_{i,j}(v_{i,j+1}')) n_{i,j+1}(v_{i,j+1}') + m_i(v_{i,j+1})n_i(v_{i,j+1})$$
$$+ \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{n} (m_{i,j}(v_{i,j+1}''')) n_{i,j+1}(v_{i,j+1}''') + m_i(v_{i,j+1})n_i(v_{i,j+1})$$
$$r^2 \left[ 1 \times 1 + (3n + 2nr - 3) \times (r + (r + 1)(2n - 1)) \right]$$
$$+ \frac{1}{2} (2(2r + 1) \times (2r + 1) + (2nr + 3n - 2r - 5) \times (2n - 2)(r + 1))$$
$$(n - 2)(3r + 2) \times (r + 1) + (2nr + 3n - 3r - 7) \times (2n - 3)(r + 1))$$
$$n^2 r \left[ 1 \times 1 + (3n + 2nr - 3) \times (2n(r + 1) - 1) \right]$$
$$+ \frac{n - 1}{2} \left[ (2nr - 3r + 3n - 7) \times (2n - 3)(r + 1) + (3r + 2) \times (r + 1) \right]$$

$$= \frac{n - 1}{2} \left[ (2nr - 3r + 3n - 7) \times (2n - 3)(r + 1) + (3r + 2) \times (r + 1) \right]$$

$$= \frac{n - 1}{2} \left[ (3n + 2nr - 3) \times (2n(r + 1) - 1) + 1 \times 1 \right]$$

$$= r^3 (4n^3) + r^2 (16n^3 - 30n^2 + 43n - 28)$$
$$+ r(2n^3 - \frac{143}{2} n^2 + 117n - \frac{139}{2}) + (9n^3 - \frac{81}{2} n^2 + \frac{141}{2} n - 42)$$

$$= r^3 (n^3 + \frac{5}{2} n^2 + 2n + \frac{1}{2})$$

$$+ r^2 \left( \frac{19}{2} n^3 - \frac{17}{2} n^2 - 19n - \frac{1}{2} \right) + (r(18n^3 - 34n^2 + \frac{85}{2} n))$$
$$+ (9n^3 - 24n^2 + 24n).$$
\[
\frac{n}{2} (3r+2) \times (r+1) + (2nr+3n-2r-5) \times (2n-2)(r+1)
\]
\[
+ \frac{nr}{2} [1 \times 1 + (3n+2nr+r-1) \times ((2n+1)(r+1) - 1)]
\]
\[
+ \frac{n}{2} [(2nr-2r+3n-5) \times (2n-2)(r+1) + (3r+2) \times 3(r+1)]
\]
\[
+ n \frac{nr}{2} [1 \times 1 + (3n+2nr+r-1) \times ((2n+1)(r+1) - 1)]
\]

\[
= r^3 (n^3 + \frac{5}{2} n^2 + 2n + \frac{1}{2}) + r^2 (\frac{19}{2} n^3 - \frac{17}{2} n^2 - 19n - \frac{1}{2})
\]
\[
+ r(18n^3 - 34n^2 + \frac{85}{2} n) + (9n^3 - 24n^2 + 24n)
\]

**Corollary 8.** \(S_{ze}(\bar{W}_n) = 9n^3 - 24n^2 + 24n\).

**III. CONCLUSION AND DISCUSSION**

**Corollary 7.** \(S_{ze}(\bar{F}_n) = 9n^3 - \frac{81}{2} n^2 + \frac{141}{2} n - 42\).

**Theorem 10.** \(S_{ze}(I_r(\bar{W}_n)) =
\]
\[
r^3 (n^3 + \frac{5}{2} n^2 + 2n + \frac{1}{2})
\]
\[
+ r(18n^3 - 34n^2 + \frac{85}{2} n) + (9n^3 - 24n^2 + 24n).
\]

**Proof.** In view of the definition of vertex-edge Szeged index, we deduce

\[
S_{ze}(I_r(\bar{W}_n)) = \frac{1}{2} \sum_{i=1}^{n} (m_{i,v} (v,v')) n_{i,v} (v,v') + m_{i,v} (v,v') n_{i,v} (v,v')
\]
\[
+ \frac{1}{2} \sum_{i=1}^{n} (m_{i,v} (v,v')) n_{i,v} (v,v') + m_{i,v} (v,v') n_{i,v} (v,v')
\]
\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (m_{i,v} (v,v')) n_{i,v} (v,v') + m_{i,v} (v,v') n_{i,v} (v,v')
\]

**3. DISCUSSION**

Theorems 1 and 2 establish that the graph with \(n\) vertices, connectivity \(k\), and minimum Hyper-Wiener index is same in the case of vertex and edge-connectivity. One may wonder whether Theorem 1 implies Theorem 2, or vice versa. It appears (at least within the present considerations) that the proofs of these two theorems are independent.

As already mentioned, the 1- and 2-connected graphs with maximum Hyper-Wiener indices are known. The natural question at this point is to ask for \(k\)-connected (\(k \geq 2\)), \(n\)-vertex graphs having maximum Hyper-Wiener index. This problem seems to be much
more difficult, and, at this moment, we cannot offer any solution of it, not even for the case $k=3$.

Another related question is whether $n$-vertex, $k$-vertex connected and $n$-vertex, $k$-edge connected graphs with maximum Hyper-Wiener index differ at all, and if yes, for which values of $k$ and $n$.

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