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Integral Type Common Fixed Point Theorems for Cone Metric space

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***Abstract*— In this paper, we prove some fixed point and common fixed point theorems for cone metric space for Integral type mappings. Our main result is the generalized version of some known results.**

Keywords—Common fixed point, compatible maps, complete cone metric space, self mappings

I. Introduction

Branciari [1] obtained a fixed point result for a single mapping satisfying an analogue of Banach's contraction principle for an integral-type inequality. The authors in [2–6] proved some fixed point theorems involving more general contractive conditions. Also in [7], Suzuki shows that Meir-Keeler contractions of integral type are still Meir-Keeler contractions. In this paper, we establish a fixed point theorem for weakly compatible maps satisfying a general contractive inequality of integral type.

Let X be a Real Banach space and P a subset of X.P is called a cone if P satisfy followings conditions:

i. P is closed, non-empty and P≠0

ii. Ax + By Є P for all x, y Є P and non- negative real numbers a, b.

iii. P ∩ (-P) = { 0 }

Given a cone $P \subset X$, we define a partial ordering ≤ on X with respect to P by y-x Є P.

We shall write x << y if $y - x \in \operatorname{int} P$, denoted by $\|.\|$ the norm on X. the cone P is called normal if there is a number k > 0 such that for all x, y Є X . $0 \le x \le y \Rightarrow \|x\| \le k\|y\|$ (1.1)

The least positive number k satisfying the above condition (1.1) is called the normal constant of P.

The authors showed that there is no normal cone with normal constant M < 1 and for each K > 1. There are cone with normal constant M > k.

The cone P is called regular if every increasing sequence which is bounded from the above is convergent, that is if $\{x_n\}_{n\ge1}$ is a sequence such that $x_1 \le x_2 \le ..... \le y$ for some y Є X,

Then there is x Є X such that $\lim_{n\to\infty}\|x_n - x\| = 0$.

The cone P is regular iff every decreasing sequence which is bounded from below is convergent.

Definition 1.1: Let X a non empty set and X is a real Banach space, d is a mapping from X into itself such that d satisfies following conditions:

i. $\int_0^{d(x,y)} \varphi(t)dt \ge 0, \forall x, y \in X$

ii. $\int_0^{d(x,y)} \varphi(t)dt = 0 \Leftrightarrow x = y$

iii. $\int_0^{d(x,y)} \varphi(t)dt = \int_0^{d(y,x)} \varphi(t)dt$

iv. $\int_0^{d(x,y)} \varphi(t)dt \le \int_0^{d(x,z)} \varphi(t)dt + \int_0^{d(z,y)} \varphi(t)dt$

Then d is called a cone metric on X and (X, d) is called cone metric space.

Definition 1.2: Let A & S be a two mappings of a cone metric space (X, d) then it is said to be compatible if $\lim_{n\to\infty}\int_0^{d(ASx_n,SAx_n)} \varphi(t)dt = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = t$ and $\lim_{n\to\infty} Sx_n = t$ for some $t \in X$ of cone metric space

Let A and S be a two self mappings of (X, d) then it is said to be weakly compatible, if they commute at coincidence point, that is $Ax = Sx$ implies that $ASx = SAx$, for $x \in X$. It is easy to see that compatible mapping commute at their coincidence points. It is note that compatible maps are weakly compatible but converse need not be true. In this Paper we used the mapping of [22].

II MAIN RESULTS

Theorem 2.1: Let (X, d) be a complete cone metric space and P a normal cone with constant K. Suppose that the mapping $T : X \to X$ satisfies the condition

$$\int_0^{d(Tx,Ty)} \varphi(t)dt$$
$$\leq \alpha \int_0^{\frac{d(x,Tx)d(y,Ty)d(x,Ty)+d(x,y)d(y,Tx)d(y,Ty)}{[d(x,y)]^2+d(x,Ty)d(y,Ty)}} \varphi(t)dt$$
$$+\beta \int_0^{d(x,Tx)+d(y,Ty)} \varphi(t)dt$$
$$+\gamma \int_0^{d(x,Ty)+d(y,Tx)} \varphi(t)dt$$
$$+\eta \int_0^{\frac{d(x,Tx)d(y,Ty)}{d(x,y)}} \varphi(t)dt$$
$$+\delta \int_0^{d(x,y)} \varphi(t)dt \qquad 2.1.1$$

for all $x, y \in X$ and $\alpha,\beta,\gamma,\eta,\delta \geq 0$ such that $0 \leq \alpha+2\beta+2\gamma+\eta+\delta<1$, then T has a unique fixed point in X.

Proof: For any x_0 in X, we choose $x_1, x_2 \in X$ such that

$$Tx_0 = x_1 \,\&\, Tx_1 = x_2$$

In general we can define a sequence of elements of X such that $x_{2n+1} = Tx_{2n} \,\&\, x_{2n+2} = Tx_{2n+1}$

Now from (2.1.1)

$$\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt = \int_0^{d(Tx_{2n},Tx_{2n+1})} \varphi(t)dt$$
$$\leq \alpha \int_0^{\frac{d(x_{2n},Tx_{2n})d(x_{2n+1},Tx_{2n+1})d(x_{2n},Tx_{2n+1}) + d(x_{2n},x_{2n+1})d(x_{2n+1},Tx_{2n})d(x_{2n+1},Tx_{2n+1})}{[d(x_{2n},x_{2n+1})]^2+d(x_{2n},Tx_{2n+1})d(x_{2n+1},Tx_{2n+1})}} \varphi(t)dt$$
$$+\beta \int_0^{d(x_{2n},Tx_{2n})+d(x_{2n+1},Tx_{2n+1})} \varphi(t)dt$$
$$+\gamma \int_0^{d(x_{2n},Tx_{2n+1})+d(x_{2n+1},Tx_{2n})} \varphi(t)dt$$
$$+\eta \int_0^{\frac{d(x_{2n},Tx_{2n})d(x_{2n+1},Tx_{2n+1})}{d(x_{2n},x_{2n+1})}} \varphi(t)dt$$
$$+\delta \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt$$

$$\leq \alpha \int_0^{\frac{d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2})d(x_{2n},x_{2n+2}) + d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+1})d(x_{2n+1},x_{2n+2})}{[d(x_{2n},x_{2n+1})]^2+d(x_{2n},x_{2n+2})d(x_{2n+1},x_{2n+2})}} \varphi(t)dt$$
$$+\beta \int_0^{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})} \varphi(t)dt$$
$$+\gamma \int_0^{d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})} \varphi(t)dt$$
$$+\eta \int_0^{\frac{d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2})}{d(x_{2n},x_{2n+1})}} \varphi(t)dt$$
$$+\delta \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt$$

$$\leq \alpha \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt$$
$$+\beta \int_0^{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})} \varphi(t)dt$$
$$+\gamma \int_0^{d(x_{2n},x_{2n+2})} \varphi(t)dt$$
$$+\eta \int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt$$
$$+\delta \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt$$

By using triangle inequality, we get

$$\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt \leq \left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta}\right)\int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt$$

Similarly we can show that

$$\int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt \leq \left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta}\right)\int_0^{d(x_{2n-1},x_{2n})} \varphi(t)dt$$

in general we can write

$$\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt \leq \left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta}\right)^{2n+1}\int_0^{d(x_0,x_1)} \varphi(t)dt$$

on taking $\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta} = h$, we have

$$\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt \leq h^{2n+1}\int_0^{d(x_0,x_1)} \varphi(t)dt$$

for $n \leq m$ we have

$$\int_0^{d(x_{2n},x_{2m})} \varphi(t)dt \leq \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt + \int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt + \ldots + \int_0^{d(x_{2m-1},x_{2m})} \varphi(t)dt$$

$$\leq \left(h^n + h^{n+1} + h^{n+2} + h^{n+3} + \ldots + h^m\right)\int_0^{d(x_0,x_1)} \varphi(t)dt$$

$$\leq \frac{h^n}{1-h}\int_0^{d(x_0,x_1)} \varphi(t)dt$$

$$\left\|\int_0^{d(x_{2n},x_{2m})} \varphi(t)dt\right\| \leq \frac{h^n}{1-h}k\left\|\int_0^{d(x_0,x_1)} \varphi(t)dt\right\| \text{ as } \to \infty ,$$

$$\lim_{n\to\infty}\left\|\int_0^{d(x_{2n},x_{2m})} \varphi(t)dt\right\| \to 0$$

Hence $\lim_{n\to\infty}\left\|\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt\right\|\to 0$ as $n\to\infty$

Hence $\{x_n\}$ is a Cauchy sequence which converges to $u\in X$. Hence (X,d) is a complete cone metric space. Thus $x_n\to u$ as $n\to\infty$ and $Tx_{2n}\to u\ \&\ Tx_{2n+1}\to u$ as $n\to\infty$,u is fixed point of T in X.

Uniqueness: Let us assume that, v is another fixed point of T in X different from u then $Tu=u\ \&\ Tv=v$, from (2.1.1)

$$\int_0^{d(u,v)}\varphi(t)dt=\int_0^{d(Tu,Tu)}\varphi(t)dt$$

$$\le\alpha\int_0^{\frac{d(u,Tu)d(v,Tv)d(u,Tv)+d(u,v)d(v,Tu)d(v,Tv)}{[d(u,v)]^2+d(u,Tv)d(v,Tv)}}\varphi(t)dt$$

$$+\beta\int_0^{d(u,Tu)+d(v,Tv)}\varphi(t)dt$$

$$+\gamma\int_0^{d(u,Tv)+d(v,Tu)}\varphi(t)dt$$

$$+\eta\int_0^{\frac{d(u,Tu)d(v,Tv)}{d(x,y)}}\varphi(t)dt$$

$$+\delta\int_0^{d(u,v)}\varphi(t)dt$$

$$\le\alpha\int_0^{\frac{d(u,u)d(v,v)d(u,v)+d(u,v)d(v,u)d(v,v)}{[d(u,v)]^2+d(u,v)d(v,v)}}\varphi(t)dt$$

$$+\beta\int_0^{d(u,u)+d(v,v)}\varphi(t)dt$$

$$+\gamma\int_0^{d(u,v)+d(v,u)}\varphi(t)dt$$

$$+\eta\int_0^{\frac{d(u,u)d(v,v)}{d(u,v)}}\varphi(t)dt$$

$$+\delta\int_0^{d(u,v)}\varphi(t)dt$$

$$\le(2\gamma+\delta)\int_0^{d(u,v)}\varphi(t)dt$$

Which is a contradiction since $(2\gamma+\delta)<1$. Hence u is unique fixed point of S &T in X.

Theorem 2.2: Let (X,d) be a complete cone metric space and P a normal cone with constant K. suppose that S & T be mapping from X into itself satisfies the condition

$$\int_0^{d(Sx,Ty)}\varphi(t)dt\le\alpha\int_0^{\frac{d(x,Sx)d(y,Ty)d(x,Ty)+d(x,y)d(y,Sx)d(y,Ty)}{[d(x,y)]^2+d(x,Ty)d(y,Ty)}}\varphi(t)dt$$

$$+\beta\int_0^{d(x,Sx)+d(y,Ty)}\varphi(t)dt$$

$$+\gamma\int_0^{d(x,Ty)+d(y,Sx)}\varphi(t)dt$$

$$+\eta\int_0^{\frac{d(x,Sx)d(y,Ty)}{d(x,y)}}\varphi(t)dt \qquad (2.2.1)$$

$$+\delta\int_0^{d(x,y)}\varphi(t)dt$$

for all $x,y\in X$ and $\alpha,\beta,\gamma,\eta,\delta\ge 0$ such that $\alpha+2\beta+2\gamma+\eta+\delta<1$.Then S &T has a unique fixed point in X. Further moreover either ST=TS then it have unique common fixed point in X.

Proof: For any arbitrary x_0 in X we choose $x_1,x_2\in X$ such that

$$Sx_0=x_1\ \&\ Tx_1=x_2$$

In general we can define a sequence of elements of X such that

$$x_{2n+1}=Sx_{2n}\ \&\ x_{2n+2}=Tx_{2n+1}$$

Now from (2.2.1)

$$\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt\le\int_0^{d(Sx_{2n},Tx_{2n+1})}\varphi(t)dt$$

$$\le\alpha\int_0^{\frac{d(x_{2n},Sx_{2n})d(x_{2n+1},Tx_{2n+1})d(x_{2n},Tx_{2n+1})+d(x_{2n},x_{2n+1})d(x_{2n+1},Sx_{2n})d(x_{2n+1},Tx_{2n+1})}{[d(x_{2n},x_{2n+1})]^2+d(x_{2n},Tx_{2n+1})d(x_{2n+1},Tx_{2n+1})}}\varphi(t)dt$$

$$+\beta\int_0^{d(x_{2n},Sx_{2n})+d(x_{2n+1},Tx_{2n+1})}\varphi(t)dt$$

$$+\gamma\int_0^{d(x_{2n},Tx_{2n+1})+d(x_{2n+1},Sx_{2n})}\varphi(t)dt$$

$$+\eta\int_0^{\frac{d(x_{2n},Sx_{2n})d(x_{2n+1},Tx_{2n+1})}{d(x_{2n},x_{2n+1})}}\varphi(t)dt$$

$$+\delta\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt$$

$$\le\alpha\int_0^{\frac{d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2})d(x_{2n},x_{2n+2})+d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+1})d(x_{2n+1},x_{2n+2})}{[d(x_{2n},x_{2n+1})]^2+d(x_{2n},x_{2n+2})d(x_{2n+1},x_{2n+2})}}\varphi(t)dt$$

$$+\beta\int_0^{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})}\varphi(t)dt$$

$$+\gamma\int_0^{d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})}\varphi(t)dt$$

$$+\eta\int_0^{\frac{d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2})}{d(x_{2n},x_{2n+1})}}\varphi(t)dt$$

$$+\delta\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt$$

$$\leq \alpha \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt$$
$$+ \beta \int_0^{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})} \varphi(t)dt$$
$$+ \gamma \int_0^{d(x_{2n},x_{2n+2})} \varphi(t)dt$$
$$+ \eta \int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt$$
$$+ \delta \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt$$

By using triangle inequality, we get

$$\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt \leq \left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta}\right)\int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt$$

Similarly we can show that

$$\int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt \leq \left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta}\right)\int_0^{d(x_{2n-1},x_{2n})} \varphi(t)dt$$

in general we can write

$$\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt \leq \left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta}\right)^{2n+1}\int_0^{d(x_0,x_1)} \varphi(t)dt$$

on taking $\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta} = h$, we have

$$\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt \leq h^{2n+1}\int_0^{d(x_0,x_1)} \varphi(t)dt$$

for $n \leq m$ we have

$$\int_0^{d(x_{2n},x_{2m})} \varphi(t)dt \leq \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt + \int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt + \ldots + \int_0^{d(x_{2m-1},x_{2m})} \varphi(t)dt$$

$$\leq \left(h^n + h^{n+1} + .. + h^m\right)\int_0^{d(x_0,x_1)} \varphi(t)dt$$

$$\leq \frac{h^n}{1-h}\int_0^{d(x_0,x_1)} \varphi(t)dt$$

$$\left\|\int_0^{d(x_{2n},x_{2m})} \varphi(t)dt\right\| \leq \frac{h^n}{1-h} k \left\|\int_0^{d(x_0,x_1)} \varphi(t)dt\right\| \text{ as } \to \infty ,$$

$$\lim_{n\to\infty}\left\|\int_0^{d(x_{2n},x_{2m})} \varphi(t)dt\right\| \to 0$$

Hence $\lim_{n\to\infty}\left\|\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt\right\| \to 0$ as $n \to \infty$

Hence $\{x_n\}$ is a Cauchy sequence, which converges to $u \in X$. Hence (X,d) is a complete cone metric space. Thus $x_n \to u$ as $n \to \infty$ and $Sx_{2n} \to u$ & $Tx_{2n+1} \to u$ as $n \to \infty$,u is fixed point of S & T in X.

Since $ST = TS$ this gives $u = Tu = TSu = STu = Su = u$

Hence u is common fixed point of S & T.

Uniqueness: Let us assume that, v is another fixed point of T in X different from u then $Tu = u$ & $Tv = v$, from (2.2.1)

$$\int_0^{d(u,v)} \varphi(t)dt = \int_0^{d(Su,Tv)} \varphi(t)dt$$

$$\leq \alpha \int_0^{\frac{d(u,Su)d(v,Tv)d(u,Tv)+d(u,v)d(v,Su)d(v,Tv)}{[d(u,v)]^2+d(u,Tv)d(v,Tv)}} \varphi(t)dt$$
$$+ \beta \int_0^{d(u,Su)+d(v,Tv)} \varphi(t)dt$$
$$+ \gamma \int_0^{d(u,Tv)+d(v,Su)} \varphi(t)dt$$
$$+ \eta \int_0^{\frac{d(u,Su)d(v,Tv)}{d(u,v)}} \varphi(t)dt$$
$$+ \delta \int_0^{d(u,v)} \varphi(t)dt$$

$$\leq \alpha \int_0^{\frac{d(u,u)d(v,v)d(u,v+d(u,v)d(v,u)d(v,v)}{[d(u,v)]^2+d(u,v)d(v,v)}} \varphi(t)dt$$
$$+ \beta \int_0^{d(u,u)+d(v,v)} \varphi(t)dt$$
$$+ \gamma \int_0^{d(u,v)+d(v,u)} \varphi(t)dt$$
$$+ \eta \int_0^{\frac{d(u,u)d(v,v)}{d(u,v)}} \varphi(t)dt$$
$$+ \delta \int_0^{d(u,v)} \varphi(t)dt$$

$$\leq (2\gamma+\delta)\int_0^{d(u,v)} \varphi(t)dt$$

which is a contradiction since $(2\gamma+\delta) < 1$. Hence u is unique fixed point of S &T in X.

Theorem 2.3: Let (X,d) be a complete cone metric space and P a normal cone with normal constant K. Suppose that S, R & T be mapping from X into itself satisfying the condition:

$$\int_0^{d(SRx,TRy)}\varphi(t)dt$$
$$\leq\alpha\int_0^{\frac{d(x,SRx)d(y,TRy)d(x,TRy)+d(x,y)d(y,SRx)d(y,TRy)}{[d(x,y)]^2+d(x,TRy)d(y,TRy)}}\varphi(t)dt$$
$$+\beta\int_0^{d(x,SRx)+d(y,TRy)}\varphi(t)dt$$
$$+\gamma\int_0^{d(x,TRy)+d(y,SRx)}\varphi(t)dt$$
$$+\eta\int_0^{\frac{d(x,SRx)d(y,TRy)}{d(x,y)}}\varphi(t)dt \quad \text{.... (2.3.1)}$$
$$+\delta\int_0^{d(x,y)}\varphi(t)dt$$

for all $x,y\in X$ and $\alpha,\beta,\gamma,\eta,\delta\geq 0$ such that $\alpha+2\beta+2\gamma+\eta+\delta<1$. Then S,R &T has a unique fixed point in X. Further moreover either SR=TR or TR=RT then it have unique common fixed point in X.

Proof: For any arbitrary x_0 in X we choose $x_1,x_2\in X$ such that

$SRx_0=x_1\ \&\ TRx_1=x_2$

In general we can define a sequence of elements of X such that

$x_{2n+1}=SRx_{2n}\ \&\ x_{2n+2}=TRx_{2n+1}$

Now from (2.3.1)

$$\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt=\int_0^{d(SRx_{2n},TRx_{2n+1})}\varphi(t)dt$$
$$\leq\alpha\int_0^{\frac{d(x_{2n},SRx_{2n})d(x_{2n+1},TRx_{2n+1})d(x_{2n},TRx_{2n+1})+d(x_{2n},x_{2n+1})d(x_{2n+1},SRx_{2n})d(x_{2n+1},TRx_{2n+1})}{[d(x_{2n},x_{2n+1})]^2+d(x_{2n},TRx_{2n+1})d(x_{2n+1},TRx_{2n+1})}}\varphi(t)dt$$
$$+\beta\int_0^{d(x_{2n},SRx_{2n})+d(x_{2n+1},TRx_{2n+1})}\varphi(t)dt$$
$$+\gamma\int_0^{d(x_{2n},TRx_{2n+1})+d(x_{2n+1},SRx_{2n})}\varphi(t)dt$$
$$+\eta\int_0^{\frac{d(x_{2n},SRx_{2n})d(x_{2n+1},TRx_{2n+1})}{d(x_{2n},x_{2n+1})}}\varphi(t)dt$$
$$+\delta\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt$$
$$\leq\alpha\int_0^{\frac{d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2})d(x_{2n},x_{2n+2})+d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+1})d(x_{2n+1},x_{2n+2})}{[d(x_{2n},x_{2n+1})]^2+d(x_{2n},x_{2n+2})d(x_{2n+1},x_{2n+2})}}\varphi(t)dt$$
$$+\beta\int_0^{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})}\varphi(t)dt$$
$$+\gamma\int_0^{d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})}\varphi(t)dt$$
$$+\eta\int_0^{\frac{d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2})}{d(x_{2n},x_{2n+1})}}\varphi(t)dt$$
$$+\delta\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt$$

$$\leq\alpha\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt$$
$$+\beta\int_0^{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})}\varphi(t)dt$$
$$+\gamma\int_0^{d(x_{2n},x_{2n+2})}\varphi(t)dt$$
$$+\eta\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt$$
$$+\delta\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt$$

By using triangle inequality, we get

$$\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt\leq\left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta}\right)\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt$$

Similarly we can show that

$$\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt\leq\left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta}\right)\int_0^{d(x_{2n-1},x_{2n})}\varphi(t)dt$$

in general we can write

$$\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt\leq\left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta}\right)^{2n+1}\int_0^{d(x_0,x_1)}\varphi(t)dt$$

on taking $\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta}=h$, we have

$$\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt\leq h^{2n+1}\int_0^{d(x_0,x_1)}\varphi(t)dt$$

for $n\leq m$ we have

$$\int_0^{d(x_{2n},x_{2m})}\varphi(t)dt\leq\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt$$
$$+\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt$$
$$+\ldots$$
$$+\int_0^{d(x_{2m-1},x_{2m})}\varphi(t)dt$$

$$\leq\left(h^n+h^{n+1}+h^{n+2}+h^{n+3}+..+h^m\right)\int_0^{d(x_0,x_1)}\varphi(t)dt$$
$$\leq\frac{h^n}{1-h}\int_0^{d(x_0,x_1)}\varphi(t)dt$$

$$\left\|\int_0^{d(x_{2n},x_{2m})}\varphi(t)dt\right\|\leq\frac{h^n}{1-h}k\left\|\int_0^{d(x_0,x_1)}\varphi(t)dt\right\| \text{ as }\to\infty,$$

$$\lim_{n\to\infty}\left\|\int_0^{d(x_{2n},x_{2m})}\varphi(t)dt\right\|\to 0$$

Hence $\lim_{n\to\infty}\left\|\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt\right\|\to 0$ as $n\to\infty$

Hence $\{x_n\}$ is a Cauchy sequence, which converges to $u\in X$. Hence (X,d) is a complete cone metric space. Thus $x_n\to u$ as $n\to\infty$ and $SRx_{2n}\to u$ & $TRx_{2n+1}\to u$ as $n\to\infty$ then u is fixed point of S & T in X.

Since $ST=TS$ this gives

$u=Tu=TSu=STu=Su=u.$

Hence u is common fixed point of S & T.

Uniqueness: Let us assume that, v is another fixed point of T in X different from u then $Tu=u\,\&\,Tv=v$,

from (2.3.1)

$$\int_0^{d(u,v)}\varphi(t)dt=\int_0^{d(Su,Tv)}\varphi(t)dt$$

$$\le\alpha\int_0^{\frac{d(u,Su)d(v,Tv)d(u,Tv)+d(u,v)d(v,Su)d(v,Tv)}{[d(u,v)]^2+d(u,Tv)d(v,Tv)}}\varphi(t)dt$$

$$+\beta\int_0^{d(u,Su)+d(v,Tv)}\varphi(t)dt$$

$$+\gamma\int_0^{d(u,Tv)+d(v,Su)}\varphi(t)dt$$

$$+\eta\int_0^{\frac{d(u,Su)d(v,Tv)}{d(u,v)}}\varphi(t)dt$$

$$+\delta\int_0^{d(u,v)}\varphi(t)dt$$

$$\le\alpha\int_0^{\frac{d(u,u)d(v,v)d(u,v)+d(u,v)d(v,u)d(v,v)}{[d(u,v)]^2+d(u,v)d(v,v)}}\varphi(t)dt$$

$$+\beta\int_0^{d(u,u)+d(v,v)}\varphi(t)dt$$

$$+\gamma\int_0^{d(u,v)+d(v,u)}\varphi(t)dt$$

$$+\eta\int_0^{\frac{d(u,u)d(v,v)}{d(u,v)}}\varphi(t)dt$$

$$+\delta\int_0^{d(u,v)}\varphi(t)dt$$

$$\le(2\gamma+\delta)\int_0^{d(u,v)}\varphi(t)dt$$

which is a contradiction since $(2\gamma+\delta)<1$. Hence u is unique fixed point of S &T in X.

Theorem 2.4: Let (X,d) be a complete cone metric space and P a normal cone with normal constant K. Suppose that A, B, S & T be mapping from X into itself satisfies the condition:

i. $A(X)\subseteq T(X)$, $B(X)\subseteq S(X)$
ii. {A, S} and {B, T}are weakly compatible
iii. S or T is continuous.

$$\int_0^{d(Ax,By)}\varphi(t)dt\le\alpha\int_0^{\frac{\left[\begin{array}{l}d(Sx,Ax)d(Ty,By)d(Sx,By)\\+d(Sx,Ty)d(Ty,Ax)d(Ty,By)\end{array}\right]}{[d(Sx,Ty)]^2+d(Sx,By)d(Ty,By)}}\varphi(t)dt$$

$$+\beta\int_0^{d(Sx,Ax)+d(Ty,By)}\varphi(t)dt$$

$$+\gamma\int_0^{d(Sx,By)+d(Ty,Ax)}\varphi(t)dt$$

$$+\eta\int_0^{\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)}}\varphi(t)dt \qquad . (2.4.1)$$

$$+\delta\int_0^{d(Sx,Ty)}\varphi(t)dt$$

for all $x,y\in X$ and $\alpha,\beta,\gamma,\eta,\delta\ge 0$ such that $\alpha+2\beta+2\gamma+\eta+\delta<1$.Then A, B S &T has a unique fixed point in X. Further moreover either SA=AS or BT=TB then it have unique common fixed point in X.

Proof: For any arbitrary x_0 in X we define sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Ax_{2n}=Tx_{2n+1}=y_{2n}$ & $Bx_{2n+1}=Sx_{2n+2}=y_{2n+1}$ for all n= 0, 1, 2….

Now from (2.4)

$$\int_0^{d(y_{2n},y_{2n+1})}\varphi(t)dt=\int_0^{d(Ax_{2n},Bx_{2n+1})}\varphi(t)dt$$

$$\le\alpha\int_0^{\frac{d(Sx_{2n},Ax_{2n})d(Tx_{2n+1},Bx_{2n+1})d(Sx_{2n},Bx_{2n+1})+d(Sx_{2n},Tx_{2n+1})d(Tx_{2n+1},Ax_{2n})d(Tx_{2n+1},Bx_{2n+1})}{[d(Sx_{2n},Tx_{2n+1})]^2+d(Sx_{2n},Bx_{2n+1})d(Tx_{2n+1},Bx_{2n+1})}}\varphi(t)dt$$

$$+\beta\int_0^{d(Sx_{2n}Ax_{2n})+d(Tx_{2n+1},Bx_{2n+1})}\varphi(t)dt$$

$$+\gamma\int_0^{d(Sx_{2n},Bx_{2n+1})+d(Tx_{2n+1},Ax_{2n})}\varphi(t)dt$$

$$+\eta\int_0^{\frac{d(Sx_{2n},Ax_{2n})d(Tx_{2n+1},Bx_{2n+1})}{d(Sx_{2n},Tx_{2n+1})}}\varphi(t)dt$$

$$+\delta\int_0^{d(Sx_{2n},Tx_{2n+1})}\varphi(t)dt$$

$$\le\alpha\int_0^{\frac{\begin{array}{l}d(y_{2n-1},y_{2n})d(y_{2n},y_{2n+1})d(y_{2n-1},y_{2n+1})\\+d(y_{2n-1},y_{2n})d(y_{2n},y_{2n})d(y_{2n},y_{2n+1})\end{array}}{[d(y_{2n-1},y_{2n})]^2+d(y_{2n-1},y_{2n+1})d(y_{2n},y_{2n+1})}}\varphi(t)dt$$

$$+\beta\int_0^{d(y_{2n-1,}y_{2n})+d(y_{2n},y_{2n+1})}\varphi(t)dt$$

$$+\gamma\int_0^{d(y_{2n-1},y_{2n+1})+d(y_{2n},y_{2n})}\varphi(t)dt$$

$$+\eta\int_0^{\frac{d(y_{2n-1},y_{2n})d(y_{2n},y_{2n+1})}{d(y_{2n-1},y_{2n})}}\varphi(t)dt$$

$$+\delta\int_0^{d(y_{2n-1},y_{2n})}\varphi(t)dt$$

$$\leq \alpha \int_0^{d(y_{2n-1},y_{2n})} \varphi(t)dt$$
$$+\beta \int_0^{d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})} \varphi(t)dt$$
$$+\gamma \int_0^{d(y_{2n-1},y_{2n+1})} \varphi(t)dt$$
$$+\eta \int_0^{d(y_{2n},y_{2n+1})} \varphi(t)dt$$
$$+\delta \int_0^{d(y_{2n-1},y_{2n})} \varphi(t)dt$$

By using triangle inequality, we get

$$\int_0^{d(y_{2n},y_{2n+1})} \varphi(t)dt \leq \left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta}\right)\int_0^{d(y_{2n-1},y_{2n})} \varphi(t)dt$$

in general we can write

$$\int_0^{d(y_{2n},y_{2n+1})} \varphi(t)dt \leq \left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta}\right)^{2n+1}\int_0^{d(y_0,y_1)} \varphi(t)dt$$

on taking $\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta}=h$, we have

$$\int_0^{d(y_{2n},y_{2n+1})} \varphi(t)dt \leq h^{2n+1}\int_0^{d(y_0,y_1)} \varphi(t)dt$$

for $n \leq m$ we have

$$\int_0^{d(y_{2n},y_{2m})} \varphi(t)dt \leq \int_0^{d(y_{2n},y_{2n+1})} \varphi(t)dt + \int_0^{d(y_{2n+1},y_{2n+2})} \varphi(t)dt + \ldots + \int_0^{d(y_{2m-1},y_{2m})} \varphi(t)dt$$

$$\leq \left(h^n+h^{n+1}+h^{n+2}+h^{n+3}+..+h^m\right)\int_0^{d(y_0,y_1)} \varphi(t)dt$$

$$\leq \frac{h^n}{1-h}\int_0^{d(y_0,y_1)} \varphi(t)dt$$

$$\left\|\int_0^{d(y_{2n},y_{2m})} \varphi(t)dt\right\| \leq \frac{h^n}{1-h}k\left\|\int_0^{d(y_0,y_1)} \varphi(t)dt\right\|$$

as $\to \infty$, $\lim_{n\to\infty}\left\|\int_0^{d(y_{2n},y_{2m})} \varphi(t)dt\right\| \to 0$

Hence $\lim_{n\to\infty}\left\|\int_0^{d(y_{2n+1},y_{2n+2})} \varphi(t)dt\right\| \to 0$ as $n\to\infty$

Hence $\{y_n\}$is a Cauchy sequence, which converges to $u \in X$. By the continuity of S &T, sequence $\{x_n\}$ is also convergent to $u \in X$. Hence $(X,d)$is a complete cone metric space. u is fixed point of A, B, S & T. Since $\{A,S\}$and $\{B,T\}$ are weakly compatible implies that u is common fixed point of A, B, S &T.

Uniqueness: Let us assume that, v is another fixed point of A, B, S &T in X different from u then $Au=u, Av=v$ & $Bu=u, Bv=v$, from (2.4.1)

$$\int_0^{d(u,v)} \varphi(t)dt$$
$$=\int_0^{d(Au,Bv)} \varphi(t)dt$$

$$\leq \alpha \int_0^{\frac{d(Su,Au)d(Tv,Bv)d(Su,Bv+d(Su,Tv)d(Tv,Au)d(Tv,Bv)}{[d(Su,Tv)]^2+d(Su,Bv)d(Tv,Bv)}} \varphi(t)dt$$
$$+\beta \int_0^{d(Su,Au)+d(Tv,Bv)} \varphi(t)dt$$
$$+\gamma \int_0^{d(Su,Bv)+d(Tv,Au)} \varphi(t)dt$$
$$+\eta \int_0^{\frac{d(Su,Au)d(Tv,Bv)}{d(Su,Tv)}} \varphi(t)dt$$
$$+\delta \int_0^{d(Su,Tv)} \varphi(t)dt$$

$$\leq \alpha \int_0^{\frac{d(u,u)d(v,v)d(u,v)+d(u,v)d(v,u)d(v,v)}{[d(u,v)]^2+d(u,v)d(v,v)}} \varphi(t)dt$$
$$+\beta \int_0^{d(u,u)+d(v,v)} \varphi(t)dt$$
$$+\gamma \int_0^{d(u,v)+d(v,u)} \varphi(t)dt$$
$$+\eta \int_0^{\frac{d(u,u)d(v,v)}{d(u,v)}} \varphi(t)dt$$
$$+\delta \int_0^{d(u,v)} \varphi(t)dt$$

$$\leq (2\gamma+\delta)\int_0^{d(u,v)} \varphi(t)dt$$

which is a contradiction since $(2\gamma+\delta)<1$. Hence u is unique fixed point of A, B, S &T in X.

Conclusion: In this paper we proved some fixed point & common fixed point theorems for cone metric space in integral type mappings which shows that our main theorem is the generalized version of some known theorems.

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