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Approximations of pi and squaring the circle

Mieczyslaw Szyszkowicz
e-mail: mszyszkowicz@yahoo.ca

***Abstract*—In this paper 18 different methods to estimate the number pi are presented. Some of the methods are the results from simple combinations of other classical approaches. The combinations usually generate faster convergence to pi. Among the presented methods some of them can be used to construct approximate squaring the circle.**

***Keywords*—algorithm, area, circle, perimeter, polygon, quadrature, number pi, approximation**

I. INTRODUCTION

As it is well known a mathematical universal constant π represents the ratio of a circle's circumference to its diameter. (π =3.141592+.). Archimedes of Syracuse around 255 B.C. calculated the perimeters of inscribed and circumscribed regular polygons of 6, 12, 24, 48, and 96 sides to determine upper and lower boundary for the perimeter of the circle [1, 2]. He developed a method to estimate perimeters and by this approach to find bounds for π value. Using two 96 side regular polygons he provided the following estimations for π,

$$3.14084507\ldots = 3 + \frac{10}{71} < \pi < 3 + \frac{1}{7} = 3.14285714\ldots.$$

We have 96*sin(π/96) < π < 96*tan(π/96).

Fig. 1. The mathematical constant π.

II. METHODS

Here we consider simple methods to estimate π. Some of these methods are very well known but still we are able to improve even these traditional formulae. We assume that our circle has radius 1. As a main goal of this work is to calculate π, we consider half of the circle when calculate its perimeter. Thus rather than estimate 2πr, we get the value πr or just π, if r=1. We are using the following notation x= π/n, where n is the number of sides in the considered regular polygon. In practice x value represents angle. For example in the case of an equilateral triangle (n=3), we have x=60 degrees. We will use the notation M<X>, X=number used to identify the proposed methods. Table 1 summarizes all of them and provides short descriptions and used formula. Also it lists the results generated by these methods for n=3, i.e. for an equilateral triangle, where π~M<X>*3.

Algorithms of Archimedes, Snell, and Huygens

Let p_n and P_n be the perimeters of the inscribed and circumscribed n –gon in the circle. Using simple trigonometry we are able to derive the following formulae – two methods: M1 = p_n/2=n*sin(π/n) and M2 = P_n/2=n*tan(π/n). We have the following Maclaurin series for the used trigonometric functions (i.e. sin x and tan x, where x=π/n),

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots,\ \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots.$$

Willebrord Snell (Snellius) [3] observed that the perimeter of the inscribed polygons of n sides approaches π twice as fast as the perimeter of the circumscribed. This fact and many others related to the circle were proved by Christian Huygens in 1654 [4]. Using this information we define new method by the formula M4= M1+(M2-M1)/3. This simple combination eliminates the terms with x to power 3 in the corresponding Maclaurin series. The obtained method generates values which converge faster to π, than M1 or M2.

Chakrabarti and Hudson [5, 6] proposed method M6 based on the three methods M1, M2, and M3. Their method has the corresponding Maclaurin series of the following form,

$$M6\colon x + \frac{x^7}{105} + \frac{x^9}{360} + \frac{x^{11}}{39600} + \ldots.$$

As we see in this case the lowest power of x is 7, thus the method converges faster than their three components, where in each x is in power 3.

Snell and Huygens developed another approach. They estimated the length of the arc (rectification) which corresponds to the angle x. They provide two formulae-methods to obtain overestimated (M7) and under estimated (M8) values for the constructed rectification of the arc:

$$M7 = \left(2\cos\left(\frac{x}{3}\right) + 1\right)\tan\left(\frac{x}{3}\right),$$

$$M8 = \frac{3\sin(x)}{2+\cos(x)}.$$

Szyszkowicz [7] proposed to combine these two methods and as the result to produce faster method. His construction has the following simple form

$$M13 = M7 + (M8 - M7)/10.$$

This method has the following Maclaurin series

$$M13: x - \frac{x^7}{22680} - \frac{x^9}{349920} + \frac{437x^{11}}{6235574400} + \ldots.$$

This method is faster than M6 proposed by Chakrabarti and Hudson (both methods have x^7).

A. Algorithms of Archimedes, Snell, and Huygens

In the year 1800 Gauss' teacher Pfaff discovered the algorithm to iteratively realize doubling of side of n-gon in Archimedes' approach. We may start with a_0=2sqrt(3) and b_0=3, which correspond to the original Archimedes' technique, where he used a hexagon (k=6). Let a_n and b_n be the perimeters of the circumscribed and inscribed k -gon and a_{n+1} and b_{n+1} the perimeters of the circumscribed and inscribed 2k – gon, respectively. Thus the next step of the iteration doubles the number of sides.

$$a_{n+1} = \frac{2a_n b_n}{a_n + b_n},\ b_{n+1} = \sqrt{a_{n+1} b_n}.$$

A German mathematician Heinrich Dörrie improved this process (Problem #38 in his book [7]) in such sense that he provided narrower interval which sandwiches π. Dörrie generated his sequence using these elements a and b from Pfaff's method. Here, in addition we explicitly write these formulae in the corresponding trigonometric terms

$$A = \sqrt[3]{ab^2} = \sqrt[3]{\tan x \sin^2 x},$$
$$B = \frac{3ab}{2a+b} = \frac{3\tan x \sin x}{2\tan x + \sin x} = \frac{3\sin x}{2+\cos x}.$$

It is interesting that the method M8 was first time proposed in the XV century by the cardinal Nicholas of Kues (1401 – 1464) also known as Nicolaus Cusanus and Nicholas of Cusa. This method was later (XVII century) developed again by Snell and Huygens. Here Szyszkowicz proposed to combine two values A and B generated by Dörrie's sequence and as a new approximation he defined the method M10=B+(A-B)/5=M8+(M9-M8)/5. The method has the following Maclaurin series,

$$M10: x + \frac{x^7}{114} + \frac{x^9}{2160} + \frac{227x^{11}}{1283040} + \ldots.$$

Table 1 shows other methods which have high order of convergence. Higher power x means better speed, as x=π/n is small number. Comparing the presented numerical results we conclude that the method called Newton-Szyszkowicz (M16) provided the highest accuracy. It's justified by its corresponding Maclaurin series

$$M16: x + \frac{x^9}{396900} + \frac{4009x^{11}}{7072758000} + \ldots.$$

Table 1. The methods and numerical results.

Description	Method: π~n*M<X>	n=3
sin(x), arc	M1	2.598076
tan(x),arc, area	M2	5.196152
sin(2x)/2, area	M3	1.299038
SnellHuyg	M4=M1+(M2-M1)/3	3.464101
SnellHuyg	M5=M2+(M3-M2)/3	3.897114
ChakHudson	M6=(32M1+4M2-6M3)/30	3.204293
Snell, arc	M7=(2 cos x/3+1) tan x/3	3.144031
SnellDörrie	M8=3sin x/(2+cos x)	3.117691
A- Dörrie	M9=(M2*M1*M1)^1/3	3.273370
SzyszDörrie	M10=M8+(M9-M8)/5	3.148827
Newton	M11	3.139342
Castellanos	M12	3.141310
Szyszkowicz	M13=M7+(M8-M7)/10	3.141397
SzyszChHud	M14=M13+(M6-M13)/217	3.141687
NewtSzyszDö	M15=M11+(M10-M11)/3	3.142503
NewtSzysz	M16=(54M13-5M11)/49	3.141607
NewtChHud	M17=M11+(M6-M11)/21	3.142435
SnellChHud	M18=M6=0.2(8M4-3M5)	3.204293

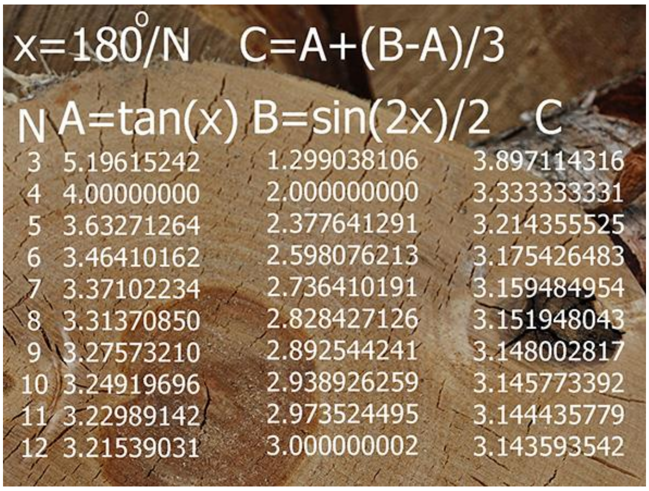
Note: M11=sin(x)(14+cos(x))/(9+6cos(x)); M12=sin(x)x(187+24cos x – cos 2x)/(10+90cos x). We have the following Maclaurin series for M11 and M12,

$$M11: x - \frac{x^7}{2100} - \frac{x^9}{18000} - \ldots,$$

$$M12: x - \frac{x^9}{17640} - \frac{x^{11}}{226385} - \ldots.$$

B. Squaring the circle - geometrically interpreted methods

Archimedes' approach allows to construct an approximate squaring the circle. In case of the methods M1 and M2 it is easy to obtain segments which are close to the value πr. For example, for the triangle (n=3), the approximated segment is close to πr/3. As we see from Table 1, in this case the approximation is not very accurate. We still can create the segment 3 times πr/3 (with π approximated) and build a rectangle of sides close to πr and r. A classical geometrical construction allows to squaring this rectangle. The obtained square will approximate the circle. In similar way the methods M7 and M8 may be used. They will create more accurate approximation to the circumference. The best method among presented here to use in squaring the circle, is Szyszkowicz's approach (M13) [7], and as we see from the table already for the triangle (n=3) this method results with π=3.14139… for n=6 we have π=3.14158975.



$x=180^{\circ}/N \quad C=A+(B-A)/3$

N	A=tan(x)	B=sin(2x)/2	C
3	5.19615242	1.299038106	3.897114316
4	4.00000000	2.000000000	3.333333331
5	3.63271264	2.377641291	3.214355525
6	3.46410162	2.598076213	3.175426483
7	3.37102234	2.736410191	3.159484954
8	3.31370850	2.828427126	3.151948043
9	3.27573210	2.892544241	3.148002817
10	3.24919696	2.938926259	3.145773392
11	3.22989142	2.973524495	3.144435779
12	3.21539031	3.000000002	3.143593542

Fig. 2. The results from methods M2, M3, and M5.

We may use Aitken's transformation to generate better approximation to π. For a given sequence $\{x_n\}$ generated by one of the presented methods, we produce new sequence A(n) using the following formulae [9]. (For example, the method M11 gives 3.13934209 (n=3, triangle), 3.14121486 (n=4, square), 3.14149622 (n=5, pentagon), and Aitken: 3.141546). M13 alone for n=5 gives 3.14158392.

Aitken's process for $\{x_n\}$

$$\Delta x(n) = x(n+1) - x(n),$$

$$\Delta^2 x(n) = x(n) - 2x(n+1) + x(n+2),$$

$$A(x(n)) = x(n) - \frac{(\Delta x(n))^2}{\Delta^2 x(n)}.$$

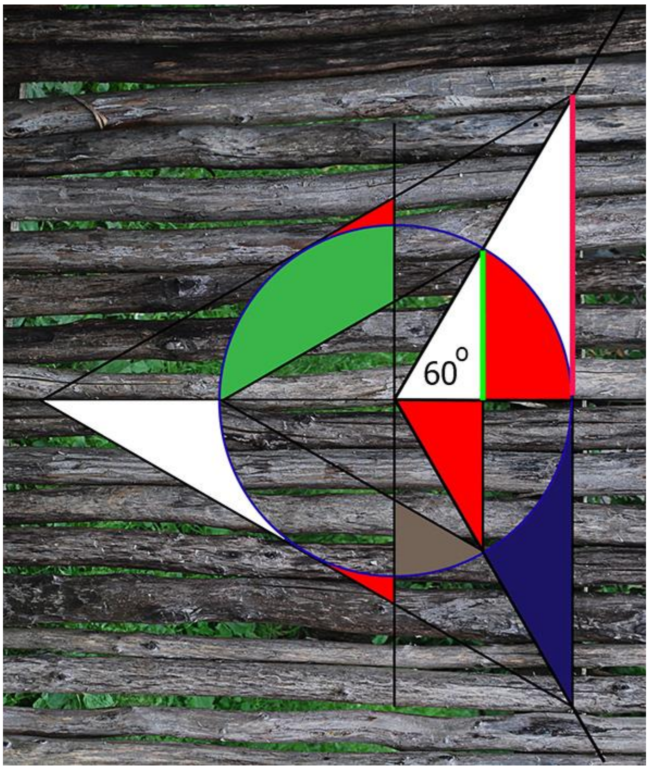


Fig. 3. The simplest regular polygons (equilateral triangles) used to approximate a circle.

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