Mathematical Analysis of Self-synchronous Theory of Vibrating System

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***Abstract*—Self-synchronous theory of vibrating system driven by three motors in the same direction is studied through mathematical analysis in this paper. The mathematics model of electromechanical coupling of the vibrating system is established, and the self-synchronization and stability conditions of the vibrating system are deduced by using Hamilton principle. The characteristic of vibration synchronization was studied by computer simulation. The vibrating system can implement stable operation of vibration synchronization when the simulation program is based on the self-synchronization and stability conditions of synchronous operation deduced in the mathematical analysis. Simulation results to the vibrating system validate the correctness of the mathematical analysis, it provides theoretical basis for the design of the vibrating system driven by several motors.**

Keywords—vibrating system; self-synchronization; mathematical analysis; stability

I. Introduction

Scientific studies show that macroscopic dynamic behavior of complex system not only depends on motional characteristics of each subsystem, but has close relationship with interaction among subsystems.

The change of motional characteristics and forms of complex system can be affected directly by interaction among the subsystem's motion, it can result in new motional structure. To study the characteristics of interaction among the subsystem's motion and the influence of the interaction among the subsystem's motion to dynamic behavior of complex system is important content of complex system science.

Zhang Yimin presented a generalized probabilistic perturbation finite element method and employed the method to solve the response analysis of multi-degree-of -freedom nonlinear vibrating systems with random parameters and vector-valued and matrix-valued functions [1].

Zhang Tianxia studied coupling effect in a synchronous vibrating system, he derived a differential equation with an analytical method of nonlinear vibration for the coupling motion of two eccentric rotors to describe mathematically the coupling parameters of the system. Then he studied the synchronization development course of two eccentric rotors in depth, and deducted the necessary coupling conditions to for a synchronization state [2].

Zhao Chunyu analyzed the dynamic characteristic of a vibrating system with two-motor drives rotating in the reverse direction by using the dynamic theory and established the equations of frequency capture, he obtained the conditions of implementing frequency capture and the equations of calculating capturing frequency and the phase [3].

Wang Degang studied the dynamic coupling feature of a vibrating system driven by two motors in the same direction, converted the problem of synchronization into that of existence and stability of zero solution for the equation of frequency capture, and then obtain the condition of frequency capture and that of stable self-synchronous operation [4].

These studies mostly analyzed motional rules of synchronous vibrating system driven by two motors, few studies analyzed systems driven by three or more motors.

To large vibrating machine, it is not fit to adopt two motors to drive vibrating system because of the large structure. It should adopt three or more motors to drive based on the structure need. There are no effective methods to analyze the self-synchronization and stability conditions of synchronous operation of the vibrating system driven by three or more motors. Because of complexity of dynamic model and restriction of mathematics method available, the research to self-synchronization and stability conditions of synchronous operation of the vibrating system is carried out mostly around approximate synchronous state. There are no further studies to nonlinear dynamic mechanism of synchronous system and synchronous stability problems.

In this paper, the self-synchronous theory of the vibrating system driven by three motors is studied through mathematical analysis in depth, the self-synchronization and stability conditions of synchronous operation of the vibrating system are deduced by using Hamilton principle. Simulation results to the vibrating system verify the correctness of the mathematical analysis.

II. Self-synchronous Theory of Vibrating System Driven by Three Motors

A. Mathematical Model of a Vibrating System Driven by Three Motors

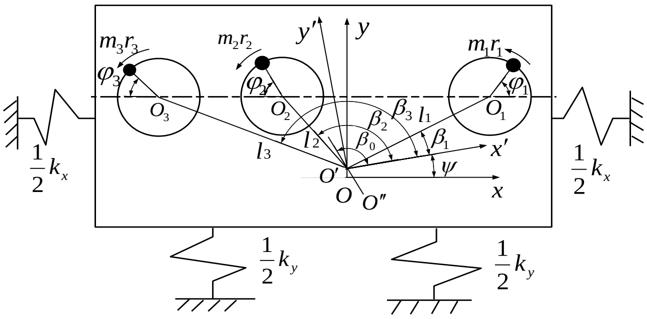


Fig. 1. *Dynamic model of a vibrating system driven by three motors in the same direction*

Dynamic model of a vibrating system driven by three motors in the same direction is shown in *Fig. 1*. O'' in the figure is the system centroid, O' and O are its synthesized centroid. Oxy is fixed coordinates, $O'x'y'$ is moving coordinates. $O'O''$ is the distance from synthesized centroid to system centroid, $O'O''=l_0$. O_1、O_2、O_3 are rotative centers of the three exciting motors, and they are in one lines. $O'O_1=l_1$, $O'O2=l_2$, $O'O3=l_3$.

In working process, the three motors rotate in counter-clockwise direction. The three motors which supply by the same power drive three eccentric lumps each other. The three eccentric lumps excite the system to vibrate. The motions of the system are vibrations in horizontal direction x, in vertical direction y and in rocking direction ψ (ψ<<1) [5-8]. With aid of the Langrange's equations, in addition, choosing x, y, ψ, φ_1, φ_2 and φ_3 as variable parameters, the vibration equation can be established through the expression of kinetic energy and potential energy of the system.

The kinetic energy expression of the vibrating system is shown in (1).

$$
\begin{aligned}
T =& \frac{1}{2}m_0\left\{\begin{array}{l}\left[\dot{x}-l_0\dot{\psi}\sin(\beta_0+\psi+\pi)\right]^2\\ +\left[\dot{y}+l_0\dot{\psi}\cos(\beta_0+\psi+\pi)\right]^2\end{array}\right\}\\
&+\frac{1}{2}m_1\left\{\left[\dot{x}-l_1\dot{\psi}\sin(\beta_1+\psi)-r_1\dot{\varphi}_1\sin\varphi_1\right]^2\right.\\
&\left.+\left[\dot{y}+l_1\dot{\psi}\cos(\beta_1+\psi)+r_1\dot{\varphi}_1\cos\varphi_1\right]^2\right\}\\
&+\frac{1}{2}m_2\left\{\begin{array}{l}\left[\dot{x}-l_2\dot{\psi}\sin(\beta_2+\psi)-r_2\dot{\varphi}_2\sin\varphi_2\right]^2\\ +\left[\dot{y}+l_2\dot{\psi}\cos(\beta_2+\psi)+r_2\dot{\varphi}_2\cos\varphi_2\right]^2\end{array}\right\}\\
&+\frac{1}{2}m_3\left\{\left[\dot{x}-l_3\dot{\psi}\sin(\beta_3+\psi)-r_3\dot{\varphi}_3\sin\varphi_3\right]^2\right.\\
&\left.+\left[\dot{y}+l_3\dot{\psi}\cos(\beta_3+\psi)+r_3\dot{\varphi}_3\cos\varphi_3\right]^2\right\}\\
&+\frac{1}{2}J_0\dot{\psi}^2+\frac{1}{2}\sum_{i=1}^{3}J_i\dot{\varphi}_i^2
\end{aligned}
\tag{1}
$$

where

x, y and ψ are the displacements of the vibrating body in the directions x, y and ψ, respectively,

$\dot{x}, \dot{y}, \dot{\psi}$ are speeds of the vibrating body in the directions x, y and ψ, respectively,

m_0 is mass of the vibrating body (not including the three eccentric lumps),

m_i (i=1, 2, 3) are masses of the three eccentric lumps, respectively,

r_i(i=1, 2, 3) are eccentricities of the three eccentric lumps, respectively,

φ_i (i=1, 2, 3) are angular displacements of the three eccentric lumps, respectively,

$\dot{\varphi}_i$ (i=1, 2, 3) are angular velocities of the three eccentric lumps, respectively,

J_0 is the moment of inertia of the vibrating body (not including the three eccentric lumps),

J_i (i=1, 2, 3) are the moments of inertia of the three eccentric lumps encircled respective rotative centers.

The potential energy expression of the vibrating system is shown in (2).

$$V=\frac{1}{2}k_x x^2+\frac{1}{2}k_y y^2+\frac{1}{2}k_\psi\psi^2 \tag{2}$$

where k_x, k_y, and k_ψ are the stiffness of spring in the directions x, y, ψ, respectively.

The expression of energy dissipate function of the vibrating system is shown in (3).

$$
\begin{aligned}
D=&\frac{1}{2}f_x x^2+\frac{1}{2}f_y y^2+\frac{1}{2}f_\psi\psi^2-\frac{1}{2}f_1(\dot{\varphi}_1-\dot{\psi})^2\\
&-\frac{1}{2}f_2(\dot{\varphi}_2-\dot{\psi})^2-\frac{1}{2}f_3(\dot{\varphi}_3-\dot{\psi})^2
\end{aligned}
\tag{3}
$$

where

f_x, f_y and f_ψ are the damping coefficient in the directions x, y, ψ, respectively,

f_i (i=1, 2, 3) are the damping coefficient of the three rotors, respectively.

Assuming x, y, ψ and φ_i are generalized coordinates q_i, then we can get the expression of generalized force which is shown in (4).

$$
Q_i=\begin{bmatrix}
-f_x\dot{x}\\
-f_y\dot{y}\\
f_1(\dot{\varphi}_1-\dot{\psi})+f_2(\dot{\varphi}_2-\dot{\psi})+f_3(\dot{\varphi}_3-\dot{\psi})-f_\psi\dot{\psi}\\
T_{e1}-T_{f1}\\
T_{e2}-T_{f2}\\
T_{e3}-T_{f3}
\end{bmatrix}
\tag{4}
$$

where

T_{ei} (i=1, 2, 3) are the electromagnetic torque of the three motors, respectively,

T_{fi} (i=1, 2, 3) are the load torque of the three motors, respectively.

The Langrange's equations is shown in (5).

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_i}-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}+\frac{\partial D}{\partial \dot{q}_i}=Q_i \tag{5}$$

Applying the expressions of kinetic energy, potential energy, energy dissipate function and generalized force into (5), we can obtain the rotation differential equations of the vibrating system in three directions and that of the three eccentric lumps which are shown in (6).

$$\begin{aligned}
M\ddot{x}+f_x\dot{x}+k_x x &= m_1r_1(\ddot{\varphi}_1\sin\varphi_1+\dot{\varphi}_1^{\,2}\cos\varphi_1)\\
&+m_3r_3(\ddot{\varphi}_3\sin\varphi_3+\dot{\varphi}_3^{\,2}\cos\varphi_3)\\
&+m_2r_2(\ddot{\varphi}_2\sin\varphi_2+\dot{\varphi}_2^{\,2}\cos\varphi_2)\\
M\ddot{y}+f_y\dot{y}+k_y y &= m_1r_1(-\ddot{\varphi}_1\cos\varphi_1+\dot{\varphi}_1^{\,2}\sin\varphi_1)\\
&+m_3r_3(-\ddot{\varphi}_3\cos\varphi_3+\dot{\varphi}_3^{\,2}\sin\varphi_3)\\
&++m_2r_2(-\ddot{\varphi}_2\cos\varphi_2+\dot{\varphi}_2^{\,2}\sin\varphi_2)\\
J\ddot{\psi}+f_\psi\dot{\psi}+k_\psi\psi &= f_1(\dot{\varphi}_1-\dot{\psi})+f_3(\dot{\varphi}_3+\dot{\psi})\\
&+f_2(\dot{\varphi}_2+\dot{\psi})\\
&+m_1l_1r_1[-\ddot{\varphi}_1\cos(\varphi_1-\beta_1-\psi)\\
&+\dot{\varphi}_1^{\,2}\sin(\varphi_1-\beta_1-\psi)]\\
&+m_3l_3r_3[-\ddot{\varphi}_3\cos(\varphi_3-\beta_3-\psi)\\
&+\dot{\varphi}_3^{\,2}\sin(\varphi_3-\beta_3-\psi)]\\
&+m_2l_2r_2[-\ddot{\varphi}_2\cos(\varphi_2-\beta_2-\psi)\\
&+\dot{\varphi}_2^{\,2}\sin(\varphi_2-\beta_2-\psi)]\\
J_{0i}\ddot{\varphi}_i &= T_{ei}-T_{fi}-f_i(\dot{\varphi}_i-\dot{\psi})\\
&+m_ir_i[\ddot{x}\sin\varphi_i-\ddot{y}\cos\varphi_i]\\
&-m_il_ir_i[\ddot{\psi}_i\cos(\varphi_i-\beta_i-\psi)\\
&+\dot{\psi}_i^{\,2}\sin(\varphi_i-\beta_i-\psi)] \qquad i=1,2,3
\end{aligned} \tag{6}$$

where

M is mass of the vibrating system (including the three exciting motors and the three eccentric lumps),

J is the moment of inertia of the vibrating system surround O (including the three exciting motors and the three eccentric lumps), $J=J_0+m_0l_0^2+\sum_{i=1}^{3}m_il_i^{\,2}$,

where

$\ddot{x}$, $\ddot{y}$ and $\ddot{\psi}$ are accelerations of the vibrating body in the directions x, y and ψ, respectively,

$\ddot{\varphi}_i$ (i=1, 2, 3) are angular accelerations of the three eccentric lumps.

Assuming that the phase of eccentric rotor 1 leads that of eccentric rotor 2 by α_{12} and that of eccentric rotor 2 leads that of eccentric rotor 3 by α_{23}, i.e. $\varphi_1-\varphi_2=\alpha_{12}$, $\varphi_2-\varphi_3=\alpha_{23}$. Assuming that the average phase of the three eccentric rotors is φ when the vibrating system operates at stable state, and then we have:

$$\begin{aligned}
\varphi_1 &= \varphi+\frac{1}{2}\alpha_{12}\\
\varphi_2 &= \varphi-\frac{1}{2}\alpha_{12}\\
\varphi_3 &= \varphi-\frac{1}{2}\alpha_{12}-\alpha_{23}\\
\varphi &= \omega t
\end{aligned} \tag{7}$$

where ω is the average angular velocity of the three motors when the vibrating system operates at stable state.

Assuming the instantaneous variation coefficients of φ, α_{12} and α_{23} are ε_1, ε_2 and ε_3 (ε_1, ε_2 and ε_3 are the functions of time t),

$$\begin{aligned}
\dot{\varphi} &= \omega(1+\varepsilon_1)\\
\dot{\alpha}_{12} &= \omega\varepsilon_2\\
\dot{\alpha}_{23} &= \omega\varepsilon_3
\end{aligned} \tag{8}$$

From (7) and (8), we can obtain the angular velocities of the three motors:

$$\begin{aligned}
\dot{\varphi}_1 &= \omega_1=\omega(1+\varepsilon_1+\frac{1}{2}\varepsilon_2)\\
\dot{\varphi}_2 &= \omega_2=\omega(1+\varepsilon_1-\frac{1}{2}\varepsilon_2)\\
\dot{\varphi}_3 &= \omega_3=\omega(1+\varepsilon_1-\frac{1}{2}\varepsilon_2-\varepsilon_3)
\end{aligned} \tag{9}$$

The angular accelerations of the three motors can be obtained by (9).

$$\begin{aligned}
\ddot{\varphi}_1 &= \omega(\dot{\varepsilon}_1+\frac{1}{2}\dot{\varepsilon}_2)\\
\ddot{\varphi}_2 &= \omega(\dot{\varepsilon}_1-\frac{1}{2}\dot{\varepsilon}_2)\\
\ddot{\varphi}_3 &= \omega(\dot{\varepsilon}_1-\frac{1}{2}\dot{\varepsilon}_2-\dot{\varepsilon}_3)
\end{aligned} \tag{10}$$

For the operating speed of the induction motor is slightly lower than synchronous speed, the influence of $\ddot{\varphi}_1$, $\ddot{\varphi}_2$ and $\ddot{\varphi}_3$ can be ignored when the system operate in a stable state. And then the prior three equations of (6) can be rewritten to (11).

$$\begin{aligned}
M\ddot{x}+f_x\dot{x}+k_x x &= m_1r_1\dot{\varphi}_1^{\,2}\cos\varphi_1+m_3r_3\dot{\varphi}_3^{\,2}\cos\varphi_3\\
&+m_2r_2\dot{\varphi}_2^{\,2}\cos\varphi_2\\
M\ddot{y}+f_y\dot{y}+k_y y &= m_1r_1\dot{\varphi}_1^{\,2}\sin\varphi_1+m_3r_3\dot{\varphi}_3^{\,2}\sin\varphi_3\\
&+m_2r_2\dot{\varphi}_2^{\,2}\sin\varphi_2\\
J\ddot{\psi}+f_\psi\dot{\psi}+k_\psi\psi &= f_1(\dot{\varphi}_1-\dot{\psi})+f_3(\dot{\varphi}_3-\dot{\psi})\\
&+f_2(\dot{\varphi}_2-\dot{\psi})\\
&+m_1l_1r_1\dot{\varphi}_1^{\,2}\sin(\varphi_1-\beta_1-\psi)\\
&+m_3l_3r_3\dot{\varphi}_3^{\,2}\sin(\varphi_3-\beta_3-\psi)\\
&+m_2l_2r_2\dot{\varphi}_2^{\,2}\sin(\varphi_2-\beta_2-\psi)
\end{aligned} \tag{11}$$

The effect of the damping to the amplitude of the vibrating system can be neglected because the damping constant is very small ($\xi < 0.07$) [6, 7]. So the response of x, y and ψ in (11) can be expressed approximately as

$$x \approx -\frac{A\cos\alpha_x}{m_x'\omega^2}\sin(\varphi+\alpha_x+\gamma_1)$$

$$y \approx -\frac{B\cos\alpha_y}{m_y'\omega^2}\sin(\varphi+\alpha_y+\gamma_2) \qquad (12)$$

$$\psi \approx -\frac{C\cos\alpha_\psi}{J_\psi'\omega^2}\sin(\varphi+\alpha_\psi+\gamma_3)$$

where

$$\alpha_x = \arctan(\frac{f_x}{m_x'\omega}),$$

$$\alpha_y = \arctan(\frac{f_y}{m_y'\omega}),$$

$$\alpha_\psi = \arctan(\frac{f_\psi}{J_\psi'\omega}),$$

$$m_x' = M - \frac{k_x}{\omega^2},$$

$$m_y' = M - \frac{k_y}{\omega^2},$$

$$J_\psi' = J - \frac{k_\psi}{\omega^2},$$

$$A = -\sqrt{\begin{aligned}&[\sqrt{(A_1 + A_2\cos\alpha_{12})^2 + A_2^2\sin^2\alpha_{12}}\\&- A_3\sin(\alpha_{12}+\alpha_{23}+a_1)]^2\\&+ A_3^{\,2}\cos^2(\alpha_{12}+\alpha_{23}+a_1)\end{aligned}},$$

$$B = \sqrt{\begin{aligned}&[\sqrt{(B_1 + B_2\cos\alpha_{12})^2 + B_2^2\sin^2\alpha_{12}}\\&+ B_3\cos(\alpha_{12}+\alpha_{23}+b_1)]^2\\&+ B_3^{\,2}\sin^2(\alpha_{12}+\alpha_{23}+b_1)\end{aligned}},$$

$$C = \sqrt{\begin{aligned}&[\sqrt{\begin{aligned}&[C_1 + C_2\cos(\alpha_{12}-\beta_1+\beta_2)]^2\\&+ C_2^2\sin^2(\alpha_{12}-\beta_1+\beta_2)\end{aligned}}\\&+ C_3\cos(\alpha_{12}+\alpha_{23}-\beta_1+\beta_3+c_1)]^2\\&+ C_3^{\,2}\sin^2(\alpha_{12}+\alpha_{23}-\beta_1+\beta_3+c_1)\end{aligned}},$$

$$\gamma_1 = 0.5\alpha_{12} + a_1 + a_2,$$

$$\gamma_2 = 0.5\alpha_{12} + b_1 + b_2,$$

$$\gamma_3 = 0.5\alpha_{12} + c_1 + c_2 - \beta_1,$$

$$A_1 = B_1 = m_1 r_1 \dot\varphi_1^{\,2},$$

$$A_2 = B_2 = m_2 r_2 \dot\varphi_2^{\,2},$$

$$A_3 = B_3 = m_3 r_3 \dot\varphi_3^{\,2},$$

$$C_1 = m_1 l_1 r_1 \dot\varphi_1^2,$$

$$C_2 = m_2 l_2 r_2 \dot\varphi_2^2,$$

$$C_3 = m_3 l_3 r_3 \dot\varphi_3^2,$$

$$a_1 = -\arctan\frac{A_1 + A_2\cos\alpha_{12}}{A_2\sin\alpha_{12}},$$

$$a_2 = -\arctan\frac{-A_3\cos(\alpha_{12}+\alpha_{23}+a_1)}{\begin{aligned}&[\sqrt{(A_1 + A_2\cos\alpha_{12})^2 + A_2^2\sin^2\alpha_{12}}\\&+ A_3\sin(\alpha_{12}+\alpha_{23}+a_1)]\end{aligned}},$$

$$b_1 = -\arctan\frac{B_2\sin\alpha_{12}}{B_1 + B_2\cos\alpha_{12}},$$

$$b_2 = -\arctan\frac{B_3\sin(\alpha_{12}+\alpha_{23}+b_1)}{\begin{aligned}&[\sqrt{(B_1 + B_2\cos\alpha_{12})^2 + B_2^2\sin^2\alpha_{12}}\\&+ B_3\cos(\alpha_{12}+\alpha_{23}+b_1)]\end{aligned}},$$

$$c_1 = -\arctan\frac{C_2\sin(\alpha_{12}-\beta_1+\beta_2)}{C_1 + C_2\cos(\alpha_{12}-\beta_1+\beta_2)},$$

$$c_2 = -\arctan\frac{C_3\sin(\alpha_{12}+\alpha_{23}-\beta_1+\beta_3+c_1)}{\begin{aligned}&[\sqrt{\begin{aligned}&[C_1 + C_2\cos(\alpha_{12}-\beta_1+\beta_2)]^2\\&+ C_2^2\sin^2(\alpha_{12}-\beta_1+\beta_2)\end{aligned}}\\&+ C_3\cos(\alpha_{12}+\alpha_{23}-\beta_1+\beta_3+c_1)]\end{aligned}}.$$

B. Self-synchronization Conditions of the Vibrating System

The overall kinetic energy of the vibrating system includes motional kinetic energy of the vibrating body, rotatory kinetic energy of the vibrating body surround its centroid, rotatory kinetic energy of the three eccentric lumps. The expression of the overall kinetic energy is shown in (13).

$$T = \frac{1}{2}M\dot x^2 + \frac{1}{2}M\dot y^2 + \frac{1}{2}J\dot\psi^2 + T_i \qquad i = 1, 2, 3 \qquad (13)$$

where T_i ($i = 1, 2, 3$) is rotatory kinetic energy of the three eccentric lumps, respectively. Rotatory kinetic energy of the eccentric lumps can be seen constant when motor operates in a stable state.

The expression of the potential energy is shown in (2).

By using Hamilton principle [9, 10], we can get the Hamilton action over a single period when the vibrating system operates in a stable state.

$$H = \int_0^{2\pi}(T - V)\mathrm{d}(\omega t) = \frac{\pi A^2\cos^2\alpha_x}{2m_x'\omega^2} + \frac{\pi B^2\cos^2\alpha_y}{2m_y'\omega^2} + \frac{\pi C^2\cos^2\alpha_\psi}{2J_\psi'\omega^2} + 2\pi T_i \qquad (14)$$

The elastic force and damping force of the vibrating system is far less than the inertia force and exciting force of the vibrating body. If the influence of elastic force and damping force is neglected, the dynamical system is affected by driving moment M_{gi} (i=1, 2, 3) and friction torque M_{fi} (i=1, 2, 3), besides potential force, i.e. gravitation. So the system is a holonomic nonconservative dynamical system. By using Hamilton principle of holonomic nonconservative dynamical system, we can obtain

$$\delta H + \int_0^{2\pi} \sum_{i=1}^{2} Q_i \delta q_i \mathrm{d}(\omega t) = 0 \qquad (15)$$

where Q_i is generalized force of the system, q_i is generalized coordinates.

In this system, α_{12} and α_{23} are generalized coordinates, so the expression of generalized force Q_1 and Q_2 can be shown in (16).

$$\begin{aligned} Q_1 &= \sum_{i=1}^{3}(M_{gi} - M_{fi})\frac{\partial \varphi_i}{\partial \alpha_{12}} = \frac{1}{2}(M_{g1} - M_{f1}) \\ &\quad -\frac{1}{2}(M_{g2} - M_{f2}) - \frac{1}{2}(M_{g3} - M_{f3}) \\ Q_2 &= \sum_{i=1}^{3}(M_{gi} - M_{fi})\frac{\partial \varphi_i}{\partial \alpha_{23}} = -(M_{g3} - M_{f3}) \end{aligned} \qquad (16)$$

Considering the independence of phase difference α_{12} and α_{23}, applying (14) and (16) into (15), we have

$$\begin{aligned} &-(E_1E_2 \sin\alpha_{12} - E_2E_3 \sin\alpha_{23})\omega^2\kappa_x \\ &+(E_1E_2 \sin\alpha_{12} + E_2E_3 \sin\alpha_{23})\omega^2\kappa_y \\ &+[E_1l_1E_2l_2 \sin(\alpha_{12} + \beta_1 + \beta_3) \\ &+E_2l_2E_3l_3 \sin(\alpha_{23} - \beta_2 - \beta_3)]\omega^2\kappa_\psi + W \\ &=0 \end{aligned} \qquad (17)$$

where

$E_1 = m_1r_1$, $E_2 = m_2r_2$, $E_3 = m_3r_3$,

$\kappa_x = \frac{\cos^2\alpha_x}{m'_x}$, $\kappa_y = \frac{\cos^2\alpha_y}{m'_y}$, $\kappa_\psi = \frac{\cos^2\alpha_\psi}{m'_\psi}$,

$W = (M_{g1} - M_{f1}) - (M_{g2} - M_{f2}) + (M_{g3} - M_{f3})$.

Equation (17) can be rewritten as

$$\sqrt{H_1^2 + H_2^2}\sin(\alpha_{12} + \delta_1) + \sqrt{H_3^2 + H_4^2}\sin(\alpha_{23} + \delta_2) - W = 0 \qquad (18)$$

where

$H_1 = E_1E_2\omega^2[\kappa_x + \kappa_y - \kappa_\psi l_1l_2\cos(\beta_1 + \beta_2)]$,

$H_2 = E_1l_1E_2l_2\omega^2\kappa_\psi l_1l_2\sin(-\beta_1 - \beta_2)$,

$H_3 = E_2E_3\omega^2[\kappa_x - \kappa_y - \kappa_\psi l_2l_3\cos(\beta_2 + \beta_3)]$,

$H_4 = E_2l_2E_3l_3\omega^2\kappa_\psi\sin(-\beta_2 - \beta_3)$.

Form (18), we can obtain

$$\sin(\alpha_{12} + \delta_1) = \frac{W - \sqrt{H_3^2 + H_4^2}\sin(\alpha_{23} + \delta_2)}{\sqrt{H_1^2 + H_2^2}} \qquad (19)$$

If the phase difference α_{12} and α_{23} can be stable over a single period $T = 2\pi/\omega$, the three motors of the vibrating system operate at the same rotational speed, the system is in a vibratory synchronization state. At this time, the phase difference α_{12} keep stability over a single period, thus, (19) must have a solution of α_{12}. Consequently, we have

$$\left|\frac{W - \sqrt{H_3^2 + H_4^2}\sin(\alpha_{23} + \delta_2)}{\sqrt{H_1^2 + H_2^2}}\right| \le 1 \qquad (20)$$

We can get (21) easily

$$\left|\frac{W - \sqrt{H_3^2 + H_4^2}\sin(\alpha_{23} + \delta_2)}{\sqrt{H_1^2 + H_2^2}}\right| \le \frac{|W| + \sqrt{H_3^2 + H_4^2}}{\sqrt{H_1^2 + H_2^2}} \qquad (21)$$

In addition, we define the ratio in (21) as D_1:

$$D_1 = \frac{|W| + \sqrt{H_3^2 + H_4^2}}{\sqrt{H_1^2 + H_2^2}} \qquad (22)$$

If we order

$$D_1 \le 1 \qquad (23)$$

then (20) will come into existence without fail. Inequation (23) is one of the synchronization conditions of the vibrating system.

By the same method, we can get

$$D_2 = \frac{|W| + \sqrt{H_1^2 + H_2^2}}{\sqrt{H_3^2 + H_4^2}} \le 1 \qquad (24)$$

Inequations (23) in company with (24), i.e. $D_1 \le 1$ and $D_2 \le 1$, make up of synchronization conditions of the vibrating system driven by three motors in reverse direction.

C. Stability Conditions of Synchronous Operation

The stability conditions of the vibrating system driven by three motors is analyzed based on extreme value theory of Hamilton action, stability criterion of system with function of many variables, extreme value theory of function. From stability of constrained system, we can know that there is minimum value in Hamilton action of true motion. Consequently, we can get stability conditions of synchronous operation of the vibrating system driven by three motors.

$$\begin{aligned} &\frac{\partial^2 H}{\partial \alpha_{12}^2} > 0 \\ &\frac{\partial^2 H}{\partial \alpha_{12}^2}\frac{\partial^2 H}{\partial \alpha_{23}^2} - \left[\frac{\partial^2 H}{\partial \alpha_{12}\partial \alpha_{23}}\right]^2 > 0 \end{aligned} \qquad (25)$$

Applying (15) into (25), we have

$$
\begin{aligned}
&E_3\sqrt{\begin{aligned}&[\kappa_x-\kappa_y-\kappa_\psi l_1 l_3\cos(\beta_1-\beta_3)]^2\\&+[\kappa_\psi l_1 l_3\sin(\beta_1-\beta_3)]^2\end{aligned}}\cos(\alpha_{12}+\alpha_{23}+\eta_2)\\
&-E_2\sqrt{\begin{aligned}&[\kappa_x+\kappa_y+\kappa_\psi l_1 l_2\cos(\beta_1++\beta_2)]^2\\&+[\kappa_\psi l_1 l_2\sin(\beta_1+\beta_2)]^2\end{aligned}}\cos(\alpha_{12}+\eta_1)\\
&>0
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
&E_1\sqrt{\begin{aligned}&[\kappa_x+\kappa_y+\kappa_\psi l_1 l_2\cos(\beta_1+\beta_2)]^2\\&+[\kappa_\psi l_1 l_2\sin(\beta_1+\beta_2)]^2\end{aligned}}\\
&\times\sqrt{\begin{aligned}&[\kappa_x-\kappa_y-\kappa_\psi l_1 l_3\cos(\beta_1-\beta_3)]^2\\&+[\kappa_\psi l_1 l_3\sin(\beta_1-\beta_3)]^2\end{aligned}}\\
&\times\cos(\alpha_{12}+\alpha_{23}+\eta_2)\cos(\alpha_{12}+\eta_1)\\
&-E_3\sqrt{\begin{aligned}&[\kappa_x-\kappa_y-\kappa_\psi l_1 l_3\cos(\beta_1-\beta_3)]^2\\&+[\kappa_\psi l_1 l_3\sin(\beta_1-\beta_3)]^2\end{aligned}}\\
&\times\sqrt{\begin{aligned}&[\kappa_x-\kappa_y-\kappa_\psi l_2 l_3\cos(\beta_2+\beta_3)]^2\\&+[\kappa_\psi l_2 l_3\sin(\beta_2+\beta_3)]^2\end{aligned}}\\
&\times\cos(\alpha_{12}+\alpha_{23}+\eta_2)\\
&-E_2\sqrt{\begin{aligned}&[\kappa_x+\kappa_y+\kappa_\psi l_1 l_2\cos(\beta_1+\beta_2)]^2\\&+[\kappa_\psi l_1 l_2\sin(\beta_1+\beta_2)]^2\end{aligned}}\\
&\times\sqrt{\begin{aligned}&[\kappa_x-\kappa_y-\kappa_\psi l_2 l_3\cos(\beta_2+\beta_3)]^2\\&+[\kappa_\psi l_2 l_3\sin(\beta_2+\beta_3)]^2\end{aligned}}\\
&\times\cos(\alpha_{12}+\eta_1)\cos(\alpha_{23}+\eta_3)\\
&>0
\end{aligned}
\tag{27}
$$

From Equations (26) and (27), we can obtain

$$E_3 P_3\cos(\alpha_{12}+\alpha_{23}+\eta_2)>E_2 P_2\cos(\alpha_{12}+\eta_1) \tag{28}$$

$$
\begin{aligned}
&P_3[E_1 P_2\cos(\alpha_{12}+\eta_1)-E_3 P_1]\cos(\alpha_{12}+\alpha_{23}+\eta_2)\\
&>E_2 P_1 P_2\cos(\alpha_{12}+\eta_1)\cos(\alpha_{23}+\eta_3)
\end{aligned}
\tag{29}
$$

where

$$P_1=\sqrt{\begin{aligned}&[\kappa_x-\kappa_y-\kappa_\psi l_2 l_3\cos(\beta_2+\beta_3)]^2\\&+[\kappa_\psi l_2 l_3\sin(\beta_2+\beta_3)]^2\end{aligned}},$$

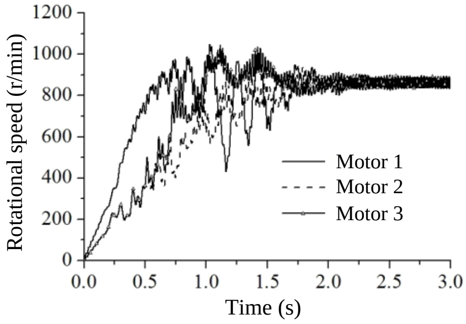
$$P_2=\sqrt{\begin{aligned}&[\kappa_x+\kappa_y+\kappa_\psi l_1 l_2\cos(\beta_1+\beta_2)]^2\\&+[\kappa_\psi l_1 l_2\sin(\beta_1+\beta_2)]^2\end{aligned}},$$

$$P_3=\sqrt{\begin{aligned}&[\kappa_x-\kappa_y-\kappa_\psi l_1 l_3\cos(\beta_1+\beta_3)]^2\\&+[\kappa_\psi l_1 l_3\sin(\beta_1-\beta_3)]^2\end{aligned}}.$$

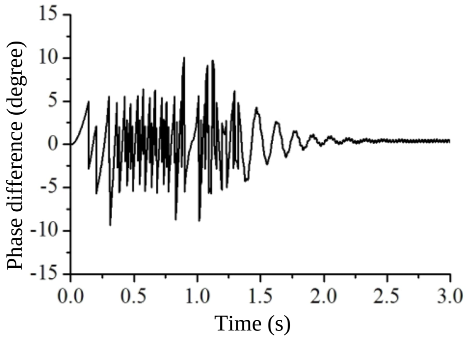
Inequations (28) in company with (29) make up of stability conditions of synchronous operation of the vibrating system driven by three motors.

III. Results of Computer Simulation and Discussion

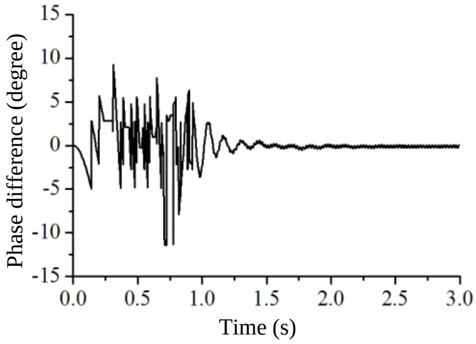
The vibrating system driven by three motors in the same direction is simulated by a computer program based on the self-synchronization and stability conditions of synchronous operation which are deduced in the above mathematical analysis. The results of computer simulation are shown in *Fig. 2*.



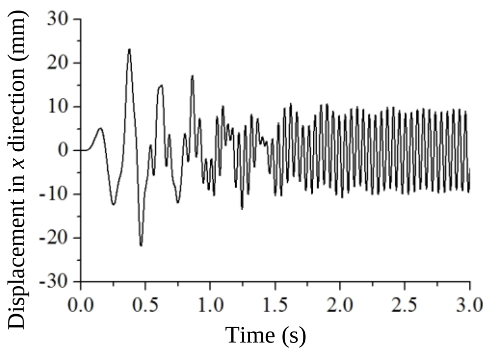
(a) Rotational speeds of the three motors



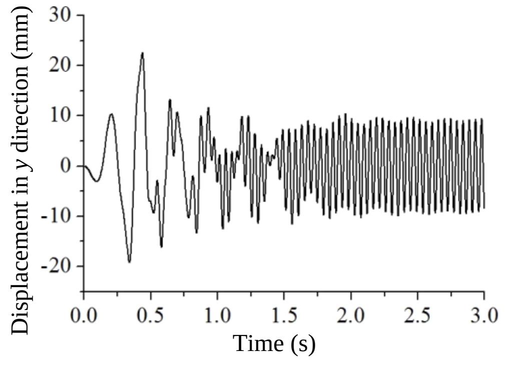
(b) Phase difference between motor 1 and motor 2



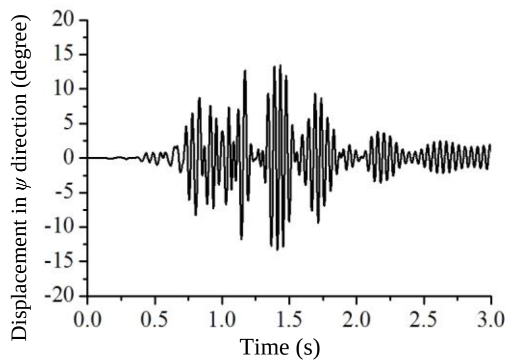
(c) Phase difference between motor 2 and motor 3



(d) Displacement in horizontal direction



(e) Displacement in vertical direction



(f) Displacement in rocking direction.

Fig. 2. *Results of computer simulation of a vibrating system driven by three motors in the same direction*

Fig. 2 (a) is rotational speeds of the three motors. *Fig. 2* (b) is phase difference between the first motor and the second one, and *Fig. 2* (c) is phase difference between the second motor and the third one. *Fig. 2* (d), (e) and (f) are the displacements of the system in x, y and ψ directions.

From *Fig. 2* we can see that there are great fluctuations in the displacement of the vibrating system when the system starts to operate. After operating a few seconds, the system is in a relatively stable synchronous operation state. This phenomenon result from that the phase difference between the motors does not reach a stable state at initial stage after the system's operating. At this time, the vibrating system has not reach the stability conditions of synchronous operation. Because the structure parameters of the vibrating system meet the self-synchronization and stability conditions of synchronous operation, consequently, we can see from *Fig. 2* that the phase difference between the motors reach a stable state and meet the self-synchronization and stability conditions of synchronous operation. Therefore, there are great fluctuations in the motion of the vibrating system at initial stage, but it reach periodic stable operation soon, and the vibrating system is in a stable synchronous operation state.

Because of the periodic change of the exciting force and the load torque, the phase difference between the motors is in a periodic fluctuating state. Phase difference between motor 1 and 2 fluctuate around 0.4°, and phase difference between motor 2 and 3 fluctuate around -0.1°. To sum up, the vibrating system reach a stable synchronous state, the self-synchronization and stability conditions of synchronous operation of the vibrating system are validated by simulation results.

IV. Conclusion

From *Fig. 2* we can see that the vibrating system driven by three motors in the same direction can operate in a stable synchronous state when the computer simulation is programmed based on the self-synchronization and stability conditions of synchronous operation which are deduced in the above mathematical analysis. When the three motors operate respectively at initial stage, the rotational speed and phase difference is not stable and the vibrating system can not operate in a stable state. After operating a few seconds, the phase difference get stable under interaction of the dynamic effect, and then the vibrating system reach a stable synchronous state. The results of computer simulation demonstrate that the vibrating system realizes speed synchronization and phase synchronization after operating a few seconds when the computer simulation is programmed based on the self-synchronization and stability conditions of synchronous operation. The computer simulation verifies the correctness of the mathematical analysis.

Self-synchronous vibrating system driven by three motors in the same direction is analyzed based on dynamic theory and computer simulation. The self-synchronization and stability conditions of synchronous operation of the vibrating system are deduced by using Hamilton principle.

The dynamic analysis of the self-synchronous vibrating system has significant theory and engineering value for solving engineering application problem, it can provide a good theoretical basis for the design of the self-synchronous vibrating system driven by several motors.

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