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Lossless Transmission Lines with Josephson Junction – Approximated Continuous Generalized Solutions

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***Abstract* — The present paper is devoted to the investigation of lossless transmission lines with Josephson junction. Such lines are described by first order hyperbolic system partial differential equations with sine nonlinearity. We formulate a mixed problem for this system with boundary conditions generated by a circuit corresponding to Josephson junction. We present the transformed mixed problem in an operator form, introduce approximated solution and obtain a sequence convergent to a generalized solution.**

Keywords — lossless transmission lines; Josephson junction; superconductivity; fixed point theorem; mixed problem; nonlinear hyperbolic system; sine Gordon equation.

I. Introduction

A lot of papers have been devoted to the investigation of lossless transmission lines terminated by nonlinear loads and their applications to RF-circuits (cf. for instance [1]-[13]). Here we consider a lossless transmission line with Josephson junction (cf. [14], [15]) arising in the problems of superconductivity. From mathematical point of view the lossless transmission line system with Josephson junction is a nonlinear hyperbolic system plus a relation between Josephson flux and voltage:

$$\frac{\partial u(x,t)}{\partial x} = -L\frac{\partial i(x,t)}{\partial t},$$
$$\frac{\partial i(x,t)}{\partial x} = -C\frac{\partial u(x,t)}{\partial t} - j_0 \sin\frac{2\pi\Phi(x,t)}{\Phi_0}, \qquad (1)$$
$$\frac{\partial \Phi(x,t)}{\partial t} = u(x,t)$$

$(x,t)\in\Pi = \{(x,t)\in R^2 : (x,t)\in[0,\Lambda]\times[0,T]\}.$

Here $u(x,t)$, $i(x,t)$ and $\Phi(x,t)$ are unknown functions − voltage, current and Josephson flux, L and C are prescribed specific parameters of the line, $\Lambda > 0$ is its length, $v = \frac{1}{\sqrt{LC}}$ − the speed of propagation, j_0 − maximal Josephson current per unit length, $\Phi_0 = \hbar/(2e) = 2.10^{15} W/m^2$ − flux induction quant, $K_J = 1/\Phi_0$ − Josephson constant.

The commonly accepted approach to solve the above system is to obtain sine Gordon equation which can be derived in the following way:

$$\frac{\partial^2\Phi(x,t)}{\partial x\partial t} = \frac{\partial u(x,t)}{\partial t} \Rightarrow \frac{\partial^2\Phi(x,t)}{\partial x\partial t} = -L\frac{\partial i(x,t)}{\partial t} \Rightarrow$$
$$\frac{\partial^3\Phi(x,t)}{\partial x^2\partial t} + L\frac{\partial^2 i(x,t)}{\partial t\partial x} = 0,$$
$$\frac{\partial i(x,t)}{\partial x} = -C\frac{\partial^2\Phi(x,t)}{\partial t^2} - j_0\sin\frac{2\pi\Phi(x,t)}{\Phi_0} \Rightarrow$$
$$L\frac{\partial^2 i(x,t)}{\partial x\partial t} = -LC\frac{\partial^3\Phi(x,t)}{\partial t^3} - Lj_0\frac{\partial}{\partial t}\left(\sin\frac{2\pi\Phi(x,t)}{\Phi_0}\right).$$

Therefore

$$\frac{\partial^3\Phi(x,t)}{\partial x^2\partial t} - LC\frac{\partial^3\Phi(x,t)}{\partial t^3} - Lj_0\frac{\partial}{\partial t}\left(\sin\frac{2\pi\Phi(x,t)}{\Phi_0}\right) = 0$$

or

$$\frac{\partial^2\Phi(x,t)}{\partial x^2} - LC\frac{\partial^2\Phi(x,t)}{\partial t^2} - Lj_0\sin\frac{2\pi\Phi(x,t)}{\Phi_0} = 0. \qquad (2)$$

The above transformations prove that if (1) has a solution then (2) is satisfied. It is quite obvious that the inverse assertion cannot be proved without additional assumptions. That is why we consider the original first order lossless transmission line system with sine nonlinearity (cf. Fig. 1):

$$\frac{\partial u(x,t)}{\partial t} + \frac{1}{C}\frac{\partial i(x,t)}{\partial x} = -\frac{j_0}{C}\sin\left(2\pi K_J\int_0^t u(x,s)ds\right)$$
$$\frac{\partial i(x,t)}{\partial t} + \frac{1}{L}\frac{\partial u(x,t)}{\partial x} = 0. \qquad (3)$$

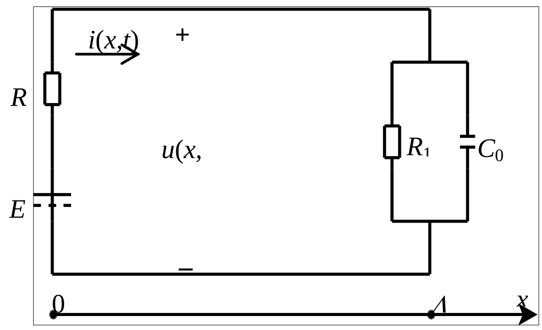

Fig. 1 *Losssless transmission line with Josephson junction*

First we assume that the resistive element R_1 (at the right end) is a linear one in order to show of how to overcome the difficulty generated by sine nonlinearity. We investigate the case of nonlinear resistive element in a next paper.

For (3) one can formulate the following mixed (initial-boundary value) problem: to find the unknown functions $u(x,t)$ and $i(x,t)$ in domain

$$\Pi=\{(x,t): x\in[0,\Lambda], t\in[0,T]\}$$

satisfying initial conditions

$$u(x,0)=u_0(x),\ i(x,0)=i_0(x),\ x\in[0,\Lambda] \tag{4}$$

and boundary conditions obtained from the loads (cf. Fig.1):

$$E(t)-u(0,t)-R_0 i(0,t)=0,\ t\in[0,T], \tag{5}$$

$$C_0\frac{du(\Lambda,t)}{dt}=i(\Lambda,t)-\frac{1}{R_1}u(\Lambda,t),\ t\in[0,T]. \tag{6}$$

Here $i_0(x), u_0(x)$ are prescribed initial functions − the current and voltage at the initial instant, $E(t)$ is a prescribed source function, R_0, R_1 and C_0 − specific parameters of the elements of the circuits.

First we transform the hyperbolic system in a diagonal form and then present the mixed problem for the new obtained hyperbolic system in an operator form (cf. [16]). The operator formulated is not strict contraction. Theorem 2.1 (cf. [17]) cannot be applied because the operator is not non-expansive. That is why we introduce a notion n-approximated solution. It is obtained as a unique fixed point of an operator with domain $\Pi_n=\{(x,t): x\in[0,\Lambda-(1/n)], t\in[0,T]\}$. This operator is strict contractive on Π_n and therefore has a unique fixed point in the space of continuous functions, namely n-approximated solution. Then we extend functions from Π_n on Π and choose a convergent subsequence. Its limit we call generalized solution of the mixed problem.

II. Diagonalization of the Hyperbolic System

Introducing denotations

$$U=\begin{bmatrix}u\\ i\end{bmatrix},\quad \frac{\partial U}{\partial t}\equiv U_t=\begin{bmatrix}u_t\\ i_t\end{bmatrix},\quad \frac{\partial U}{\partial x}\equiv U_x=\begin{bmatrix}u_x\\ i_x\end{bmatrix},$$

$$A=\begin{bmatrix}0 & 1/C\\ 1/L & 0\end{bmatrix},\quad \Gamma=\begin{bmatrix}-\frac{j_0}{C}\sin\left(2\pi K_J\int_0^t u(x,s)ds\right)\\ 0\end{bmatrix}$$

we can rewrite (3) in the form

$$\begin{bmatrix}u_t\\ i_t\end{bmatrix}+\begin{bmatrix}0 & 1/C\\ 1/L & 0\end{bmatrix}\begin{bmatrix}u_x\\ i_x\end{bmatrix}=\begin{bmatrix}-\frac{j_0}{C}\sin\left(2\pi K_J\int_0^t u(x,s)ds\right)\\ 0\end{bmatrix}$$

or

$$U_t+AU_x=\Gamma. \tag{7}$$

In order to transform the matrix $A=\begin{bmatrix}0 & 1/C\\ 1/L & 0\end{bmatrix}$ in diagonal form we solve the characteristic equation: $\begin{vmatrix}-\lambda & 1/C\\ 1/L & -\lambda\end{vmatrix}=0$ with roots $\lambda_1=1/\sqrt{LC},\ \lambda_2=-1/\sqrt{LC}$.

Eigen-vectors are solutions of the systems:

$$\left|\begin{matrix}-1/\left(\sqrt{LC}\right)\xi_1^{(1)} & +1/C\xi_2^{(1)}=0\\ 1/L\xi_1^{(1)} & -1/\left(\sqrt{LC}\right)\xi_2^{(1)}=0\end{matrix}\right.,\quad \left|\begin{matrix}1/\left(\sqrt{LC}\right)\xi_1^{(2)} & +1/C\xi_2^{(2)}=0\\ 1/L\xi_1^{(2)} & +1/\left(\sqrt{LC}\right)\xi_2^{(2)}=0\end{matrix}\right.,$$

that is:

$$\left(\xi_1^{(1)},\xi_2^{(1)}\right)=\left(\sqrt{C},\sqrt{L}\right),\quad \left(\xi_1^{(2)},\xi_2^{(2)}\right)=\left(-\sqrt{C},\sqrt{L}\right).$$

Denote by H the matrix formed by eigen-vectors $H=\begin{bmatrix}\sqrt{C} & \sqrt{L}\\ -\sqrt{C} & \sqrt{L}\end{bmatrix}$. Its inverse is $H^{-1}=\frac{1}{2}\begin{bmatrix}1/\sqrt{C} & -1/\sqrt{C}\\ 1/\sqrt{L} & 1/\sqrt{L}\end{bmatrix}$.

Then $A^{\text{can}}=HAH^{-1}=\begin{bmatrix}1/\sqrt{LC} & 0\\ 0 & -1/\sqrt{LC}\end{bmatrix}$.

Introduce new variables

$$Z=\begin{bmatrix}V(x,t)\\ I(x,t)\end{bmatrix},\quad U=\begin{bmatrix}u(x,t)\\ i(x,t)\end{bmatrix}, Z=HU, (U=H^{-1}Z)$$

or

$$\left|\begin{matrix}V(x,t)=\sqrt{C}\ u(x,t)+\sqrt{L}\ i(x,t)\\ I(x,t)=-\sqrt{C}\ u(x,t)+\sqrt{L}\ i(x,t)\end{matrix}\right.$$

and

$$\left|\begin{matrix}u(x,t)=\frac{V(x,t)}{2\sqrt{C}}-\frac{I(x,t)}{2\sqrt{C}}\\ i(x,t)=\frac{V(x,t)}{2\sqrt{L}}+\frac{I(x,t)}{2\sqrt{L}}.\end{matrix}\right. \tag{8}$$

Substituting $U=H^{-1}Z$ in (7) we obtain $\frac{\partial\left(H^{-1}Z\right)}{\partial t}+A\frac{\partial\left(H^{-1}Z\right)}{\partial x}=\Gamma$. But H^{-1} is a constant matrix that implies $H^{-1}Z_t+\left(AH^{-1}\right)Z_x=\Gamma$. Multiplying from the left by H we obtain

$$Z_t+A^{\text{can}}Z_x=H\Gamma. \tag{9}$$

Since

$$H\Gamma=\begin{bmatrix}\sqrt{C} & \sqrt{L}\\ -\sqrt{C} & \sqrt{L}\end{bmatrix}\cdot\begin{bmatrix}-\frac{j_0}{C}\sin\left(2\pi K_J\int_0^t u(x,s)ds\right)\\ 0\end{bmatrix}=$$

$$=\begin{bmatrix}-\frac{j_0}{\sqrt{C}}\sin\left(\frac{\pi K_J}{\sqrt{C}}\int_0^t\left(V(x,s)-I(x,s)\right)ds\right)\\ -\frac{j_0}{\sqrt{C}}\sin\left(\frac{\pi K_J}{\sqrt{C}}\int_0^t\left(V(x,s)-I(x,s)\right)ds\right)\end{bmatrix}$$

then (9) can be written in the form:

$$\begin{bmatrix}\frac{\partial V}{\partial t}\\ \frac{\partial I}{\partial t}\end{bmatrix}+\begin{bmatrix}\frac{1}{\sqrt{LC}} & 0\\ 0 & -\frac{1}{\sqrt{LC}}\end{bmatrix}\begin{bmatrix}\frac{\partial V}{\partial x}\\ \frac{\partial I}{\partial x}\end{bmatrix}=$$

$$=\begin{bmatrix}-\frac{j_0}{\sqrt{C}}\sin\left(\frac{\pi K_J}{\sqrt{C}}\int_0^t\left(V(x,s)-I(x,s)\right)ds\right)\\ -\frac{j_0}{\sqrt{C}}\sin\left(\frac{\pi K_J}{\sqrt{C}}\int_0^t\left(V(x,s)-I(x,s)\right)ds\right)\end{bmatrix}$$

or

$$\frac{\partial V}{\partial t}+\frac{1}{\sqrt{LC}}\frac{\partial V}{\partial x}=-\frac{j_0}{\sqrt{C}}\sin\left(\frac{\pi K_J}{\sqrt{C}}\int_0^t (V(x,s)-I(x,s))ds\right)$$
$$\frac{\partial I}{\partial t}-\frac{1}{\sqrt{LC}}\frac{\partial I}{\partial x}=-\frac{j_0}{\sqrt{C}}\sin\left(\frac{\pi K_J}{\sqrt{C}}\int_0^t (V(x,s)-I(x,s))ds\right). \quad (10)$$

The new initial conditions for $x\in[0,\Lambda]$ become:

$$V(x,0)=\sqrt{C}\,u(x,0)+\sqrt{L}\,i(x,0)=$$
$$\sqrt{C}\,u_0(x)+\sqrt{L}\,i_0(x)\equiv V_0(x),$$
$$I(x,0)=-\sqrt{C}\,u(x,0)+\sqrt{L}\,i(x,0)=$$
$$=-\sqrt{C}u_0(x)+\sqrt{L}\,i_0(x)\equiv I_0(x). \quad (11)$$

We obtain new boundary conditions substituting $u(x,t)$ and $i(x,t)$ from (8) into (5) and (6). Indeed, in view of

$$\left|\begin{aligned} u(0,t)&=\frac{V(0,t)}{2\sqrt{C}}-\frac{I(0,t)}{2\sqrt{C}}\\ i(0,t)&=\frac{V(0,t)}{2\sqrt{L}}+\frac{I(0,t)}{2\sqrt{L}}\end{aligned}\right., \quad \left|\begin{aligned} u(\Lambda,t)&=\frac{V(\Lambda,t)}{2\sqrt{C}}-\frac{I(\Lambda,t)}{2\sqrt{C}}\\ i(\Lambda,t)&=\frac{V(\Lambda,t)}{2\sqrt{L}}+\frac{I(\Lambda,t)}{2\sqrt{L}}\end{aligned}\right.$$

and $Z_0=\sqrt{L/C}$ we obtain

$$V(0,t)=\frac{2Z_0\sqrt{C}}{Z_0+R_0}E(t)+\frac{Z_0-R_0}{Z_0+R_0}I(0,t)$$
$$\frac{dI(\Lambda,t)}{dt}=\frac{dV(\Lambda,t)}{dt}-\frac{R_1-Z_0}{C_0Z_0R_1}V(\Lambda,t)-\frac{R_1+Z_0}{C_0Z_0R_1}I(\Lambda,t). \quad (12)$$

If we assume that

$$I(\Lambda,0)=I_0(\Lambda)=V_0(\Lambda)=V(\Lambda,0)=0$$

then

$$I(\Lambda,t)=V(\Lambda,t)-\frac{R_1-Z_0}{C_0Z_0R_1}\int_0^t V(\Lambda,s)ds-\frac{R_1+Z_0}{C_0Z_0R_1}\int_0^t I(\Lambda,s)ds.$$

Introducing denotations

$$\alpha=\frac{2Z_0}{Z_0+R_0},\ \beta=\frac{Z_0-R_0}{Z_0+R_0},\ \gamma_1=\frac{R_1-Z_0}{C_0Z_0R_1},\ \gamma_2=\frac{R_1+Z_0}{C_0Z_0R_1}$$

in (12) we obtain new boundary conditions:

$$V(0,t)=\alpha\sqrt{C}E(t)+\beta I(0,t),$$
$$I(\Lambda,t)=V(\Lambda,t)-\gamma_1\int_0^t V(\Lambda,s)ds-\gamma_2\int_0^t I(\Lambda,s)ds. \quad (13)$$

III. Operator Formulation of the Mixed Problem

The mixed problem is: to find a solution $(V(x,t),I(x,t))$ of the system

$$\frac{\partial V}{\partial t}+\frac{1}{\sqrt{LC}}\frac{\partial V}{\partial x}=-\frac{j_0}{\sqrt{C}}\sin\left(\frac{\pi}{\Phi_0\sqrt{C}}\int_0^t (V(x,s)-I(x,s))ds\right)$$
$$\frac{\partial I}{\partial t}-\frac{1}{\sqrt{LC}}\frac{\partial I}{\partial x}=-\frac{j_0}{\sqrt{C}}\sin\left(\frac{\pi}{\Phi_0\sqrt{C}}\int_0^t (V(x,s)-I(x,s))ds\right)$$

satisfying initial conditions

$$V(x,0)=V_0(x),\ I(x,0)=I_0(x),\ x\in[0,\Lambda] \quad (14)$$

and boundary conditions

$$V(0,t)=\alpha\sqrt{C}E(t)+\beta I(0,t),\quad t\in[0,T];$$
$$I(\Lambda,t)=V(\Lambda,t)-\gamma_1\int_0^t V(\Lambda,s)ds-\gamma_2\int_0^t I(\Lambda,s)ds;\ t\in[0,T].$$

Remark 3.1. Since we prove an existence of continuous solution we assume that the following Conformity condition **(CC)** is satisfied (cf. [16]):

$$V(0,0)=\alpha\sqrt{C}E(t)+\beta I(0,0),\ I(\Lambda,0)=V(\Lambda,0).$$

Indeed the following conditions

$$I(0,0)=0, V(0,0)=0,\ E(0)=0,\ I_0(\Lambda)=V_0(\Lambda)$$

imply **(CC)**.

We assign to mixed problem (14) the regularized problem (14-n): to find a solution $(V_n(x,t),I_n(x,t))$ of the system

$$\frac{\partial V}{\partial t}+\frac{1}{\sqrt{LC}}\frac{\partial V}{\partial x}=-\frac{j_0}{\sqrt{C}}\sin\left(\frac{\pi}{\Phi_0\sqrt{C}}\int_0^t (V(x,s)-I(x,s))ds\right),$$
$$\frac{\partial I}{\partial t}-\frac{1}{\sqrt{LC}}\frac{\partial I}{\partial x}=-\frac{j_0}{\sqrt{C}}\sin\left(\frac{\pi}{\Phi_0\sqrt{C}}\int_0^t (V(x,s)-I(x,s))ds\right)$$

for $(x,t)\in\Pi_n=\{(x,t)\in R^2:(x,t)\in[0,\Lambda-(1/n)]\times[0,T]\}$, satisfying initial conditions

$$V(x,0)=V_0(x); I(x,0)=I_0(x);\ x\in[0,\Lambda-(1/n)], n\in N \quad \text{(14-}n\text{)}$$

and boundary conditions

$$V(0,t)=\alpha\sqrt{C}E(t)+\beta I(0,t),\quad t\in[0,T];$$
$$I\left(\Lambda-\frac{1}{n},t\right)=V\left(\Lambda-\frac{1}{n},t\right)-\gamma_1\int_0^t V\left(\Lambda-\frac{1}{n},s\right)ds-$$
$$-\gamma_2\int_0^t I\left(\Lambda-\frac{1}{n},s\right)ds,\quad t\in[0,T].$$

Prior to define an operator corresponding to the mixed problem we consider Cauchy problem for the characteristics (cf. [16]) ($v=1/\sqrt{LC}$):

$$\frac{d\xi}{d\tau}=v,\ \xi(t)=x \quad \text{for each } (x,t)\in\Pi_n \Rightarrow$$
$$\varphi_V(\tau;x,t)=v\tau+x-vt, \quad (15)$$
$$\frac{d\xi}{d\tau}=-v,\ \xi(t)=x \quad \text{for each } (x,t)\in\Pi_n \Rightarrow$$
$$\varphi_I(\tau;x,t)=-v\tau+x+vt. \quad (16)$$

The functions $\lambda_V(x,t)=v>0$ and $\lambda_I(x,t)=-v<0$ are continuous ones and imply a uniqueness to the left from t_0 of the solution $x=\varphi_V(t;x_0,t_0)$ for $dx/dt=v$; $x(t_0)=x_0$ and respectively $x=\varphi_I(t;x_0,t_0)$ for $dx/dt=-v$; $x(t_0)=x_0$.

Denote by $\chi_V(x,t)$ the smallest value of τ such that the solution $\varphi_V(\tau;x,t)=v\tau+x-vt$ of (15) still belongs to Π_n and respectively by $\chi_I(x,t)$ the solution $\varphi_I(\tau;x,t)=-v\tau+x+vt$ of (16). If $\chi_V(x,t)>0$ then $\varphi_V(\chi_V(x,t);x,t)=0$ or $\varphi_V(\chi_V(x,t);x,t)=\Lambda-(1/n)$ and respectively if $\chi_I(x,t)>0$ then $\varphi_I(\chi_I(x,t);x,t)=0$ or $\varphi_I(\chi_I(x,t);x,t)=\Lambda-(1/n)$.

In our case

$$\chi_V(x,t)=\begin{cases}(vt-x)/v & \text{for } vt-x>0\\ 0 & \text{for } vt-x\le 0\end{cases};$$
$$\chi_I(x,t)=\begin{cases}(vt+x-(\Lambda-(1/n)))/v & \text{for } vt+x-(\Lambda-(1/n))>0\\ 0 & \text{for } vt+x-(\Lambda-(1/n))\le 0.\end{cases}$$

Remark 3.2. We notice that $\chi_V(x,t)$ and $\chi_I(x,t)$ are retarded functions in t, that is,

$$\chi_V(x,t) \le t - \frac{x}{v} \le t,$$

$$\chi_I(x,t) \le t + \frac{x-\Lambda}{v} \le t + \frac{\Lambda-(1/n)-\Lambda}{v} = t - \frac{1}{nv} < t.$$

It is easy to see that

$$\varphi_V(\tau;x,t) = v\tau + x - vt \;\Rightarrow\; \varphi_V(0;x,t) = x - vt,$$

$$\varphi_I(\tau;x,t) = -v\tau + x + vt \;\Rightarrow\; \varphi_I(0;x,t) = x + vt.$$

Introduce the sets

$$\Pi^n_{in,V} = \{(x,t)\in\Pi_n : \chi_V(x,t)=0\} \equiv \{(x,t)\in\Pi_n : x - vt \ge 0\};$$

$$\Pi^n_{in,I} = \{(x,t)\in\Pi_n : \chi_I(x,t)=0\} \equiv$$

$$\equiv \{(x,t)\in\Pi_n : x + vt - (\Lambda-(1/n)) \le 0\};$$

$$\Pi^n_{0V} = \{(x,t)\in\Pi_n : \chi_V(x,t) > 0,$$

$$\varphi_V(\chi_V(x,t);x,t) = \frac{v(vt-x)}{v} + x - vt = 0\};$$

$$\Pi^n_{0I} = \{(x,t)\in\Pi_n : \chi_I(x,t) > 0,$$

$$\varphi_I(\chi_I(x,t);x,t) = -\frac{v[vt + x - (\Lambda-(1/n))]}{v} + x + vt = 0\} = \varnothing;$$

$$\Pi^n_{\Lambda V} = \{(x,t)\in\Pi_n : \chi_V(x,t) > 0,$$

$$\varphi_V(\chi_V(x,t);x,t) = \frac{v(vt-x)}{v} + x - vt = \Lambda - 1/n\} = \varnothing;$$

$$\Pi^n_{\Lambda I} = \{(x,t)\in\Pi_n : \chi_I(x,t) > 0,$$

$$\varphi_I(\chi_I(x,t);x,t) = \frac{-v[vt + x - (\Lambda-(1/n))]}{v} + x + vt = \Lambda - 1/n\}.$$

Prior to present problem (14) in an operator form we introduce

$$\Phi_V(V,I)(x,t) = \begin{cases} V_0(\varphi_V(0;x,t)), & (x,t)\in\Pi^n_{in,V} \\ \Phi_{0V}(V,I)(\chi_V(x,t)), & (x,t)\in\Pi^n_{0V} \\ \Phi_{\Lambda V}(V,I)(\chi_V(x,t)), & (x,t)\in\Pi^n_{\Lambda V}\end{cases} =$$

$$= \begin{cases} V_0(x-vt)), & (x,t)\in\Pi^n_{in,V} \\ \Phi_{0V}(V,I)(\chi_V(x,t)), & (x,t)\in\Pi^n_{0V}\end{cases}$$

and

$$\Phi_I(V,I)(x,t) = \begin{cases} I_0(\varphi_I(0;x,t)), & (x,t)\in\Pi^n_{in,I} \\ \Phi_{0I}(V,I)(\chi_I(x,t)), & (x,t)\in\Pi^n_{0I} \\ \Phi_{\Lambda I}(V,I)(\chi_I(x,t)), & (x,t)\in\Pi^n_{\Lambda I}\end{cases} =$$

$$= \begin{cases} I_0(x+vt)), & (x,t)\in\Pi^n_{in,I} \\ \Phi_{\Lambda I}(V,I)(\chi_I(x,t)), & (x,t)\in\Pi^n_{\Lambda I}\end{cases}$$

or

$$\Phi_V(V,I)(x,t) = \begin{cases} V_0(x-vt), & (x,t)\in\Pi^n_{in,V} \\ \alpha\sqrt{C}E(\chi_V) + \beta I(0,\chi_V), & (x,t)\in\Pi^n_{0V}\end{cases}$$

$$\Phi_I(V,I)(x,t) =$$

$$= \begin{cases} I_0(x+vt)), & (x,t)\in\Pi^n_{in,I} \\ V(x,t) - \gamma_1\int_0^t V(x,s)ds - \gamma_2\int_0^t I(x,s)ds, & (x,t)\in\Pi^n_{\Lambda I}.\end{cases}$$

So we assign to the above mixed problem the following system of operator equations

$$V = B_V(V,I),\quad I = B_I(V,I), \tag{17}$$

where

$$B_V(V,I)(x,t) :=$$

$$= \begin{cases} V_0(x-vt), & (x,t)\in\Pi^n_{in,V} \\ \Phi_V(V,I)(x,t) - \frac{j_0}{\sqrt{C}}\int_{\chi_V(x,t)}^t \sin\left(\frac{\pi}{\Phi_0\sqrt{C}}\int_0^\tau (V(x,s)-I(x,s))ds\right)d\tau, & (x,t)\in\Pi^n_{0V}\end{cases}$$

$$B_I(V,I)(x,t) :=$$

$$= \begin{cases} I_0(x+vt)), & (x,t)\in\Pi^n_{in,I} \\ \Phi_I(V,I)(x,t) - \frac{j_0}{\sqrt{C}}\int_{\chi_I(x,t)}^t \sin\left(\frac{\pi}{\Phi_0\sqrt{C}}\int_0^\tau (V(x,s)-I(x,s))ds\right)d\tau, & (x,t)\in\Pi^n_{\Lambda I}.\end{cases}$$

Remark 3.3. We restrict ourselves on the subset

$$\Pi_n = \{(x,t)\in R^2 : (x,t)\in[0,\Lambda-(1/n)]\times[0,T]\}\subset\Pi$$

because (as we shall see below) operator (17) is non-expansive on Π, while it is a strict contraction on Π_n. So we get an existence of a unique solution defined on

$$\Pi_n = \{(x,t)\in R^2 : (x,t)\in[0,\Lambda-(1/n)]\times[0,T]\}\subset\Pi$$

for every fixed $n\in N$.

Introduce the function sets

$$M_V = \{V\in C(\Pi) : |V(x,t)| \le V_0 e^{\mu t}, x\in[0,\Lambda], t\in[0,T]\},$$

$$M_I = \{I\in C(\Pi) : |I(x,t)| \le I_0 e^{\mu t}, x\in[0,\Lambda], t\in[0,T]\},$$

$$M_{V,n} = \{V\in C(\Pi_n) : |V(x,t)| \le V_0 e^{\mu t}, x\in[0,\Lambda-(1/n)], t\in[0,T]\},$$

$$M_{I,n} = \{I\in C(\Pi_n) : |I(x,t)| \le I_0 e^{\mu t}, x\in[0,\Lambda-(1/n)], t\in[0,T]\},$$

where V_0, I_0, μ are positive constants.

It is easy to verify that the set $M_V\times M_I$ turns out into a complete metric space with respect to the metric:

$$\rho((V,I),(\bar V,\bar I)) = \max\{\rho(V,\bar V), \rho(I,\bar I)\},$$

where

$$\rho(V,\bar V) = \sup\{e^{-\mu t}|V(x,t)-\bar V(x,t)| : (x,t)\in\Pi\},$$

$$\rho(I,\bar I) = \sup\{e^{-\mu t}|I(x,t)-\bar I(x,t)| : (x,t)\in\Pi\}$$

and constant $\mu > 0$ can be chosen sufficiently large.

The set $M_{V,n}\times M_{I,n}$ is endowed with induced metric on Π_n, namely

$$\rho_n((V,I),(\bar V,\bar I)) = \max\{\rho_n(V,\bar V), \rho_n(I,\bar I)\},$$

$$\rho_n(V,\bar V) = \sup\{e^{-\mu t}|V(x,t)-\bar V(x,t)| : (x,t)\in\Pi_n\}.$$

IV. Existence-Uniqueness of an Approximated Continuous Solution

Theorem 4.1. Let the following conditions be fulfilled:

4.1) $|E(t)| \le E_0,\ t\in[0,T];\quad |V_0(x)| \le V_{00};\ |I_0(x)| \le I_{00},$ $x\in[0,\Lambda-(1/n)]$;

4.2) $\max\{V_{00};\alpha\sqrt{C}E_0 + |\beta| I_0\} + \frac{j_0\pi(V_0+I_0)}{\mu^2\Phi_0 C} \le V_0$;

4.3)

$$\max\left\{I_{00};\ V_0 e^{-\frac{\mu}{nv}} + e^{-\frac{\mu}{nv}}\frac{|\gamma_1|V_0+|\gamma_2|I_0}{\mu}\right\} + \frac{j_0\pi(V_0+I_0)}{\mu^2\Phi_0 C} \le I_0,$$

$$K_V = |\beta| + \frac{2j_0\pi}{\mu^2\Phi_0 C} < 1;\ K_I = e^{-\frac{\mu}{nv}}\left(1+\frac{|\gamma|}{\mu}\right) + \frac{2j_0\pi}{\mu^2\Phi_0 C} < 1$$

for sufficiently large $\mu > 0$.

Then there exists a unique $C(\Pi_n)$-solution of (14-n).

Proof: We establish the operator B maps the set $M_{V,n}\times M_{I,n}$ into itself.

We notice that $B_V(x,t)$ and $B_I(x,t)$ are continuous functions. We have to show that

$|B_V(V,I)(x,t)| \le V_0$, $|B_I(V,I)(x,t)| \le I_0$.

Indeed

$$|B_V(V,I)(x,t)| \le |\Phi_V(x,t)| +$$

$$+\frac{j_0}{\sqrt{C}}\int_{\chi_V(x,t)}^{t}\left|\sin\left(\frac{\pi}{\Phi_0\sqrt{C}}\int_0^{\tau}(V(x,s)+I(x,s))ds\right)\right|d\tau \le$$

$$\le\left\{\frac{|V_0(x-vt)|}{\alpha\sqrt{C}|E(\chi_V(x,t))|+|\beta||I(0,\chi_V(x,t))|}\right. +$$

$$+\frac{j_0\pi}{\Phi_0 C}\int_{\chi_V(x,t)}^{t}\int_0^{\tau}(|V(x,s)|+|I(x,s)|)ds\,d\tau \le$$

$$\le\left\{\frac{V_{00}}{\alpha\sqrt{C}E_0+|\beta|I_0}\right. + \frac{j_0\pi}{\Phi_0 C}(V_0+I_0)\int_{\chi_V(x,t)}^{t}\int_0^{\tau}e^{\mu s}ds\,d\tau \le$$

$$\le e^{\mu t}\max\left\{V_{00};\alpha\sqrt{C}E_0+|\beta|I_0\right\}+\frac{j_0\pi(V_0+I_0)}{\mu^2\Phi_0 C}\left(e^{\mu t}-e^{\mu\chi_V}\right)\le V_0 e^{\mu t}$$

and analogously

$$|B_I(V,I)(x,t)| \le \left\{\frac{|I_0(x+vt)|}{|V(\Lambda-(1/n),\chi_I)|+|\gamma_1|\int_0^{\chi_I}V_0e^{\mu s}ds+|\gamma_2|\int_0^{\chi_I}I_0e^{\mu s}ds}\right. +$$

$$+\frac{j_0}{\sqrt{C}}\left|\int_{\chi_I(x,t)}^{t}\sin\left(\frac{\pi}{\Phi_0\sqrt{C}}\int_0^{\tau}(V(x,s)-I(x,s))ds\right)d\tau\right| \le$$

$$\le\left\{\frac{I_{00}}{V_0e^{\mu\chi_I}+e^{\mu\chi_I}\frac{|\gamma_1|V_0+|\gamma_2|I_0}{\mu}}\right. + \frac{j_0\pi(V_0+I_0)}{\mu^2\Phi_0 C}e^{\mu t} \le$$

$$\le\max\left\{I_{00};\ V_0e^{\mu\left(t-\frac{1}{nv}\right)}+e^{\mu\left(t-\frac{1}{nv}\right)}\frac{|\gamma_1|V_0+|\gamma_2|I_0}{\mu}\right\}+$$

$$+\frac{j_0\pi(V_0+I_0)}{\mu^2\Phi_0 C}e^{\mu t} \le$$

$$\le e^{\mu t}\max\left\{I_{00};\ V_0e^{-\frac{\mu}{nv}}+e^{e^{-\frac{\mu}{nv}}}\frac{|\gamma_1|V_0+|\gamma_2|I_0}{\mu}\right\}+$$

$$+\frac{j_0\pi(V_0+I_0)}{\mu^2\Phi_0 C}e^{\mu t} \le I_0 e^{\mu t}.$$

Operator B is a strict contraction:

$$\left|B_V(V,I)(x,t)-B_V(\bar V,\bar I)(x,t)\right| \le \left|\Phi_V(V,I)(x,t)-\Phi_V(\bar V,\bar I)(x,t)\right|+$$

$$+\frac{j_0\pi}{\Phi_0 C}\int_{\chi_V(x,t)}^{t}\left(\int_0^{\tau}|V(x,s)-\bar V(x,s)|ds+\int_0^{\tau}|I(x,s)-\bar I(x,s)|ds\right)d\tau \le$$

$$\le|\beta|\left|I(0,\chi_V(x,t))-\bar I(0,\chi_V(x,t))\right|e^{-\mu\chi_V(x,t)}e^{\mu\chi_V(x,t)}+$$

$$+\frac{j_0\pi}{\Phi_0 C}\left(\int_{\chi_V(x,t)}^{t}\left(\int_0^{\tau}|V(x,s)-\bar V(x,s)|e^{-\mu s}e^{\mu s}ds+\right.\right.$$

$$\left.\left.+\int_0^{\tau}|I(x,s)-\bar I(x,s)|e^{-\mu s}e^{\mu s}ds\right)\right)d\tau \le$$

$$\le|\beta|\rho_n(I,\bar I)e^{\mu\chi_V(x,t)}+\frac{j_0\pi\left(\rho_n(V,\bar V)+\rho_n(I,\bar I)\right)}{\Phi_0 C}\int_{\chi_V(x,t)}^{t}\int_0^{\tau}e^{\mu s}ds\,d\tau \le$$

$$\le\left(|\beta|e^{\mu t}+\frac{2j_0\pi\left(e^{\mu t}-e^{\mu\chi_V}\right)}{\mu^2\Phi_0 C}\right)\rho_n((V,I),(\bar V,\bar I)) \le$$

$$\le e^{\mu t}\left(|\beta|+\frac{2j_0\pi}{\mu^2\Phi_0 C}\right)\rho_n((V,I),(\bar V,\bar I)).$$

It follows

$$\rho_n(B_I(V,I),B_I(\bar V,\bar I)) \le \left(|\beta|+\frac{2j_0\pi}{\mu^2\Phi_0 C}\right)\rho_n((V,I),(\bar V,\bar I))$$

$$\equiv K_V\rho_n((V,I),(\bar V,\bar I)).$$

For the second component we obtain

$$\left|B_I(V,I)(x,t)-B_I(\bar V,\bar I)(x,t)\right| \le \left|\Phi_I(V,I)(x,t)-\Phi_I(\bar V,\bar I)(x,t)\right|+$$

$$+\frac{j_0\pi}{\Phi_0 C}\left(\int_{\chi_I(x,t)}^{t}\left(\int_0^{\tau}|V(x,s)-\bar V(x,s)|ds+\int_0^{\tau}|I(x,s)-\bar I(x,s)|ds\right)d\tau\right) \le$$

$$\le\left|V(\Lambda-(1/n),\chi_I)-\bar V(\Lambda-(1/n),\chi_I)\right|e^{-\mu\chi_I}e^{\mu\chi_I}+$$

$$+|\gamma_1|\int_0^{\chi_I}\left|V(\Lambda-(1/n),s)-\bar V(\Lambda-(1/n),s)\right|e^{-\mu s}e^{\mu s}ds+$$

$$+|\gamma_2|\int_0^{\chi_I}\left|I(\Lambda-(1/n),s)-\bar I(\Lambda-(1/n),s)\right|e^{-\mu s}e^{\mu s}ds+$$

$$+\frac{j_0\pi}{\Phi_0 C}\left(\int_{\chi_I(x,t)}^{t}\left(\int_0^{\tau}|V(x,s)-\bar V(x,s)|e^{-\mu s}e^{\mu s}ds+\right.\right.$$

$$\left.\left.+\int_0^{\tau}|I(x,s)-\bar I(x,s)|e^{-\mu s}e^{\mu s}ds\right)\right)d\tau \le$$

$$\le\rho_n(V,\bar V)e^{\mu\chi_I}+$$

$$+|\gamma_1|\rho_n(V,\bar V)\int_0^{\chi_I}e^{\mu s}ds+|\gamma_2|\rho_n(I,\bar I)\int_0^{\chi_I}e^{\mu s}ds+$$

$$+\frac{j_0\pi\left(\rho_n(V,\bar V)+\rho_n(I,\bar I)\right)}{\Phi_0 C}\int_{\chi_I(x,t)}^{t}\int_0^{\tau}e^{\mu s}ds\,d\tau \le$$

$$\leq e^{\mu t}\rho_n(V,\bar{V})e^{-\frac{\mu}{nv}}+|\gamma_1|\rho_n(V,\bar{V})\frac{e^{\mu\chi_I}-1}{\mu}+$$

$$+|\gamma_2|\rho_n(I,\bar{I})\frac{e^{\mu\chi_I}-1}{\mu}+$$

$$+\frac{j_0\pi\left(\rho_n(V,\bar{V})+\rho_n(I,\bar{I})\right)}{\Phi_0 C}\int_{\chi_I(x,t)}^{t}\frac{e^{\mu\tau}-1}{\mu}d\tau\leq$$

$$\leq\left(e^{\mu t}e^{-\frac{\mu}{nv}}+e^{\mu t}e^{-\frac{\mu}{nv}}\frac{|\gamma_1|+|\gamma_2|}{\mu}+\frac{2j_0\pi}{\mu^2\Phi_0 C}\left(e^{\mu t}-e^{\mu\chi_I}\right)\right)\times$$

$$\times\rho_n((V,I),(\bar{V},\bar{I}))\leq$$

$$\leq e^{\mu t}\left(e^{-\frac{\mu}{nv}}+e^{-\frac{\mu}{nv}}\frac{|\gamma_1|+|\gamma_2|}{\mu}+\frac{2j_0\pi}{\mu^2\Phi_0 C}\right)\rho_n((V,I),(\bar{V},\bar{I})).$$

It follows

$$\rho_n(B_I(V,I),B_I(\bar{V},\bar{I}))\leq\left(e^{-\frac{\mu}{nv}}+\frac{|\gamma_1|+|\gamma_2|}{\mu}e^{-\frac{\mu}{nv}}+\frac{2j_0\pi}{\mu^2\Phi_0 C}\right)\times$$

$$\times\rho_n((V,I),(\bar{V},\bar{I}))\equiv K_I\rho_n((V,I),(\bar{V},\bar{I})).$$

Consequently

$$\rho_n\left((B_V(V,I),B_I(V,I)),(B_V(\bar{V},\bar{I}),B_I(\bar{V},\bar{I}))\right)\leq$$

$$\leq\max\{K_V;K_I\}\rho_n((V,I),(\bar{V},\bar{I}))\equiv K\rho_n((V,I),(\bar{V},\bar{I})).$$

Operator B is contractive one for sufficiently large $\mu>0$ and (14-n) has a unique solution (V_n,I_n) belonging to $M_{V,n}\times M_{I,n}$.

Theorem 4.1 is thus proved.

V. Existence of Solution of the Mixed Problem

Let us introduce the subsets

$$M_V^l=\{V\in M_V:|V(x,t)-V(\bar{x},\bar{t})|\leq l_V(|x-\bar{x}|+|t-\bar{t}|),$$
$$x,\bar{x}\in[0,\Lambda],t,\bar{t}\in[0,T]\};$$

$$M_I^l=\{I\in M_I:|I(x,t)-I(\bar{x},\bar{t})|\leq l_I(|x-\bar{x}|+|t-\bar{t}|),$$
$$x,\bar{x}\in[0,\Lambda],t,\bar{t}\in[0,T]\}.$$

For sufficiently large fixed n in accordance of Theorem 4.1 we obtain a unique solution (V_n,I_n) on

$$M_{V,n}^l=$$
$$=\{V\in M_{V,n}:|V(x,t)-V(\bar{x},\bar{t})|\leq l_V(|x-\bar{x}|+|t-\bar{t}|)\}\times M_{I,n}^l=$$
$$=\{I\in M_{I,n}:|I(x,t)-I(\bar{x},\bar{t})|\leq l_I(|x-\bar{x}|+|t-\bar{t}|)\}$$

for every $n\in N$. We extend the functions (V_n,I_n) on the whole domain $\Pi=[0,\Lambda]\times[0,T]$ such that extensions $(\tilde{V}_n,\tilde{I}_n)\in M_V^l\times M_I^l$.

For instance

$$\tilde{V}_n(x,t)=\begin{cases}V_n(x,t), & (x,t)\in[0;\Lambda-(1/n)]\times[0;T]\\ V_n(\Lambda-(1/n),t), & (x,t)\in[\Lambda-(1/n);\Lambda]\times[0;T]\end{cases},$$

$$\tilde{I}_n(x,t)=\begin{cases}I_n(x,t), & (x,t)\in[0;\Lambda-(1/n)]\times[0;T]\\ I_n(\Lambda-(1/n),t), & (x,t)\in[\Lambda-(1/n);\Lambda]\times[0;T]\end{cases}.$$

Now we are able to state the problem for existence of $\lim_{n\to\infty}(\tilde{V}_n,\tilde{I}_n)$ in the topology of $M_V^l\times M_I^l$. The sequence $(\tilde{V}_n,\tilde{I}_n)$ is bounded, but we are not sure that it is convergent. In view of Arzela-Ascoli theorem, we can choose a convergent subsequence provided $(\tilde{V}_n,\tilde{I}_n)$ is equicontinuous family of functions.

Indeed, if we add conditions

E1) $|\beta|l_I+\frac{2j_0}{\sqrt{C}}\leq l_V$;

E2) $l_V+|\gamma_1|V_0+|\gamma_2|I_0+\frac{2j_0}{\sqrt{C}}\leq l_I$

then $\{B_V(V,I)(x,t),B_I(V,I)(x,t)\}$ forms an equi-continuous family of functions.

In fact

$$|V_n(x,t)-V_n(\bar{x},\bar{t})|=|B_V(V_n,I_n)(x,t)-B_V(V_n,I_n)(\bar{x},\bar{t})|\leq$$

$$\leq|\Phi_V(V_n,I_n)(x,t)-\Phi_V(V_n,I_n)(\bar{x},\bar{t})|+$$

$$+\frac{j_0}{\sqrt{C}}\left|\int_{\chi_V(x,t)}^{t}\sin\left(\frac{\pi}{\Phi_0\sqrt{C}}\int_0^{\tau}(V_n(x,s)-I_n(x,s))ds\right)d\tau-\right.$$

$$\left.-\int_{\chi_V(\bar{x},\bar{t})}^{\bar{t}}\sin\left(\frac{\pi}{\Phi_0\sqrt{C}}\int_0^{\tau}(V_n(x,s)-I_n(x,s))ds\right)d\tau\right|\leq$$

$$\leq|\beta||I(0,\chi_V(x,t))-I(0,\chi_V(\bar{x},\bar{t}))|+$$

$$+\frac{j_0}{\sqrt{C}}\left(|t-\bar{t}|+|\chi_V(x,t)-\chi_V(\bar{x},\bar{t})|\right)\leq$$

$$\leq|\beta|l_I|\chi_V(x,t)-\chi_V(\bar{x},\bar{t})|+\frac{j_0}{\sqrt{C}}|t-\bar{t}|+$$

$$+\frac{j_0}{\sqrt{C}}|\chi_V(x,t)-\chi_V(\bar{x},\bar{t})|\leq$$

$$\leq\left(|\beta|l_I+\frac{j_0}{\sqrt{C}}\right)|\chi_V(x,t)-\chi_V(\bar{x},\bar{t})|+\frac{j_0}{\sqrt{C}}|t-\bar{t}|\leq$$

$$\leq\left(|\beta|l_I+\frac{j_0}{\sqrt{C}}\right)\left(|t-\bar{t}|+\frac{1}{v}|x-\bar{x}|\right)+\frac{j_0}{\sqrt{C}}|t-\bar{t}|\leq$$

$$\leq\left(|\beta|l_I+\frac{2j_0}{\sqrt{C}}\right)|t-\bar{t}|+\frac{1}{v}\left(|\beta|l_I+\frac{j_0}{\sqrt{C}}\right)|x-\bar{x}|\leq$$

$$\leq\max\left\{|\beta|l_I+\frac{2j_0}{\sqrt{C}};\frac{1}{v}\left(|\beta|l_I+\frac{2j_0}{\sqrt{C}}\right)\right\}\left(|t-\bar{t}|+|x-\bar{x}|\right)\leq$$

$$\leq\left(|\beta|l_I+\frac{2j_0}{\sqrt{C}}\right)\left(|t-\bar{t}|+|x-\bar{x}|\right)\leq l_V\left(|t-\bar{t}|+|x-\bar{x}|\right)$$

and

$$|I_n(x,t)-I_n(\bar{x},\bar{t})|=|B_I(V_n,I_n)(x,t)-B_I(V_n,I_n)(\bar{x},\bar{t})|\leq$$

$$\leq|\Phi_I(V_n,I_n)(x,t)-\Phi_I(V_n,I_n)(\bar{x},\bar{t})|+$$

$$+\frac{j_0}{\sqrt{C}}\left|\int_{\chi_I(x,t)}^{t}\sin\left(\frac{\pi}{\Phi_0\sqrt{C}}\int_0^{\tau}(V_n(x,s)-I_n(x,s))ds\right)d\tau-\right.$$

$$\left.-\int_{\chi_I(\bar{x},\bar{t})}^{\bar{t}}\sin\left(\frac{\pi}{\Phi_0\sqrt{C}}\int_0^{\tau}(V_n(x,s)-I_n(x,s))ds\right)d\tau\right|\leq$$

$$\le \left|V_n(\Lambda-(1/n),\chi_I(x,t))-V_n(\Lambda-(1/n),\chi_I(\bar{x},\bar{t}))\right|+$$
$$+|\gamma_1|\left|\int_{\chi_I(\bar{x},\bar{t})}^{\chi_I(x,t)}V_n(\Lambda-(1/n),s)ds\right|+|\gamma_2|\left|\int_{\chi_I(\bar{x},\bar{t})}^{\chi_I(x,t)}I_n(\Lambda-(1/n),s)ds\right|+$$
$$+\frac{j_0}{\sqrt{C}}\left|\int_{\chi_I(x,t)}^{t}\sin\left(\frac{\pi}{\Phi_0\sqrt{C}}\int_0^{\tau}(V_n(x,s)-I_n(x,s))ds\right)d\tau-\right.$$
$$\left.-\int_{\chi_I(\bar{x},\bar{t})}^{\bar{t}}\sin\left(\frac{\pi}{\Phi_0\sqrt{C}}\int_0^{\tau}(V_n(x,s)-I_n(x,s))ds\right)d\tau\right|\le$$
$$\le l_V|\chi_I(x,t)-\chi_I(\bar{x},\bar{t})|+|\gamma_1|\left|\int_{\chi_I(\bar{x},\bar{t})}^{\chi_I(x,t)}V_0ds\right|+|\gamma_2|\left|\int_{\chi_I(\bar{x},\bar{t})}^{\chi_I(x,t)}I_0ds\right|+$$
$$+\frac{j_0}{\sqrt{C}}\left(|t-\bar{t}|+|\chi_I(x,t)-\chi_I(\bar{x},\bar{t})|\right)\le$$
$$\le l_V|\chi_I(x,t)-\chi_I(\bar{x},\bar{t})|+$$
$$+(|\gamma_1|V_0+|\gamma_2|I_0)|\chi_I(x,t)-\chi_I(\bar{x},\bar{t})|+$$
$$+\frac{j_0}{\sqrt{C}}\left(|t-\bar{t}|+|\chi_I(x,t)-\chi_I(\bar{x},\bar{t})|\right)\le$$
$$\le\left(l_V+(|\gamma_1|V_0+|\gamma_2|I_0)+\frac{j_0}{\sqrt{C}}\right)\left(|t-\bar{t}|+\frac{1}{v}|x-\bar{x}|\right)+$$
$$+\frac{j_0}{\sqrt{C}}|t-\bar{t}|\le\left(l_V+|\gamma_1|V_0+|\gamma_2|I_0+\frac{2j_0}{\sqrt{C}}\right)|t-\bar{t}|+$$
$$+\frac{1}{v}\left(l_V+|\gamma_1|V_0+|\gamma_2|I_0+\frac{j_0}{\sqrt{C}}\right)|x-\bar{x}|\le$$
$$\le\left(l_V+|\gamma_1|V_0+|\gamma_2|I_0+\frac{2j_0}{\sqrt{C}}\right)(|t-\bar{t}|+|x-\bar{x}|)\le$$
$$\le l_I(|t-\bar{t}|+|x-\bar{x}|).$$

It is easy to see that functions $(\tilde{V}_n,\tilde{I}_n)$ are extended in such a way that they remain equicontinuous family on Π. Therefore $(\tilde{V}_n,\tilde{I}_n)$ contains a convergent sub-sequence whose limit we call a generalized solution of (14).

VI. Conclusion

In fact we have no uniqueness of solution but we have obtained a sequence tending to solution which is difficult for compact and densifying operators. A specific algorithm of constructing such a sequence we will give in a next paper.

Here we collect all inequalities from the proof of the above theorems and show they are satisfied for real data. Indeed for sufficiently small V_{00}, I_{00} we have:

$$\max\{V_{00};\alpha\sqrt{C}E_0+|\beta|I_0\}+\frac{j_0\pi(V_0+I_0)}{\mu^2\Phi_0C}\le V_0;$$
$$\max\left\{I_{00};V_0e^{-\frac{\mu}{nv}}+e^{-\frac{\mu}{nv}}\frac{|\gamma_1|V_0+|\gamma_2|I_0}{\mu}\right\}+\frac{j_0\pi(V_0+I_0)}{\mu^2\Phi_0C}\le I_0;$$
$$K_V=|\beta|+\frac{2j_0\pi}{\mu^2\Phi_0C}<1;$$
$$K_I=e^{-\frac{\mu}{nv}}+\frac{|\gamma_2|}{\mu}e^{-\frac{\mu}{nv}}+\frac{2}{\mu^2}\frac{j_0\pi}{\Phi_0C}<1;$$
$$|\beta|l_I+\frac{2j_0}{\sqrt{C}}\le l_V;\ l_V+|\gamma_1|V_0+|\gamma_2|I_0+\frac{2j_0}{\sqrt{C}}\le l_I.$$

Let us consider a Josephson transmission line (cf. [19], [20]) with $L=2{,}5.10^{-9}\,H/m,\ C=1{,}3.10^{-6}\,F/m$, $\sqrt{C}=1{,}14.10^{-3},\sqrt{L}=5.10^{-5},v=1/(1{,}14.10^{-3}.5.10^{-5})=1{,}75.10^{7}$, characteristic impedance $Z_0=\sqrt{L/C}=\sqrt{(2{,}5.10^{-9})/(1{,}3.10^{-6})}$ $\approx 0{,}044\Omega$, $\Phi_0=2.10^{-15}W/m^2$, $j_0=1{,}9\,A/m$. Let us take $R_1=R_0=Z_0,\ C_0=10^{-11}F$. Then

$$\alpha=\frac{2Z_0}{Z_0+R_0}=1,\ \beta=\frac{Z_0-R_0}{Z_0+R_0}=0,\ \gamma_1=\frac{R_1-Z_0}{C_0Z_0R_1}=0,$$
$$\gamma_2=\frac{R_1+Z_0}{C_0Z_0R_1}\approx 4{,}5.10^{12};\ j_0\pi=5{,}97;\ j_0/\sqrt{C}=1{,}9/(1{,}14.10^{-3})\approx$$
$$\approx 1{,}66.10^{3};\ (\pi j_0)/(\Phi_0C)\approx 2{,}6.10^{21}.$$

For length $\Lambda=10^{-1}m$ we have to choose the accuracy at least $1/n=10^{-5}$. Obviously we must choose at least $\mu=10^{13}$ and therefore $\exp(-\mu/nv)=$ $=\exp(-(10^{13}.10^{-5})/(1{,}75.10^{7}))\approx\exp(-5{,}71)\approx 3{,}3.10^{-3}$.

Then the above inequalities for sufficiently small initial data become:

$$1{,}14.10^{-3}E_0+\frac{2{,}6.10^{21}}{10^{26}}(V_0+I_0)\le V_0;$$
$$\left(3{,}3.10^{-3}+\frac{2{,}6.10^{21}}{10^{26}}\right)V_0+\left(\frac{4{,}5.10^{12}}{10^{13}}+\frac{2{,}6.10^{21}}{10^{26}}\right)I_0\le I_0;$$
$$K_V=\frac{2}{10^{26}}2{,}6.10^{21}<1;$$
$$K_I=3{,}3.10^{-3}\left(1+\frac{4{,}5.10^{12}}{10^{13}}\right)+\frac{2}{10^{26}}2{,}6.10^{21}\approx 3{,}3.10^{-3}.1{,}45<1;$$
$$2.1{,}66.10^{3}\le l_V;$$
$$l_V+4{,}5.10^{12}I_0+2.1{,}66.10^{3}\le l_I.$$

It should be noted that the actual physical quantities must be calculated by the formulas

$$u(x,t)=V(x,t)/(2\sqrt{L})+I(x,t)/(2\sqrt{L})$$
$$i(x,t)=\ V(x,t)/(2\sqrt{C})-I(x,t)/(2\sqrt{C})$$

or

$$u(x,t)=10^4V(x,t)+10^4I(x,t)$$
$$i(x,t)=\ 4{,}4.10^2\ V(x,t)-4{,}4.10^2I(x,t).$$

So if we want to get voltage order 10^0 we have to choose $V_0=I_0\approx 10^{-4}$.

The above example shows that in contrast to [16] we obtain a generalized solution on the whole rectangle $[0,\Lambda]\times[0,T]$.

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