Fixed Points of Expansion Mappings in Fuzzy Menger Spaces with Property (E.A)

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***Abstract*—The aim of this paper is to prove a common fixed point theorem for non-surjective expansion mappings in Fuzzy Menger space employing the property (E.A).**

Keywords—Fuzzy Menger space, non-surjective mappings, weakly compatible mappings, expansion mappings, property (E.A).

1. Introduction

Fréchet [3] introduced the concept of metric space in which notion of distance appears. An essential feature is the fact that, for any two points in the space, there is defined a positive number called the distance between the two points. However, in practice we find very often that this association of a single number for each pair is, strictly speaking, an over-idealization. Therefore, Menger [8] introduced the concept of probabilistic metric space (briefly, PM-space) as a generalization of metric space.

Banach contraction principle [1] is an important tool in the theory of metric spaces. Due to its simplicity and usefulness, it became a very popular tool in solving existence problems in pure and applied sciences such as biology, medicine, physics, and computer science. Probabilistic contractions were first defined and studied by Sehgal [12]. Banach contraction principle [1] also yields a fixed point theorem for a diametrically opposite class of mappings, viz. expansion mappings. The study of metrical fixed point theorem for expansion mapping is initiated by Wang et al. [17] . Since then, Pant et al. [10] studied fixed point theorem for expansion mappings in framework of probabilistic metric spaces . and so many authors [2], [4], [12],[14] and[16] worked on this topic. Rajesh Shrivastav, Vivek Patel and Vanita Ben Dhagat[15] have given the definition of fuzzy probabilistic metric space and proved fixed point theorem for such space.

2. Preliminaries

Definition 2.1 A fuzzy probabilistic metric space (FPM space) is an ordered pair (X,F_α) consisting of a nonempty set X and a mapping F_α from XxX into the collections of all fuzzy distribution functions $F_\alpha \in R$ for all α. ∈ [0,1]. For x, y ∈ X we denote the fuzzy distribution function F_α (x,y) by $F_{\alpha(x,y)}$ and $F_{\alpha(x,y)}(u)$ is the value of $F_{\alpha(x,y)}$ at u in R.

The functions $F_{\alpha(x,y)}$ for all α. ∈ [0,1] assumed to satisfy the following conditions:

(a) $F_{\alpha(x,y)}(u) = 1 \ \forall\ u > 0$ iff x = y,

(b) $F_{\alpha(x,y)}(0) = 0 \ \forall$ x , y in X,

(c) $F_{\alpha(x,y)} = F_{\alpha(y,x)} \ \forall$ x , y in X,

(d) If $F_{\alpha(x,y)}(u) = 1$ and $F_{\alpha(y,z)}(v) = 1 \Rightarrow F_{\alpha(x,z)}(u+v) = 1 \ \forall$ x , y ,z ∈X and u, v > 0.

Definition 2.2 A commutative, associative and non-decreasing mapping t: [0,1] × [0,1]→ [0,1] is a t-norm if and only if t(a,1)= a ∀a∈[0,1] , t(0,0)=0 and t(c,d) ≥ t(a,b) for c ≥ a, d ≥ b .

Definition 2.3 A Fuzzy Menger space is a triplet (X,F_α,t), where (X,F_α) is a FPM-space, t is a t-norm and the generalized triangle inequality

$$F_{\alpha(x,z)}(u+v) \geq t\,(F_{\alpha(x,z)}(u), F_{\alpha(y,z)}(v))$$

holds for all x, y, z in X u, v > 0 and α. ∈ [0,1].

The concept of neighborhoods in Fuzzy Menger space is introduced as

Definition 2.4 Let (X,F_α,t) be a Fuzzy Menger space. If x ∈ X, ε > 0 and λ ∈(0,1), then (ε,λ) - neighborhood of x, called $U_x(\varepsilon,\lambda)$, is defined by

$$U_x(\varepsilon,\lambda) = \{y\in X: F_{\alpha(x,y)}(\varepsilon)>(1-\lambda)\}.$$

An (ε,λ)-topology in X is the topology induced by the family $\{U_x(\varepsilon,\lambda): x \in X, \varepsilon > 0, \alpha \in [0,1] \text{ and } \lambda\in(0,1)\}$ of neighborhood.

Remark: If t is continuous, then Fuzzy Menger space (X,F_α,t) is a Housdroff space in (ε,λ)-topology.

Let (X,F_α,t) be a complete Fuzzy Menger space and A⊂X. Then A is called a bounded set if

$$\lim_{u\to\infty}\ \inf_{x,y\in A} F_{\alpha(x,y)}(u) = 1$$

Definition 2.5 A sequence $\{x_n\}$ in (X,F_α,t) is said to be convergent to a point x in X if for every ε>0and λ>0, there exists an integer N=N(ε,λ) such that $x_n \in U_x(\varepsilon,\lambda)$ ∀n ≥ N or equivalently $F_\alpha(x_n, x; \varepsilon) > 1-\lambda$ for all n ≥ N and α∈[0,1].

Definition 2.6 A sequence $\{x_n\}$ in (X,F_α, t) is said to be cauchy sequence if for every ε > 0 and λ > 0, there exists an integer N=N(ε,λ) such that for all α∈[0,1] $F_\alpha(x_n,x_m; \varepsilon) > 1-\lambda$ ∀ n, m ≥ N.

Definition 2.7 A Fuzzy Menger space (X,F_α,t) with the continuous t-norm is said to be complete if every

Cauchy sequence in X converges to a point in X for all α∈[0,1].

Following lemmas are selected from [8] and [12] respectively in fuzzy menger space.

Lemma 2.1. Let $\{x_n\}$ be a sequence in a Fuzzy Menger space (X, F_α,t) with continuous t-norm ∗ and t ∗ t ≥ t. If there exists a constant k ∈ (0, 1) such that

$F_{\alpha(x_n, x_{n+1})}(kt) \geq F_{\alpha(x_{n-1}, x_n)}(t)$ for all t > 0 , α∈[0,1].and n = 1, 2, . . .,

then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.2 . Let (X, F_α, t) be a Fuzzy Menger space. If there exists k ∈ (0, 1) such that

$F_{\alpha(x,y)}(kt) \geq F_{\alpha(x,y)}(t)$ for all x, y ∈ X , for all α∈[0,1] and t > 0, then x = y.

Definition 2.8[5] A pair (A, S) of self mappings of a non-empty set X is said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if $Az = Sz$ some $z \in X$, then $ASz = SAz$.

Two compatible self-maps are weakly compatible, but the converse is not true (see [13, Example 1]). Therefore the concept of weak compatibility is more general than that of compatibility.

Definition 2.9[6] A pair (A, S) of self mappings of a Fuzzy Menger space (X,F_α,t) is said to satisfy the property (E.A), if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$.

3. Main Results

Now we prove our main result:

Theorem 3.1. Let A, B, S and T be four self mappings of a Fuzzy Menger space (X,F_α,t). Suppose that

(3.1) (A, S) (or (B, T)) satisfies the property (E.A);

(3.2) $T(X) \subseteq A(X)$, $S(X) \subseteq B(X)$;

(3.3) (A, S) and (B, T) are weak compatible

(3.4) One of the range of the mappings A, B, S or T is a closed subspace of X. (3.5) There exists a constant $k > 1$ such that

$$F_{\alpha(Ax,By)}(kt) \leq F_{\alpha(Sx,Ty)}(t),$$

for all $x, y \in X$, for all α∈[0,1] and $t > 0$.

Then A, B, S and T have a unique common fixed point in X.

Proof. If the pair (B, T) satisfies the property (E.A), then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z,$$

for some $z \in X$ as $n \to \infty$.

Since $S(X) \subseteq B(X)$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$. Hence, $\lim_{n\to\infty} Sy_n = z$. Also, since $T(X) \subseteq A(X)$, there exists a sequence $\{y'_n\}$ in X such that $Ay'_n = Tx_n$ and so $\lim_{n\to\infty} Ay'_n = z$.

Assume that $S(X)$ is a closed subspace of X, then there exists a point $u \in X$ such that $z = Su$. By inequality (3.4), we have

$$F_{\alpha(Au,\, Bx_n)}(kt) \leq F_{\alpha(Su,\, Tx_n)}(t).$$

On letting $n \to \infty$, we get

$$F_{\alpha(Au,\, z)}(kt) \leq F_{\alpha(z,\, z)}(t) = 1,$$

for all $t > 0$, α∈[0,1]. and $k > 1$. By Lemma 2.2 we have $Au = z$ and hence $Au = Su = z$.

The weak compatibility of A and S implies that $Az = ASu = SAu = Sz.$ Now, we assert that z is a common fixed point of A and S. From inequality (3.4), we have

$$F_{\alpha(Az,\, Bx_n)}(kt) \leq F_{\alpha(Sz,\, Tx_n)}(t).$$

On letting $n \to \infty$, we get

$$F_{\alpha(Az,\, z)}(kt) \leq F_{\alpha(Az,\, z)}(t),$$

By Lemma 2.2, we have $Az = Sz = z$. On other hand, since $S(X) \subseteq B(X)$, there exists a $v \in X$ such that $Bv = Su = Au = z.$ On using inequality (3.4), we have

$$F_{\alpha(Au,\, Bv)}(kt) \leq F_{\alpha(Su,\, Tv)}(t),$$

or equivalently,

$$F_{\alpha(z,\, Bv)}(kt) \leq F_{\alpha(z,\, z)}(t),$$

for all $t > 0$, α∈[0,1] and $k > 1$. In view of Lemma 2.2, we get $Bv. = Tv = z$.

Similarly, the weak compatibility of B and T implies that $Bz = BTv = TBv = Tz$. By inequality (3.4), we have

$$F_{\alpha(Au,Bz)}(kt) \le F_{\alpha(Su,Tz)}(t),$$

and so

$$F_{\alpha(z,Bz)}(kt) \le F_{\alpha(z,Bz)}(t).$$

Owing to Lemma 2.2, we have $Bz = Tz = z$. Thus in all, we have $Az = Bz = Sz = Tz = z$ which shows that z is a common fixed point of mappings A, B, S and T.

Finally, we prove the uniqueness of z. Let $w(\ne z)$ be another common fixed point of involved mappings A, B, S and T then using (3.4), we have

$$F_{\alpha(Az,Bw)}(kt) \le F_{\alpha(Sz,Tw)}(t),$$

or, equivalently,

$$F_{\alpha(z,w)}(kt) \le F_{\alpha(z,w)}(t).$$

Appealing to Lemma 2.2, it follows that $z = w$. This completes the proof.

The proof is similar if we assume that one of the subspace $B(X), S(X)$ or $T(X)$ is closed.

Remark 3.1. The conclusion of Theorem 3.1 remains true if we replace inequality (3.4) by one of the following: for all $k > 1$, $x, y > 0$, α∈[0,1] and $t > 0$

(3.5) $F_{\alpha(Ax,By)}(kt) \le \min\{F_{\alpha(Sx,Ty)}(t), F_{\alpha(Ax,Sx)}(t), F_{\alpha(By,Ty)}(t)\}$,

(3.6) $(F_{\alpha(Ax,By)}(kt))^2 \le F_{\alpha(Ax,Sx)}(t)\, F_{\alpha(By,Ty)}(t)$

By setting $A = B$ and $S = T$ in Theorem 3.1, we can obtain a natural result for a pair of self mappings.

Corollary 3.1. Let A and S be two self mappings of a Fuzzy Menger space (X, F_α, t). Suppose that

(3.10) $S(X) \subseteq A(X)$;

(3.11) (A, S) satisfies the property (E.A);

(3.12) One of the range of the mappings A or S is a closed subspace of X,

(3.13) There exists a constant $k > 1$ such that

$$F_{\alpha(Ax,Ay)}(kt) \le F_{\alpha(Sx,Sy)}(t),$$

for all $x, y \in X$, α∈[0,1]. and $t > 0$.

Then A and S have a unique common fixed point in X.

Conclusion

Theorem 3.1 is a generalization of some results in the sense it is proved for non-surjective mappings under weak compatibility which is more general than compatibility and without any requirement of completeness of the whole space and continuity of the involved mappings.

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